DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **6**, Number **4**, August **2013**

doi:10.3934/dcdss.2013.6.909

pp. **909–923**

FATIGUE ACCUMULATION IN AN OSCILLATING PLATE

Michela Eleuteri

Dipartimento di Matematica, Università degli Studi di Milano
via Saldini 50,
20133 Milano. Italy

Jana Kopfová

Mathematical Institute of the Silesian University
Na Rybníčku 1
746 01 Opava, Czech Republic

Pavel Krejčí

Institute of Mathematics, Academy of Sciences of the Czech Republic
Žitná 25
11567 Praha 1, Czech Republic

Abstract. A thermodynamic model for fatigue accumulation in an oscillating elastoplastic Kirchhoff plate based on the hypothesis that the fatigue accumulation rate is proportional to the dissipation rate, is derived for the case that both the elastic and the plastic material characteristics change with increasing fatigue. We prove the existence of a unique solution in the whole time interval before a singularity (material failure) occurs under the simplifying hypothesis that the temperature history is a priori given.

Introduction. In this paper, we pursue the study of cyclic fatigue accumulation in oscillating elastoplastic systems started in [6, 7]. The main goal of our project is to model the most important experimental features of fatigue, such as *material softening*, *heat release*, and *material failure in finite time*. The analysis of the so-called rainflow method of cyclic damage evaluation carried out in [2] has shown a qualitative and quantitative correspondence between the damage accumulation rule and the energy dissipation. On the other hand, experimental measurements at the point of material failure confirm strong temperature increase, which manifests an energy dissipation peak. In fact, temperature tests are regularly used in engineering practice for damage analysis in high frequency regimes (e.g. in aircraft industry). Our substantial modeling hypothesis thus consists in introducing a scalar fatigue parameter m, assuming that its time derivative (the fatigue rate) is proportional to the dissipation rate, and that the material parameters depend on m. We believe that this assumption is realistic. Plastic deformations are driven by moving dislocations and ruptures of interatomic connections, which at the same time dissipate energy, and reduce the cohesion of the solid. Note that in the Gurson model for void nucleation and growth in elastoplastic materials, see [10], the elasticity

2010 *Mathematics Subject Classification.* Primary: 74R20; Secondary: 74H35, 74N30.

Key words and phrases. Elastoplastic plate, Prandtl-Ishlinskii operator, material fatigue.

Supported by GAČR Grant P201/10/2315. The work of J.K. was also supported by RVO: 47813059 and by the project MSM4781305904 of the Czech Ministry of Education. The work of M.E. was partially supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase).

domain shrinks, being parameterized by the plastic dissipation rate. For a more detailed discussion about the modeling issues, see [6]. In the previous papers, we have considered the case that only the hardening modulus depends on m, and that the plastic characteristics are not altered by fatigue (but may possibly depend on temperature). Here, we include the fatigue dependence into the plastic constitutive law as well.

The PDE system of momentum and energy balance equations for transversal oscillations of an elastoplastic plate under fatigue is derived in Section 1. The unknowns of the full problem are w (the transversal displacement), θ (absolute temperature), and m (fatigue). In this paper, however, we do not prove the well-posedness of the complete system resulting from a thermodynamic analysis. We only make a first step in this direction and solve the momentum balance equation coupled with the fatigue accumulation equation, assuming that the temperature history is known. An existence and uniqueness theory for the full system will be the subject of a subsequent paper.

It cannot be expected that solutions of the system with fatigue exist globally in time. The material failure in finite time is an integral part of the model. We give an efficient lower bound for the existence time.

The paper is organized as follows. In Section 1, we derive the model equations from thermodynamic principles. The mathematical problem is stated in Section 2. Some new properties of the vectorial Prandtl-Ishlinskii operator are proved in Section 3, and the proof of the main existence and uniqueness theorems is carried out in Sections 4–6.

1. **The model.**

An elastoplastic plate subject to bending exhibits plasticized zones occurring first on the boundary and propagating towards the interior. Assuming a single yield von Mises plasticity criterion in the original 3D setting (cf. [16]), the Kirchhoff dimensional reduction to 2D carried out in [9] was shown to give rise to a Prandtl-Ishlinskii (also called "generalized St. Venant", see [16]) constitutive relation with a continuum of yield surfaces that are successively activated in agreement with natural expectations: The midsurface of the plate is subject to smaller deformations than eccentric layers, therefore plastic yielding occurs earlier far from the midsurface.

This emerging multiyield character of the elastoplastic plate bending problem does not seem to have been taken into consideration before. For instance in [1] there is no direct reference to plates; in [3, 13, 18] the yield condition is still described by one sharp surface of plasticity. The methods of the papers [8, 19, 14] based on Γ-convergence of energy minimizers have been only recently refined in [15] to obtain the Prandtl-Ishlinskii model in the Γ-limit as well. The drawback of the Γ-limit technique is that it cannot be extended to the study of oscillating systems, and of nonequilibrium problems in general.

After dimensional reduction and integration over the thickness, the strain and stress tensors are transformed into functions of the space variable x from a domain $\Omega \subset \mathbb{R}^2$ and the time variable $t \in (0,T)$, and have only three independent components. The elastoplastic plate problem is derived in [9] in the form

$$\int_\Omega \big(\varrho w_{tt}(x,t)(\varphi - \frac{h^2}{12}\Delta\varphi)(x) + \sigma(x,t)\cdot D_2\varphi(x)\big)\,\mathrm{d}x = \int_\Omega g(x,t)\varphi(x)\,\mathrm{d}x \qquad (1.1)$$

$$\forall \varphi \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\,,$$

$$\varepsilon = D_2 w = \begin{pmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{pmatrix}, \tag{1.2}$$

$$\sigma = \mathbf{B}\varepsilon + \int_0^\infty \gamma(r)\,\mathbf{K}\chi_r\,\mathrm{d}r\,, \tag{1.3}$$

where w is the transversal displacement, ε and σ are weighted averages of strain and stress tensors over the thickness of the plate (i.e. bulk moments), $\varrho > 0$ is the constant mass density, $h > 0$ is the thickness of the plate, $\mathbf{B}$, $\mathbf{K}$ are positive definite symmetric matrices, g is the external load, $\gamma \in L^1(0,\infty)$ is a given Prandtl-Ishlinskii density function, and χ_r for $r > 0$ are solutions of the family of variational inequalities

$$\left.\begin{array}{ll} \chi_r(x,t) \in rZ & \text{for all } \ x \in \Omega,\ t \in (0,T),\ r > 0\,, \\ \frac{\partial}{\partial t}(\varepsilon - \chi_r)\cdot \mathbf{K}(\chi_r - z) \ge 0 & \text{for all } \ z \in rZ \ \text{ a.e.,} \\ \chi_r(x,0) = Q_{rZ}(\varepsilon(x,0)), & \end{array}\right\} \tag{1.4}$$

constrained to the system of convex closed sets rZ, where $Z \subset \mathbb{R}^3$ is a referential set, and $rZ = \{rz \in X : z \in Z\}$ for $r > 0$. We denote by $Q_{rZ} : \mathbb{R}^3 \to rZ$ the projection onto rZ, orthogonal with respect to the scalar product $\langle \xi, \eta\rangle = \mathbf{K}\xi\cdot\eta$. The mapping $\mathcal{P}$ which associates to each ε (in some suitable space) the integral part of (1.3) is called the *vectorial Prandtl-Ishlinskii operator*, see (3.7).

In [6], we have included the fatigue and temperature dependence into the model by introducing a scalar fatigue parameter $m(x,t) \ge 0$, assuming that the matrix $\mathbf{B}$ depends on m and the Prandtl-Ishlinskii density γ depends on temperature. Here, we let both the matrix $\mathbf{B}$ and the function γ depend on m, and complement the constitutive law (1.3) with viscosity and thermal expansion terms to obtain

$$\sigma = \mathbf{B}(m)\varepsilon + \int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r\,\mathrm{d}r + \mathbf{C}\varepsilon_t - \beta(\theta-\theta_0)\mathbf{1}\,, \tag{1.5}$$

where $\theta > 0$ is the absolute temperature, $\theta_0 > 0$ is a given referential temperature, $\mathbf{1}$ is the vector $(1,1,0)$, β is the thermal expansion coefficient, $\mathbf{C}$ is the viscosity matrix, and $\mathbf{B}(m), \gamma(m,r)$ are functions specified below in Hypothesis 2.1. By analogy to [6, 9], we associate with (1.5) the free energy $\mathcal{F}$ defined by the formula

$$\mathcal{F}[\varepsilon,\theta] = c_0\theta(1-\log(\theta/\theta_0)) + \frac{1}{2}\mathbf{B}(m)\varepsilon\cdot\varepsilon + \frac{1}{2}\int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r - \beta(\theta-\theta_0)\varepsilon\cdot\mathbf{1}\,, \tag{1.6}$$

where c_0 is the constant specific heat capacity. The internal energy $\mathcal{U}$ and the entropy $\mathcal{S}$ thus have the form

$$\mathcal{U}[\varepsilon,\theta] = c_0\theta + \frac{1}{2}\mathbf{B}(m)\varepsilon\cdot\varepsilon + \frac{1}{2}\int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r + \beta\theta_0\varepsilon\cdot\mathbf{1}\,, \tag{1.7}$$

$$\mathcal{S}[\varepsilon,\theta] = c_0\log(\theta/\theta_0) + \beta\varepsilon\cdot\mathbf{1}\,. \tag{1.8}$$

The equations for the state variables θ and m are derived from the first and the second principles of thermodynamics in the form

$$\frac{\partial}{\partial t}\mathcal{U}[\varepsilon,\theta] + \operatorname{div}\mathbf{q} = \sigma\cdot\varepsilon_t\,, \qquad \text{(energy conservation)} \tag{1.9}$$

$$\frac{\partial}{\partial t}\mathcal{S}[\varepsilon,\theta] + \operatorname{div}\frac{\mathbf{q}}{\theta} \ge 0\,, \qquad \text{(Clausius-Duhem inequality)} \tag{1.10}$$

where $\mathbf{q}$ is the heat flux vector that we assume in the form

$$\mathbf{q} = -\kappa\nabla\theta\,, \tag{1.11}$$

with a constant heat conductivity coefficient $\kappa > 0$. Then (1.9) reads

$$\begin{aligned} c_0\theta_t - \kappa\Delta\theta = \mathbf{C}\varepsilon_t\cdot\varepsilon_t - \frac{1}{2}m_t\left(\mathbf{B}'(m)\varepsilon\cdot\varepsilon + \int_0^\infty \gamma_m(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r\right)\\ + \int_0^\infty \gamma(m,r)\frac{\partial}{\partial t}(\varepsilon-\chi_r)\cdot\,\mathbf{K}\chi_r\,\mathrm{d}r - \beta\theta\varepsilon_t\cdot\mathbf{1}\,. \end{aligned} \tag{1.12}$$

The notation is slightly ambiguous, and we hope that the reader will not get confused. For simplicity, we denote by $_t$ and $_m$ partial derivatives with respect to the corresponding variables. The index $_r$ is not a partial derivative. There is no differentiation with respect to r in the paper.

In view of (1.11), we see that the Clausius-Duhem inequality (1.10) is certainly satisfied if

$$\theta\frac{\partial}{\partial t}\mathcal{S}[\varepsilon,\theta] + \sigma\cdot\varepsilon_t - \frac{\partial}{\partial t}\mathcal{U}[\varepsilon,\theta] \ge 0\,, \tag{1.13}$$

that is,

$$\begin{aligned} \mathbf{C}\varepsilon_t\cdot\varepsilon_t - \frac{1}{2}m_t\left(\mathbf{B}'(m)\varepsilon\cdot\varepsilon + \int_0^\infty \gamma_m(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r\right)\\ + \int_0^\infty \gamma(m,r)\frac{\partial}{\partial t}(\varepsilon-\chi_r)\cdot\,\mathbf{K}\chi_r\,\mathrm{d}r \ge 0\,. \end{aligned} \tag{1.14}$$

The last integral term in (1.14) is nonnegative by virtue of (1.4). The assumption that the fatigue accumulation rate m_t is nonnegative (that is, fatigue can only increase in time) is therefore compatible with the second principle provided

$$\mathbf{B}'(m)\ \text{ is negative semidefinite}\,,\quad \gamma_m(m,r)\le 0\ \text{ a.e.} \tag{1.15}$$

In other words, *material softening takes place under increasing fatigue* in agreement with experimental evidence similarly as in [10].

We close the system by assuming that the fatigue accumulation rate m_t at a point $x\in\Omega$ is proportional to the dissipation rate averaged over a neighborhood of the point x, that is,

$$\begin{aligned} m_t(x,t) = \int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\frac{\partial}{\partial t}(\varepsilon-\chi_r)\cdot\,\mathbf{K}\chi_r(y,t)\,\mathrm{d}y\,\mathrm{d}r \\ -\frac{1}{2}\int_\Omega \lambda(x-y)m_t(y,t)\left(\mathbf{B}'(m)\varepsilon\cdot\varepsilon + \int_0^\infty \gamma_m(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r\right)(y,t)\,\mathrm{d}y\,, \end{aligned} \tag{1.16}$$

where $\lambda\in L^\infty(\mathbb{R}^2)$ is a given nonnegative function with compact support. The full system then consists of Eqs. (1.1), (1.2), (1.4), (1.5), (1.12), (1.16), and its solutions satisfy the thermodynamic principles (1.9), (1.10).

2. **Statement of the problem.**

As a first step towards the full system (1.1), (1.2), (1.4), (1.5), (1.12), (1.16), we assume that a function $\theta:\Omega\times(0,T)\to\mathbb{R}^3$ describing a combined action of thermal expansion and external load is given, set the physical constants to 1 for simplicity, and consider the problem

$$\begin{aligned} \int_\Omega \big(w_{tt}(x,t)(\varphi-\Delta\varphi)(x) + \mathbf{C}D_2w_t(x,t)\cdot D_2\varphi(x)\big)\,\mathrm{d}x \\ = \int_\Omega \big(\theta - P[m,\varepsilon]\big)(x,t)\cdot D_2\varphi(x)\,\mathrm{d}x \qquad \forall\varphi\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)\,, \end{aligned} \tag{2.1}$$

$$\varepsilon = D_2 w = \begin{pmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{pmatrix}, \tag{2.2}$$

$$P[m,\varepsilon] = \mathbf{B}(m)\varepsilon + \int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r \,\mathrm{d}r\,, \tag{2.3}$$

$$\begin{aligned} m_t(x,t) = &\int_\Omega \lambda(x-y) \int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r \cdot (\xi_r)_t \,\mathrm{d}r\,\mathrm{d}y \\ &- \frac{1}{2}\int_\Omega \lambda(x-y) m_t(y,t) \Big(\mathbf{B}'(m)\varepsilon\cdot\varepsilon + \int_0^\infty \gamma_m(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r\Big)(y,t)\,\mathrm{d}y \end{aligned} \tag{2.4}$$

$$w(x,0) = w_t(x,0) = m(x,0) = 0 \quad \forall x \in \Omega\,, \tag{2.5}$$

where we denote $\xi_r = \varepsilon - \chi_r$. The following hypotheses are assumed to hold.

Hypothesis 2.1. *We fix a Lipschitzian domain $\Omega \subset \mathbb{R}^2$, and denote by $|\cdot|_p$ the $L^p(\Omega)$ norm for $p \geq 1$. We assume that there exists a constant $\nu > 0$ such that for all $\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ we have*

$$|D_2\varphi|_2 \leq \nu |\Delta\varphi|_2\,, \tag{2.6}$$

and for $T > 0$ we denote $\Omega_T = \Omega \times (0,T)$. Furthermore,

- (i) $\mathbf{C}$, $\mathbf{K}$ *are symmetric positive definite 3×3 matrices, and there exist constants $c^*, c_* > 0$ such that $\mathbf{C}\varepsilon\cdot\varepsilon \geq c_*|\varepsilon|^2$, $\mathbf{K}\varepsilon\cdot\varepsilon \geq c_*|\varepsilon|^2$, $|\mathbf{K}\varepsilon| \leq c^*|\varepsilon|$ for all $\varepsilon \in \mathbb{R}^3$;*
- (ii) *$\lambda \in L^\infty(\mathbb{R}^2)$ is a given function with compact support, and $\Lambda > 0$ is a constant such that $0 \leq \lambda \leq \Lambda$ a.e.;*
- (iii) *$\mathbf{B}(m)$ for $m \geq 0$ is a symmetric positive semidefinite 3×3 matrix, and there exists $b_* > 0$ such that $|\mathbf{B}(m)\varepsilon| \leq b_*|\varepsilon|$ for all $\varepsilon \in \mathbb{R}^3$. Moreover, $\mathbf{B}'(m)$ is negative semidefinite and depends Lipschitz continuously on m, $\mathbf{B}'(0) = \mathbf{0}$;*
- (iv) *$\gamma : [0,\infty)\times(0,\infty) \to [0,\infty)$ is a given C^2-function such that $\gamma_m(m,r) \leq 0$ a.e., $\gamma_m(0,r) = 0$ a.e., and there exists a constant $\Gamma > 0$ such that*

$$\int_0^\infty (\gamma(m,r) + \gamma_m(m,r) + \gamma_{mm}(m,r))(1+r^2)\,\mathrm{d}r \leq \Gamma \quad \forall m > 0;$$

- (v) *$\theta \in L^2(\Omega_T;\mathbb{R}^3)$ is a given function for some $T > 0$;*
- (vi) *$Z \subset \mathbb{R}^3$ is a bounded convex closed domain with boundary of class $W^{2,\infty}$, $0 \in \mathrm{Int}\, Z$.*

We prove in the next sections the following existence and uniqueness results.

Theorem 2.2. *Let Hypothesis 2.1 hold and let $R > 0$ be given. Then there exists $T^R \in (0,T]$ and a unique solution (w,m) such that $w_{tt} \in L^2(\Omega_{T^R})$, $w_t \in L^2(0,T^R; W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))$, $m \in L^\infty(\Omega_{T^R}), m_t \in L^\infty(\Omega_{T^R})$, equations (1.4), (2.1)–(2.3), and (2.5) are satisfied almost everywhere, and (2.4) is replaced by*

$$\begin{aligned} m_t(x,t) = Q_R&\left(\int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r\cdot(\xi_r)_t\,\mathrm{d}r\,\mathrm{d}y\right) \\ &- \frac{1}{2}\int_\Omega \lambda(x-y)m_t(y,t)\Big(\mathbf{B}'(m)\varepsilon\cdot\varepsilon + \int_0^\infty \gamma_m(m,r)\,\mathbf{K}\chi_r\cdot\chi_r\,\mathrm{d}r\Big)(y,t)\,\mathrm{d}y, \end{aligned} \tag{2.7}$$

where $Q_R(z) = \min\{R, \max\{z,-R\}\}$ is the projection of $\mathbb{R}$ onto $[-R,R]$.

Theorem 2.3. *In addition to Hypothesis 2.1, assume that* $\theta_t \in L^2(\Omega_T;\mathbb{R}^3)$. *Then there exists* $T^* \in (0,T]$ *and a unique solution* (w,m) *to* (1.4), (2.1)–(2.5) *with the additional regularity* $w_{tt} \in L^2(0,T^*;W_0^{1,2}(\Omega))$, $w_t \in L^\infty(0,T^*;W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))$. *Furthermore, there exists a constant* $C>0$ *such that if* $\theta^\sharp,\theta^\flat$ *are two functions satisfying the hypotheses and if* $w^\sharp,w^\flat$ *are the corresponding solutions, then the differences* $\bar w = w^\sharp - w^\flat, \bar\theta = \theta^\sharp - \theta^\flat$ *satisfy the inequality*

$$|\bar w_t|_2^2(t) + |\nabla \bar w_t|_2^2 + \int_0^t |D_2 \bar w_t|_2^2(\tau)\,\mathrm{d}\tau \le C\int_0^t |\bar\theta|_2^2(\tau)\,\mathrm{d}\tau \qquad \forall t\in[0,T^*]. \tag{2.8}$$

3. **Vectorial Prandtl-Ishlinskii operator.** We list some basic properties of the vectorial Prandtl-Ishlinskii operator. The original Prandtl-Ishlinskii construction ([20, 11]) is one dimensional. A vector Prandtl-Ishlinskii model is based on the concept of *stop operator* introduced in [12], which we recall here in an abstract framework for the reader's convenience.

Consider a real separable Hilbert space X endowed with a scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. For each function $\varepsilon \in W^{1,1}(0,T;X)$, we define $\chi \in W^{1,1}(0,T;X)$ as the unique solution of the variational inequality

$$\left.\begin{aligned} &\chi(t)\in Z \quad \forall t\in[0,T]\,,\\ &\chi(0) = Q_Z(\varepsilon(0))\,,\\ &\langle \varepsilon_t(t)-\chi_t(t), \chi(t)-z\rangle \ge 0 \ \text{ a.e.} \quad \forall z\in Z\,, \end{aligned}\right\} \tag{3.1}$$

where $Q_Z : X\to Z$ is the orthogonal projection onto Z. The solution mapping

$$\mathfrak{s}_Z : W^{1,1}(0,T;X)\to W^{1,1}(0,T;X) : \varepsilon\mapsto\chi \tag{3.2}$$

is called the *stop with characteristic* Z.

The variational inequality (1.4) fits with the framework of (3.1), provided we choose in (1.4)

$$X=\mathbb{R}^3\,, \quad \langle\xi,\eta\rangle = \mathbf{K}\xi\cdot\eta\,, \quad \chi_r = \mathfrak{s}_{rZ}[\varepsilon]\,. \tag{3.3}$$

We rewrite Hypothesis 2.1 (vi) in this abstract setting as follows.

Hypothesis 3.1. *The set* Z *is bounded, convex, closed, and there exist* $C>c>0$ *such that*

$$B_c(0)\subset Z\subset B_C(0) \tag{3.4}$$

where $B_\rho(z)$ *for* $\rho>0$ *and* $z\in X$ *denotes the open ball centered at* z *with radius* ρ. *The boundary of* Z *is smooth in the sense that the unit outward normal mapping is Lipschitz continuous with respect to the norm in* X.

Analytical properties of the stop operator were studied in detail in [5, Chapter 2], and we list here without proof some of them that are needed in the sequel.

Proposition 3.2. *Assume that a set* Z *satisfies Hypothesis 3.1. Then the mapping* $\mathfrak{s}_Z$ *defined by (3.1)–(3.2) has the following properties.*

(i) $\mathfrak{s}_Z$ *is locally Lipschitz continuous in* $W^{1,1}(0,T;X)$;
(ii) $\langle \varepsilon_t(t) - (\mathfrak{s}_Z[\varepsilon])_t(t), (\mathfrak{s}_Z[\varepsilon])_t(t)\rangle = 0$ *a.e.;*
(iii) $\mathfrak{s}_Z$ *can be extended to a continuous mapping* $C([0,T];X)\to C([0,T];X)$;
(iv) *The mapping* $\mathfrak{s}_Z$ *is monotone in the sense that*

$$\left\langle \mathfrak{s}_Z[\varepsilon^\sharp](t) - \mathfrak{s}_Z[\varepsilon^\flat](t), \varepsilon_t^\sharp(t)-\varepsilon_t^\flat(t)\right\rangle \ge \frac12\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathfrak{s}_Z[\varepsilon^\sharp](t)-\mathfrak{s}_Z[\varepsilon^\flat](t)\right|^2 \ \text{a.e.} \tag{3.5}$$

for every $\varepsilon^\sharp,\varepsilon^\flat\in W^{1,1}(0,T;X)$.

As an immediate consequence of (3.5), we have for all $t \in [0,T]$ that

$$\left|\mathfrak{s}_Z[\varepsilon^\sharp](t) - \mathfrak{s}_Z[\varepsilon^\flat](t)\right| \le \left|\mathfrak{s}_Z[\varepsilon^\sharp](0) - \mathfrak{s}_Z[\varepsilon^\flat](0)\right| + \int_0^t |\varepsilon^\sharp_t - \varepsilon^\flat_t|(\tau)\,\mathrm{d}\tau\,. \tag{3.6}$$

Given a nonnegative function $\gamma \in L^1(0,\infty)$, we define the *vectorial Prandtl-Ishlinskii operator* $\mathcal{P}$ with characteristic Z and density γ by the formula

$$\mathcal{P}[\varepsilon](t) = \int_0^\infty \gamma(r)\,\mathfrak{s}_{rZ}[\varepsilon](t)\,\mathrm{d}r \tag{3.7}$$

for $\varepsilon \in W^{1,1}(0,T;X)$. The definition is meaningful due to the fact that, setting $\varepsilon_\infty = \max\{|\varepsilon(t)| : t \in [0,T]\}$, we have $\mathfrak{s}_{rZ}[\varepsilon](t) = \varepsilon(t)$ for all $r > \varepsilon_\infty/c$ and all $t \in [0,T]$, where c is as in (3.4).

By virtue of Proposition 3.2, both $\mathcal{P} : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ and $\mathcal{P} : C([0,T];X) \to C([0,T];X)$ are continuous with respect to the strong topologies.

We need here a special Lipschitz continuity result (Theorem 3.3 below) which is not explicitly stated in the literature. Let us introduce first some necessary concepts.

With the convex closed set Z, we associate the *Minkowski functional* $M_Z : X \to \mathbb{R}^+$ defined by the formula

$$M_Z(\chi) = \inf\left\{s > 0 : \frac{1}{s}\chi \in Z\right\}, \tag{3.8}$$

and the polar set Z^* to Z

$$Z^* = \{\eta \in X : \langle \chi, \eta\rangle \le 1,\ \forall \chi \in Z\}. \tag{3.9}$$

By [4, Lemma 3.1], we have

$$B_{\frac{1}{C}}(0) \subset Z^* \subset B_{\frac{1}{c}}(0),$$

and the inequalities

$$\frac{|\chi|}{C} \le M_Z(\chi) \le \frac{|\chi|}{c}, \qquad c|\eta| \le M_{Z^*}(\eta) \le C|\eta| \tag{3.10}$$

hold for every $\chi, \eta \in X$.

The Minkowski functional of a convex closed set containing 0 is proper, convex, and lower semicontinuous. The smoothness assumption in Hypothesis 3.1 implies that its subdifferential $\partial M_Z(\chi)$ for all $\chi \ne 0$ contains a single vector parallel to the unit outward normal vector $n_Z(\chi/M_Z(\chi))$ taken at the point $\chi/M_Z(\chi)$ on the boundary of Z. We define the *duality mapping* $J_Z : X \to X$ by the formula

$$J_Z(\chi) = M_Z(\chi)\partial M_Z(\chi)\,. \tag{3.11}$$

By Hypothesis 3.1 and [4, Lemma 3.1], there exists $L > 0$ such that

$$|J_Z(\chi) - J_Z(\eta)| \le L|\chi - \eta|, \qquad \forall \chi, \eta \in X\,. \tag{3.12}$$

The Minkowski functionals M_Z and M_{rZ} for $r > 0$ are related through a simple scaling formula. Indeed,

$$M_{rZ}(\chi) = \inf\left\{s > 0 : \frac{1}{s}\chi \in rZ\right\} = \inf\left\{s > 0 : \frac{1}{rs}\chi \in Z\right\} = \frac{1}{r}M_Z(x)\,, \tag{3.13}$$

and

$$\begin{aligned} \xi \in \partial M_{rZ}(\chi) &\Longleftrightarrow \langle \xi, \chi - \eta\rangle \ge M_{rZ}(\chi) - M_{rZ}(\eta) \quad \forall \eta \in X \\ &\Longleftrightarrow \langle r\xi, \chi - \eta\rangle \ge M_Z(\chi) - M_Z(\eta) \quad \forall \eta \in X \end{aligned}$$

$$\iff r\xi \in \partial M_Z(\chi)\,, \tag{3.14}$$

hence $\partial M_{rZ} = \frac{1}{r}\partial M_Z$. We thus conclude that

$$J_{rZ} = \frac{1}{r^2} J_Z\,. \tag{3.15}$$

Formula [5, (3.35)] can be written here in the form

$$M_{Z^*}(J_Z(\chi)) = M_Z(\chi) \quad \forall \chi \in X\,, \tag{3.16}$$

hence, by (3.10),

$$\frac{|\chi|}{|J_Z(\chi)|} = \frac{|\chi|}{M_Z(\chi)}\frac{M_{Z^*}(J_Z(\chi))}{|J_Z(\chi)|} \le C^2 \quad \forall \chi \in X \setminus \{0\}\,. \tag{3.17}$$

In terms of the Minkowski functional, putting $\xi(t) = \varepsilon(t) - \chi(t)$, we can represent the variational inequality (3.1) by the differential inclusion $\chi(t) \in \partial M_{Z^*}(\xi_t(t))$, or, equivalently, by the identity

$$\langle \varepsilon_t(t), \chi(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|\chi(t)|^2\right) + M_{Z^*}(\xi_t(t)) \quad \text{a.e.} \tag{3.18}$$

This is the so-called *energetic formulation*, see [17]. The energetic interpretation of (3.18) is that $\langle \varepsilon_t(t), \chi(t)\rangle$ is the power supplied to the system, part of which is used for the potential increase $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|\chi(t)|^2\right)$, and the other part $\langle \xi_t(t), \chi(t)\rangle = M_{Z^*}(\xi_t(t))$ is dissipated.

We now prove the main result of this section, namely the Lipschitz continuity of the dissipation functional.

Theorem 3.3. *Let $\varepsilon^\sharp, \varepsilon^\flat \in W^{1,1}(0,T;X)$ and $r > 0$ be given, and let χ_r^i, $i = \sharp, \flat$, be solutions of the variational inequalities*

$$\left.\begin{aligned} &\chi_r^i(t) \in rZ \quad \forall t \in [0,T]\,,\\ &\chi_r^i(0) = Q_{rZ}(\varepsilon^i(0))\,,\\ &\left\langle \varepsilon_t^i(t) - (\chi_r^i)_t(t), \chi_r^i(t) - z\right\rangle \ge 0 \quad \textit{a.e.} \quad \forall z \in rZ\,. \end{aligned}\right\} \tag{3.19}$$

Set $\xi_r^i = \varepsilon^i - \chi_r^i$. Then we have

$$\begin{aligned} &\left|\left\langle \chi_r^\sharp, (\xi_r^\sharp)_t\right\rangle - \left\langle \chi_r^\flat, (\xi_r^\flat)_t\right\rangle\right|(t) + r^2C^2\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{1}{2}M_{rZ}^2(\chi_r^\sharp(t)) - \frac{1}{2}M_{rZ}^2(\chi_r^\flat(t))\right|\\ &\quad \le r\frac{C^2}{c}|\varepsilon_t^\sharp - \varepsilon_t^\flat|(t) + (1+2LC^2)|\varepsilon_t^\flat|(t)\int_0^t |\varepsilon_t^\sharp - \varepsilon_t^\flat|(\tau)\,\mathrm{d}\tau \quad \textit{a.e.}, \end{aligned} \tag{3.20}$$

where c, C, L are the constants in (3.4), (3.12).

Proof. Set

$$N(t) = \left|\frac{1}{2}M_{rZ}^2(\chi_r^\sharp(t)) - \frac{1}{2}M_{rZ}^2(\chi_r^\flat(t))\right|\,. \tag{3.21}$$

Formulas (47) and (51) of [4] state that

$$\begin{aligned} &(\xi_r^i(t))_t \neq 0 \qquad (3.22)\\ &\Rightarrow \left\langle J_{rZ}(\chi_r^i(t)), (\xi_r^i(t))_t\right\rangle > 0,\ (\xi_r^i(t))_t = \frac{\left\langle J_{rZ}(\chi_r^i(t)), (\xi_r^i(t))_t\right\rangle}{|J_{rZ}(\chi_r^i(t))|^2} J_{rZ}(\chi_r^i(t)) \quad \text{a.e.,} \end{aligned}$$

$$\left|\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle - \left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| + \frac{\mathrm{d}}{\mathrm{d}t}N(t) \tag{3.23}$$

$$\leq \left|\left\langle J_{rZ}(\chi_r^\sharp), \varepsilon_t^\sharp\right\rangle - \left\langle J_{rZ}(\chi_r^\flat), \varepsilon_t^\flat\right\rangle\right| \quad \text{a.e.}$$

The substantial step consists in proving that

$$\left|\left\langle \chi_r^\sharp, (\xi_r^\sharp)_t\right\rangle - \left\langle \chi_r^\flat, (\xi_r^\flat)_t\right\rangle\right| \leq r^2C^2\left|\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle - \left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| \\ + (1+LC^2)|\varepsilon_t^\flat||\chi_r^\sharp - \chi_r^\flat| \quad \text{a.e.} \tag{3.24}$$

This is obvious if $(\xi_r^\sharp(t))_t = (\xi_r^\flat(t))_t = 0$. If for instance $(\xi_r^\sharp(t))_t \neq 0$, $(\xi_r^\flat(t))_t = 0$, then

$$\left\langle \chi_r^\sharp, (\xi_r^\sharp)_t\right\rangle = \left\langle J_{rZ}(\chi_r^\sharp), \chi_r^\sharp\right\rangle \frac{\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle}{|J_{rZ}(\chi_r^\sharp)|^2} \leq r^2C^2\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle$$

by virtue of (3.15), (3.17), and we obtain (3.24) in a straghtforward way. It remains to consider the case $(\xi_r^\sharp(t))_t \neq 0$, $(\xi_r^\flat(t))_t \neq 0$. Then

$$\begin{aligned}
&\left|\left\langle \chi_r^\sharp, (\xi_r^\sharp)_t\right\rangle - \left\langle \chi_r^\flat, (\xi_r^\flat)_t\right\rangle\right| \\
&= \left|\left\langle J_{rZ}(\chi_r^\sharp), \chi_r^\sharp\right\rangle \frac{\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle}{|J_{rZ}(\chi_r^\sharp)|^2} - \left\langle J_{rZ}(\chi_r^\flat), \chi_r^\flat\right\rangle \frac{\left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle}{|J_{rZ}(\chi_r^\flat)|^2}\right| \\
&\leq \frac{\left\langle J_{rZ}(\chi_r^\sharp), \chi_r^\sharp\right\rangle}{|J_{rZ}(\chi_r^\sharp)|^2}\left|\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle - \left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| \\
&\quad + \left|\left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| \left|\frac{\left\langle J_{rZ}(\chi_r^\sharp), \chi_r^\sharp\right\rangle}{|J_{rZ}(\chi_r^\sharp)|^2} - \frac{\left\langle J_{rZ}(\chi_r^\flat), \chi_r^\flat\right\rangle}{|J_{rZ}(\chi_r^\flat)|^2}\right| \\
&\qquad \leq \frac{|\chi_r^\sharp|}{|J_{rZ}(\chi_r^\sharp)|}\left|\left\langle J_{rZ}(\chi_r^\sharp), (\xi_r^\sharp)_t\right\rangle - \left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| \\
&\qquad\quad + \left|\left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| \frac{|\left\langle J_{rZ}(\chi_r^\flat), \chi_r^\sharp - \chi_r^\flat\right\rangle|}{|J_{rZ}(\chi_r^\flat)|^2} \\
&\qquad\quad + \left|\left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| |\chi_r^\sharp| \left|\frac{J_{rZ}(\chi_r^\sharp)}{|J_{rZ}(\chi_r^\sharp)|^2} - \frac{J_{rZ}(\chi_r^\flat)}{|J_{rZ}(\chi_r^\flat)|^2}\right|.
\end{aligned} \tag{3.25}$$

We now estimate the three terms on the right hand side of (3.25) as follows. Notice first that by Proposition 3.2 (ii), we have $|(\xi_r^\flat)_t| \leq |\varepsilon_t^\flat|$ a.e. Furthermore, we have the elementary vector identity

$$\left|\frac{z}{|z|^2} - \frac{z'}{|z'|^2}\right| = \frac{1}{|z|\,|z'|}|z - z'|, \qquad \forall z, z' \in X \setminus \{0\}.$$

Hence,

$$\left|\left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| \frac{|\left\langle J_{rZ}(\chi_r^\flat), \chi_r^\sharp - \chi_r^\flat\right\rangle|}{|J_{rZ}(\chi_r^\flat)|^2} \leq |\varepsilon_t^\flat||\chi_r^\sharp - \chi_r^\flat|,$$

and

$$\left|\left\langle J_{rZ}(\chi_r^\flat), (\xi_r^\flat)_t\right\rangle\right| |\chi_r^\sharp| \left|\frac{J_{rZ}(\chi_r^\sharp)}{|J_{rZ}(\chi_r^\sharp)|^2} - \frac{J_{rZ}(\chi_r^\flat)}{|J_{rZ}(\chi_r^\flat)|^2}\right| \\ \leq \frac{|\chi_r^\sharp|}{|J_{rZ}(\chi_r^\sharp)|}|\varepsilon_t^\flat||J_{rZ}(\chi_r^\sharp) - J_{rZ}(\chi_r^\flat)|.$$

In view of (3.15) and (3.12), we thus obtain (3.24) from (3.25).

Combining (3.24) with (3.23), we obtain

$$\Big|\big\langle\chi_r^\sharp,(\xi_r^\sharp)_t\big\rangle-\Big\langle\chi_r^\flat,(\xi_r^\flat)_t\Big\rangle\Big|+r^2C^2\frac{\mathrm{d}}{\mathrm{d}t}N(t)$$
$$\leq r^2C^2|J_{rZ}(\chi_r^\sharp)||\varepsilon_t^\sharp-\varepsilon_t^\flat|+(1+2LC^2)|\varepsilon_t^\flat||\chi_r^\sharp-\chi_r^\flat|. \tag{3.26}$$

We now refer to (3.16) which yields

$$rc|J_{rZ}(\chi_r^\sharp)|\leq M_{(rZ)^*}(J_{rZ}(\chi_r^\sharp))=M_{rZ}(\chi_r^\sharp)\leq 1\,,$$

and (3.20) follows from (3.26) and (3.6). □

We now apply this result to our plate problem.

Corollary 3.4. *Let Hypothesis 2.1 hold. Then there exists a constant $C_1>0$ such that for every $\varepsilon^\sharp,\varepsilon^\flat\in L^2(\Omega_T)$, the corresponding solutions χ_r^i of (1.4), $\xi_r^i=\varepsilon^i-\chi_r^i$, $i=\sharp,\flat$, satisfy for all $t\in(0,T]$ and $r>0$ the inequality*

$$\int_0^t\int_\Omega\Big|\mathbf{K}\chi_r^\sharp\cdot\big(\xi_r^\sharp\big)_t-\mathbf{K}\chi_r^\flat\cdot\big(\xi_r^\flat\big)_t\Big|(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \tag{3.27}$$
$$\leq C_1\Big(r+\int_0^t|\varepsilon_t^\flat|_2(\tau)\,\mathrm{d}\tau\Big)\int_0^t|\varepsilon_t^\sharp-\varepsilon_t^\flat|_2(\tau)\,\mathrm{d}\tau\,.$$

Proof. We integrate (3.20) over Ω and obtain

$$\int_\Omega\Big|\mathbf{K}\chi_r^\sharp\cdot\big(\xi_r^\sharp\big)_t-\mathbf{K}\chi_r^\flat\cdot\big(\xi_r^\flat\big)_t\Big|(x,t)\,\mathrm{d}x$$
$$+r^2C^2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big|\frac12M_{rZ}^2(\chi_r^\sharp(t))-\frac12M_{rZ}^2(\chi_r^\flat(t))\Big|(x,t)\,\mathrm{d}x$$
$$\leq r\frac{C^2}{c}|\varepsilon_t^\sharp-\varepsilon_t^\flat|_1(t)+(1+2LC^2)\int_0^t\int_\Omega|\varepsilon_t^\flat|(x,t)|\varepsilon_t^\sharp-\varepsilon_t^\flat|(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq r\frac{C^2}{c}|\varepsilon_t^\sharp-\varepsilon_t^\flat|_1(t)+(1+2LC^2)|\varepsilon_t^\flat|_2(t)\int_0^t|\varepsilon_t^\sharp-\varepsilon_t^\flat|_2(\tau)\,\mathrm{d}\tau\,, \tag{3.28}$$

and it suffices to integrate over t. □

4. **The plate equation.** In this section, we investigate first the decoupled problem (1.4), (2.1), (2.3), (2.5), (2.7), with the goal to obtain (2.2) by means of a fixed point argument.

Here, we assume that ε is known, and $\varepsilon_t\in L^2(\Omega_T;\mathbb{R}^3)$. More specifically, under Hypothesis 2.1, we define for $t\in(0,T)$ the functions

$$p(t)=(|\theta(t)|_2+\Gamma)^2\,, \tag{4.1}$$

$$q(t)=\frac{2}{c_*^2}\int_0^t\mathrm{e}^{b_*^2(t^2-\tau^2)/c_*^2}p(\tau)\,\mathrm{d}\tau\,, \tag{4.2}$$

and for $\hat T\in(0,T]$ consider the set

$$E_{\hat T}=\Big\{\varepsilon\in L^2(\Omega_{\hat T};\mathbb{R}^3):\varepsilon_t\in L^2(\Omega_{\hat T};\mathbb{R}^3), \tag{4.3}$$
$$\varepsilon(x,0)=0,\int_0^t|\varepsilon_t(\tau)|_2^2\,\mathrm{d}\tau\leq q(t)\ \text{for}\ \ t\in(0,\hat T]\Big\}.$$

The definition is meaningful, as by Hypothesis 2.1, $p \in L^1(0,T)$. We fix the number

$$A := \sup \left| \frac{1}{2} \int_\Omega \lambda(x-y) \Big(\mathbf{B}''(m)\varepsilon \cdot \varepsilon + \int_0^\infty \gamma_{mm}(m,r)\, \mathbf{K}\chi_r \cdot \chi_r \,\mathrm{d}r \Big)(y,t)\,\mathrm{d}y \right|, \tag{4.4}$$

where the supremum is taken over all $\varepsilon \in E_T$ and $m \in L^\infty(\Omega_T)$ (it is indeed finite), and define $\mu(t)$ as the solution of the ODE

$$\dot\mu(t) = A\mu(t)\dot\mu(t) + R\,, \quad \mu(0) = 0\,, \tag{4.5}$$

that is,

$$\mu(t) = \frac{1}{A} - \sqrt{\frac{1}{A^2} - \frac{2R}{A}t} \quad \text{for} \quad t \in [0, 1/(2AR)]\,. \tag{4.6}$$

We see that $\dot\mu(t)$ blows up to $+\infty$ as $t \nearrow 1/(2AR)$. We choose a small $\delta \in (0,1)$ that we keep fixed throughout the paper, and set

$$T^R = \min\left\{T, \frac{1-\delta^2}{2AR}\right\}. \tag{4.7}$$

Eq. (2.7) cannot be expected to have global solutions for the same reason as in [7]. We state the intermediate result in the following form.

Proposition 4.1. *Let $\varepsilon \in E_{T^R}$ be given. Then system (1.4), (2.1), (2.3), (2.5), (2.7) has a unique solution with the regularity as in Theorem 2.2, and we have $D_2 w \in E_{T^R}$.*

Proof. We first check by a fixed point argument that (2.7) has a unique solution m in $[0, T^R]$. We define the set

$$M_R = \{m \in L^\infty(\Omega_{T^R}) : m_t \in L^\infty(\Omega_{T^R}), m(x,0) = 0, 0 \le m_t(x,t) \le \dot\mu(t) \ \text{ a.e.}\}, \tag{4.8}$$

and for $m \in M_R$ define $\tilde m(x,0) = 0$, and

$$\begin{aligned} \tilde m_t(x,t) := Q_R \left(\int_\Omega \lambda(x-y) \int_0^\infty \gamma(m,r)\, \mathbf{K}\chi_r \cdot (\xi_r)_t \,\mathrm{d}r\,\mathrm{d}y \right) \\ - \frac{1}{2}\int_\Omega \lambda(x-y) m_t(y,t) \Big(\mathbf{B}'(m)\varepsilon \cdot \varepsilon + \int_0^\infty \gamma_m(m,r)\, \mathbf{K}\chi_r \cdot \chi_r \,\mathrm{d}r \Big)(y,t)\,\mathrm{d}y \end{aligned} \tag{4.9}$$

for $t \in (0, T^R]$. Note that $\mathbf{B}'(0) = \mathbf{0}, \gamma_m(0,r) = 0$. Hence,

$$\tilde m_t(x,t) \le A\mu(t)\dot\mu(t) + R = \dot\mu(t)\,,$$

that is, $\tilde m_t \in M_R$. Furthermore, for $m^\sharp, m^\flat \in M_R$, we have

$$\begin{aligned} |\tilde m_t^\sharp(t) - \tilde m_t^\flat(t)|_\infty \le A\mu(t)|m_t^\sharp(t) - m_t^\flat(t)|_\infty + A\dot\mu(t)|m^\sharp(t) - m^\flat(t)|_\infty \\ + \tilde C|\varepsilon_t(t)|_1 |m^\sharp(t) - m^\flat(t)|_\infty\,, \end{aligned} \tag{4.10}$$

where $\tilde C > 0$ is a constant which comes out from the following computation:

$$\begin{aligned} &\Big| Q_R \left(\int_\Omega \lambda(x-y) \int_0^\infty \gamma(m^\sharp,r)\, \mathbf{K}\chi_r \cdot (\xi_r)_t \,\mathrm{d}r\,\mathrm{d}y \right) \\ &\qquad - Q_R \left(\int_\Omega \lambda(x-y) \int_0^\infty \gamma(m^\flat,r)\, \mathbf{K}\chi_r \cdot (\xi_r)_t \,\mathrm{d}r\,\mathrm{d}y \right) \Big| \\ &\le \int_\Omega \lambda(x-y) \int_0^\infty |\gamma(m^\sharp,r) - \gamma(m^\flat,r)|\, \mathbf{K}\chi_r \cdot (\xi_r)_t \,\mathrm{d}r\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega \lambda(x-y)\left|\int_{m^\sharp(y,t)}^{m^\flat(y,t)}\int_0^\infty |\gamma_m(m,r)|rCc^*\,\mathrm{d}r\,\mathrm{d}m\right||\varepsilon_t|\,\mathrm{d}y\\
&\leq \Gamma c^* C\int_\Omega \lambda(x-y)|m^\sharp - m^\flat||\varepsilon_t|(y,t)\,\mathrm{d}y \;\leq\; \Lambda\Gamma c^* C|\varepsilon_t(t)|_1|m^\sharp(t)-m^\flat(t)|_\infty,
\end{aligned}
$$

that is, $\tilde{C} = \Lambda\Gamma c^* C$. On $[0, T^R]$, we have $A\mu(t) \leq 1-\delta$. Inequality (4.10) is thus of the form

$$\dot{v}(t) \leq (1-\delta)\dot{u}(t) + \alpha(t)u(t)\,, \tag{4.11}$$

where we set $v(t) = \int_0^t |\tilde{m}_t^\sharp(\tau) - \tilde{m}_t^\flat(\tau)|_\infty\,\mathrm{d}\tau$, $u(t) = \int_0^t |m_t^\sharp(\tau) - m_t^\flat(\tau)|_\infty\,\mathrm{d}\tau$, $\alpha(t) = A\dot{\mu}(t) + \tilde{C}|\varepsilon_t(t)|_1$.

Put

$$\hat{\alpha}(t) = \frac{1}{\delta}\int_0^t \alpha(\tau)\,\mathrm{d}\tau\,, \tag{4.12}$$

and test (4.11) by $\mathrm{e}^{-2\hat{\alpha}(t)}$. This yields

$$\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}\dot{v}(t)\,\mathrm{d}t \leq \left(1-\frac{\delta}{2}\right)\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}\dot{u}(t)\,\mathrm{d}t\,, \tag{4.13}$$

that is,

$$\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}|\tilde{m}_t^\sharp(t)-\tilde{m}_t^\flat(t)|_\infty\,\mathrm{d}t \leq \left(1-\frac{\delta}{2}\right)\int_0^{T^R} \mathrm{e}^{-2\hat{\alpha}(t)}|m_t^\sharp(t)-m_t^\flat(t)|_\infty\,\mathrm{d}t\,. \tag{4.14}$$

We see that the mapping that with m associates $\tilde{m}$ is a contraction on M_R, hence Eq. (2.7) has a unique solution m for every $\varepsilon \in E_{T^R}$.

Eq. (2.1) with a given $\varepsilon \in E_{T^R}$ is linear in w and the existence and uniqueness of a solution can be easily proved e.g. by Galerkin approximations. The required regularity follows by testing (2.1) successively by $\varphi = w_t$ and $\varphi = (I-\Delta)^{-1}w_{tt}$, using the assumption (2.6). To complete the proof, it remains to check that $D_2 w \in E_{T^R}$. Choosing again $\varphi = w_t$ in (2.1) and using Hypothesis 2.1, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|w_t|_2^2 + |\nabla w_t|^2\big) + c_*|D_2 w_t|_2^2 &\leq (|\theta|_2 + b_*|\varepsilon|_2 + \Gamma)|D_2 w_t|_2\\
&\leq \frac{1}{c_*}\big(p(t) + b_*^2|\varepsilon|_2^2\big) + \frac{c_*}{2}|D_2 w_t|_2^2,
\end{aligned} \tag{4.15}
$$

hence

$$\int_0^t |D_2 w_t(\tau)|_2^2\,\mathrm{d}\tau \leq \frac{2}{c_*^2}\int_0^t \big(p(\tau) + b_*^2|\varepsilon(\tau)|_2^2\big)\,\mathrm{d}\tau. \tag{4.16}$$

We have $|\varepsilon(x,t)| \leq \int_0^t |\varepsilon_t(x,\tau)|\,\mathrm{d}\tau$ a.e., so that $|\varepsilon(t)|_2^2 \leq t\int_0^t |\varepsilon_t(\tau)|_2^2\,\mathrm{d}\tau \leq tq(t)$. From (4.16) it follows that

$$\int_0^t |D_2 w_t(\tau)|_2^2\,\mathrm{d}\tau \leq \frac{2}{c_*^2}\int_0^t \big(p(\tau) + b_*^2\tau q(\tau)\big)\,\mathrm{d}\tau = q(t), \tag{4.17}$$

which we wanted to prove. □

5. **The coupled system.** This section is devoted to the proof of Theorem 2.2. We start with an auxiliary result on the solution mapping of (1.4), (2.3), (2.7) which with a given $\varepsilon \in E_{T^R}$ associates $m \in M_R$.

Lemma 5.1. *There exists a constant C_2 depending only on R and on the data of the problem such that for all $\varepsilon^\sharp, \varepsilon^\flat \in E_{T^R}$, the corresponding solutions $m^\sharp, m^\flat \in M_R$ to (2.3), (2.7) satisfy the inequality*

$$\int_0^{T^R} |m_t^\sharp - m_t^\flat|_\infty(t)\,\mathrm{d}t \le C_2 \int_0^{T^R} |\varepsilon_t^\sharp - \varepsilon_t^\flat|_2(t)\,\mathrm{d}t\,. \tag{5.1}$$

Proof. With the notation of Section 4 we have

$$\begin{aligned}\delta|m_t^\sharp(t) - m_t^\flat(t)|_\infty \le S\dot\mu(t)\Big(|\varepsilon^\sharp(t) - \varepsilon^\flat(t)|_2 + \int_0^t |\varepsilon_t^\sharp - \varepsilon_t^\flat|_2(\tau)\,\mathrm{d}\tau\Big) \\ + \alpha(t)|m^\sharp(t) - m^\flat(t)|_\infty + \Lambda\int_0^\infty \gamma(0,r)\left|\mathbf{K}\chi_r^\sharp\cdot\big(\xi_r^\sharp\big)_t - \mathbf{K}\chi_r^\flat\cdot\big(\xi_r^\flat\big)_t\right|_1(t)\,\mathrm{d}r\,,\end{aligned} \tag{5.2}$$

where $S > 0$ is a constant, and where we have used the fact that $\gamma(m_2, r) \le \gamma(0,r)$ by Hypothesis 2.1 (iv). Testing (5.2) by $\mathrm{e}^{-\hat\alpha(t)}$, with $\hat\alpha$ from (4.12), yields that

$$\begin{aligned}\delta\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{-\hat\alpha(t)}\int_0^t |m_t^\sharp - m_t^\flat|_\infty(\tau)\,\mathrm{d}\tau\Big) \le \mathrm{e}^{-\hat\alpha(t)}\Big(S\dot\mu(t)\Big(|\varepsilon^\sharp(t) - \varepsilon^\flat(t)|_2 \\ + \int_0^t |\varepsilon_t^\sharp - \varepsilon_t^\flat|_2(\tau)\,\mathrm{d}\tau\Big) + \Lambda\int_0^\infty \gamma(0,r)\left|\mathbf{K}\chi_r^\sharp\cdot\big(\xi_r^\sharp\big)_t - \mathbf{K}\chi_r^\flat\cdot\big(\xi_r^\flat\big)_t\right|_1(t)\,\mathrm{d}r\Big).\end{aligned} \tag{5.3}$$

Integrating from 0 to t and using (3.27) we obtain the assertion. □

Lemma 5.2. *The mapping defined in Proposition 4.1 that with $\varepsilon \in E_{T^R}$ associates $D_2 w \in E_{T^R}$ is a contraction with respect to a suitable norm.*

Proof. We test the difference of Eqs. (2.1) written for $\varepsilon^\sharp, \varepsilon^\flat$ and the corresponding solutions $w^\sharp, w^\flat$ by $\bar w_t = w_t^\sharp - w_t^\flat$, and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|\bar w_t|_2^2 + |\nabla \bar w_t|^2\big) + c_*|D_2\bar w_t|_2^2 \le \frac{1}{c_*}\big|P[m^\sharp,\varepsilon^\sharp] - P[m^\flat,\varepsilon^\flat]\big|_2^2. \tag{5.4}$$

We have

$$|P[m^\sharp,\varepsilon^\sharp] - P[m^\flat,\varepsilon^\flat]|_2(t) \le C_3\Big(|m^\sharp - m^\flat|_\infty(t) + |\varepsilon^\sharp - \varepsilon^\flat|_2(t) + \int_0^t |\varepsilon_t^\sharp - \varepsilon_t^\flat|_2(\tau)\,\mathrm{d}\tau\Big), \tag{5.5}$$

hence, by Lemma 5.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|\bar w_t|_2^2 + |\nabla \bar w_t|^2\big) + |D_2\bar w_t|_2^2 \le C_4\int_0^t |\bar\varepsilon_t|_2^2(\tau)\,\mathrm{d}\tau \tag{5.6}$$

with a constant $C_4 > 0$, and with the notation $\bar\varepsilon = \varepsilon^\sharp - \varepsilon^\flat$. Testing (5.6) by $\mathrm{e}^{-2C_4 t}$ and integrating from 0 to T^R, we get the inequality

$$\int_0^{T^R} \mathrm{e}^{-2C_4 t}|D_2\bar w_t|_2^2\,\mathrm{d}t \le \frac{1}{2}\int_0^{T^R} \mathrm{e}^{-2C_4 t}|\bar\varepsilon_t|_2^2(t)\,\mathrm{d}t\,, \tag{5.7}$$

which completes the proof. □

We are now ready to finish the proof of Theorem 2.2.

Proof. In order to prove Theorem 2.2, it suffices to combine Proposition 4.1 with Lemma 5.2 and apply the contraction principle. □

6. **Proof of Theorem 2.3.** The main goal of this section is to remove the cut-off function Q_R in (2.7). This will be done by establishing additional estimates, where the dependence on R is explicitly taken into account. The constants $C_5, \ldots, C_{10}$ which appear in the formulas below are independent of R.

We assume here the additional regularity $\theta_t \in L^2(\Omega_T)$ of the right hand side of (2.1). Testing Eq. (2.1) by w_{tt} (for the Galerkin approximations first, and then passing to the limit) and integrating by parts we obtain

$$\int_0^t \big(|w_{tt}|_2^2 + |\nabla w_{tt}|_2^2\big)(\tau)\,\mathrm{d}\tau + |D_2 w_t(t)|_2^2 \le C_5\left(|\theta(t)|_2^2 + |P[m,\varepsilon](t)|_2^2 + \Big(\int_0^t \big(|\theta_t|_2^2 + |P[m,\varepsilon]_t|_2^2\big)(\tau)\,\mathrm{d}\tau\Big)^{1/2}\right). \tag{6.1}$$

Using the inequality

$$|P[m,\varepsilon]_t|_2 \le C_6\big(|m_t|_\infty + |\varepsilon_t|_2\big),$$

we infer from (6.1) that

$$|D_2 w_t(t)|_2^2 \le C_7(1+R) \tag{6.2}$$

for all $t \in [0, T^R]$. On the other hand, we have

$$\Big|\int_\Omega \lambda(x-y)\int_0^\infty \gamma(m,r)\,\mathbf{K}\chi_r\cdot(\xi_r)_t\,\mathrm{d}r\,\mathrm{d}y\Big| \le C_8|D_2 w_t(t)|_1 \le C_9\sqrt{1+R}\,.$$

Choosing $R = R^*$ sufficiently large, we see that the truncation in (2.7) is never active in the interval $[0, T^{R^*}] =: [0, T^*]$, and the solution to (2.1)–(2.3), (2.5) (2.7) in fact satisfies (2.1)–(2.5) as well. The additional regularity follows from (6.1).

To complete the proof of Theorem 2.3, it remains to prove inequality (2.8). As in the proof of Lemma 5.2, we take the differences of Eqs. (2.1) written for $\varepsilon^\sharp, \varepsilon^\flat$ and the corresponding solutions $w^\sharp, w^\flat$, this time with possibly different $\theta^\sharp$ and $\theta^\flat$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|\bar w_t|_2^2 + |\nabla \bar w_t|^2\big) + c_*|D_2\bar w_t|_2^2 \le \frac{1}{c_*}\left(|\bar\theta|_2^2 + |F[\varepsilon^\sharp] - F[\varepsilon^\flat]|_2^2\right).$$

Exploiting (5.5) and Lemma 5.1, we obtain, for a constant $C_{10} > 0$, that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(|\bar w_t|_2^2 + |\nabla \bar w_t|^2\big)(t) + c_*|D_2\bar w_t|_2^2(t) &\le C_{10}\left(|\bar\theta|_2^2(t) + \int_0^t |\bar\varepsilon_t|_2^2(\tau)\,\mathrm{d}\tau\right)\\ &\le C_{10}\left(|\bar\theta|_2^2(t) + \int_0^t |D_2\bar w_t|_2^2(\tau)\,\mathrm{d}\tau\right),\end{aligned}$$

where in the last line we used (2.2). The Gronwall lemma now yields the assertion.

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Received October 2011; revised February 2012.

E-mail address: Michela.Eleuteri@unimi.it
E-mail address: Jana.Kopfova@math.slu.cz
E-mail address: krejci@math.cas.cz