INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

Universal n-tuples of operators

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Preprint No. 19-2013

PRAHA 2013

UNIVERSAL N-TUPLES OF OPERATORS

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Abstract. We generalize the classical result of Caradus concerning universal operators to the multioperator setting.

1. Introduction

An operator T acting on a Hilbert space K is called universal if it has the following property: for each operator A on a separable Hilbert space H there exist a constant $c \neq 0$ and a subspace $M \subset K$ invariant for T such that the restriction $T|M$ is similar to cA.

In other words, T "contains" all operators on separable Hilbert spaces.

The first example of a universal operator was given by G.-C. Rotta [R]. The notion of universal operators was introduced by Caradus [C], who gave also the following elegant sufficient condition for an operator to be universal.

Theorem 1. Every surjective operator with infinite-dimensional kernel is universal.

Clearly the condition that the kernel is infinite-dimensional is necessary since the operator must contain also the zero operator. The surjectivity is not necessary, but it is a natural condition which is easy to verify. The simplest example of a universal operator is the backward shift of infinite multiplicity (this was the operator considered by G.-C. Rota). For further examples of universal operators see [ChP].

The aim of this note is to generalize the result of Caradus for n-tuples of operators. We study both the commuting and non-commuting setting.

2. Commuting n-tuples

Denote by $B(H)$ the set of all bounded linear operators acting on a Hilbert space H. For $T \in B(H)$ denote by $N(T)$ and $R(T)$ its kernel, $N(T) = \{x \in H : Tx = 0\}$ and range $R(T) = TX$, respectively.

We say that two n-tuples $(T_1, \dots, T_n) \in B(H)^n$ and $(S_1, \dots, S_n) \in B(K)^n$ are similar if there exists an invertible operator $V : H \to K$ such that $VT_j = S_jV$ for all $j = 1, \dots, n$.

Definition 2. Let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of operators on a Hilbert space K. We say that T is universal for all commuting tuples if it has the following property: for each commuting n-tuple $A = (A_1, \dots, A_n)$ of operators on a separable Hilbert space there exist a constant $c \neq 0$ and a subspace $M \subset K$ invariant for all $T_1, \dots, T_n$ such that the n-tuples $T|M = (T_1|M, \dots, T_n|M)$ and $cA = (cA_1, \dots, cA_n)$ are similar.

1991 *Mathematics Subject Classification.* Primary 47A12, Secondary 47A13.

Key words and phrases. universal operators, universal tuples.

The research was supported by grant 201/09/0473 of GA ČR and RVO: 67985840.

Clearly any n-tuple $T = (T_1, \dots, T_n)$ universal for all commuting tuples must satisfy the condition $\dim \bigcap_{j=1}^n N(T_j) = \infty$. Furthermore, T should "contain" all n-tuples of the form $(A_1, 0, \dots, 0)$, so the restriction $T_1 | \bigcap_{j=2}^n N(T_j)$ must be a universal operator. Thus it is natural to assume that this restriction is surjective. Similar condition can be formulated for each T_j and its restrictions to the spaces $\bigcap_{i\in F} N(T_i)$, where $F \subset \{1, \dots, n\}, j \notin F$. Thus it is natural to consider the commuting n-tuples $T = (T_1, \dots, T_n) \in B(K)^n$ satisfying the condition that the restrictions $T_j | \bigcap_{i\in F} N(T_i)$ are surjective for all $F \subset \{1, \dots, n\} \setminus \{j\}$.

Note that for $F = \emptyset$ this means that the operators T_j are surjective for all j.

The main result of this section is that all n-tuples of operators satisfying these natural conditions are universal for commuting tuples.

Theorem 3. Let $n \ge 1$ and let $T_1, \dots, T_n \in B(K)$ be a commuting n-tuple of operators satisfying

(i) $\dim \bigcap_{j=1}^n N(T_j) = \infty$;
(ii) for all $F \subset \{1, \dots, n\}$ and $j \in \{1, \dots, n\} \setminus F$ the restriction $T_j | \bigcap_{i\in F} N(T_i)$ is surjective (i.e., $T_j\bigl(\bigcap_{i\in F} N(T_i)\bigr) = \bigcap_{i\in F} N(T_i)$).

Then $T_1, \dots, T_n$ is universal for all commuting tuples.

Proof. We fix an n-tuple $T_1, \dots, T_n \in B(K)$ of mutually commuting operators satisfying (i) and (ii).

For each $j = 1, \dots, n$ the operator T_j is surjective. Fix a right inverse $\hat{T}_j \in B(K)$, i.e., $T_j \hat{T}_j = I$.

We need several lemmas:

Lemma 4. Let $F \subset \{1, \dots, n\}$ and $j \in \{1, \dots, n\} \setminus F$. Then

$$\bigcap_{i\in F} N(T_i T_j) = N(T_j) + \bigcap_{i\in F} N(T_i).$$

Moreover, there exists a projection $P_{j,F} : \bigcap_{i\in F} N(T_i T_j) \to \bigcap_{i\in F} N(T_i T_j)$ such that $R(P_{j,F}) = \bigcap_{i\in F} N(T_i)$ and $N(P) \subset N(T_j)$.

Proof. The inclusion $\supset$ is clear.

Let $x \in \bigcap_{i\in F} N(T_i T_j)$. Then $T_j x \in \bigcap_{i\in F} N(T_i)$. So there exists $y \in \bigcap_{i\in F} N(T_i)$ such that $T_j y = T_j x$. Thus $x - y \in N(T_j)$ and $x = (x - y) + y \in N(T_j) + \bigcap_{i\in F} N(T_i)$.

Write for short $M = \bigcap_{i\in F} N(T_i)$ and $L = N(T_j)$. Then $M + L$ is a closed subspace. Let $X = (M + L) \ominus (M \cap L)$. We show that $M + L = M \oplus (L \cap X)$.

If $x \in M \cap (L \cap X)$ then $x \in M \cap L$ and $x \perp M \cap L$, so $x = 0$. Hence $M \cap (L \cap X) = \{0\}$.

Clearly $M \subset M + (L \cap X)$. Let $x \in L$. Then x can be written (uniquely) as $x = y + z$, where $y \in M \cap L$ and $z \perp (M \cap L)$. So $z \in L \cap X$ and $x = y + z \in M + (L \cap X)$. Hence $M + L = M \oplus (L \cap X)$ and there exists a projection $P_{j,F} : M + L \to M + L$ such that $R(P_{j,F}) = M = \bigcap_{i\in F} N(T_i)$ and $N(P) = L \cap X \subset L = N(T_j)$. □

Let $k = \max\Bigl\{2, \max\bigl\{\|P_{j,F}\| : F \subset \{1, \dots, n\}, j \in \{1, \dots, v\} \setminus F\bigr\}, \max\{\|\hat{T}_j\| : j = 1, \dots, n\}\Bigr\}$.

Lemma 5. Let H be a separable Hilbert space. Let $G \subset \{1,\dots,n\}$, $G \neq \emptyset$. Suppose that there exist linear operators $V_F : H \to K \quad (F \subset G, F \neq G)$ satisfying

$$T_j V_F = V_{F\setminus\{j\}} \qquad (F \subset G, F \neq G, j \in F).$$

Then there exists an operator $V_G : H \to K$ satisfying

$$T_j V_G = V_{G\setminus\{j\}} \qquad (j \in G)$$

and $\|V_G\| \le (2k^2)^{\operatorname{card} G} \max\{\|V_{G\setminus\{j\}}\| : j \in G\}$.

Proof. We prove the statement by induction on $\operatorname{card} G$.

If $\operatorname{card} G = 1$, $G = \{m\}$ then the statement is clear: set $V_{\{m\}} = \hat{T}_m V_\emptyset$. Then $T_m V_{\{m\}} = T_m \hat{T}_m V_\emptyset = V_\emptyset$ and $\|V_{\{m\}}\| \le \|\hat{T}_m\| \cdot \|V_\emptyset\| \le k\|V_\emptyset\|$.

Let $G \subset \{1,\dots,n\}$, $\operatorname{card} G \ge 2$ and suppose that the statement is true for each $\tilde{G} \subset \{1,\dots,n\}$, $1 \le \operatorname{card} \tilde{G} < \operatorname{card} G$. Fix an $m \in G$ and let $G' = G \setminus \{m\}$. Consider the operators $W_F = V_{F\cup\{m\}} \quad (F \subset G', F \neq G')$. By the induction assumption there exists an operator $V' : H \to K$ satisfying $T_j V' = W_{G'\setminus\{j\}} = V_{G\setminus\{j\}}$ for all $j \in G'$. Moreover, $\|V'\| \le (2k^2)^{\operatorname{card} G'} \max\{\|V_F\| : \operatorname{card} F = \operatorname{card} G - 1\}$.

Furthermore, let $V'' = \hat{T}_m V_{G'}$. For all $j \in G'$ we have

$$T_j T_m (V'' - V') = T_j V_{G'} - T_m V_{G\setminus\{j\}} = V_{G\setminus\{j,m\}} - V_{G\setminus\{j,m\}} = 0.$$

So

$$R(V'' - V') \subset \bigcap_{j\in G'} N(T_j T_m) = N(T_m) + \bigcap_{j \in G'} N(T_j).$$

Let $P_{m,G'} : \bigcap_{j\in G'} N(T_j T_m) \to \bigcap_{j\in G'} N(T_j T_m)$ be the projection considered above, i.e., $R(P_{m,G'}) = \bigcap_{j\in G'} N(T_j)$ and $N(P_{m,G'}) \subset N(T_m)$.

Set

$$V_G = V'' + (I - P_{m,G'})(V' - V'') = V' + P_{m,G'}(V'' - V'),$$

Since $R(I - P_{m,G'}) = N(P_{m,G'}) \subset N(T_m)$, we have $T_m V_G = T_m V'' = V_{G'} = V_{G\setminus\{m\}}$. For $j \in G'$ we have $T_j V_G = T_j V' = V_{G\setminus\{j\}}$. Moreover,

$$\|V_G\| \le \|V'\| + \|P_{m,G'}\| \cdot (\|V''\| + \|V'\|)$$
$$\le \Big((2k^2)^{\operatorname{card} G - 1} + k\big(k + (2k^2)^{\operatorname{card} G - 1}\big)\Big) \max\{\|V_F\| : F \subset G, \operatorname{card} F = \operatorname{card} G - 1\}$$
$$\le (2k^2)^{\operatorname{card} G} \max\{\|V_F\| : F \subset G : \operatorname{card} F = \operatorname{card} G - 1\}.$$

□

In the following we use the standard multiindex notation. Denote by $\mathbb{Z}_+$ the set of all non-negative integers. For $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}_+^n$ write $|\alpha| = \sum_{i=1}^n \alpha_i$. For $j = 1,\dots,n$ let $e_j = (\underbrace{0,\dots,0}_{j-1}, 1, 0, \dots, 0)$.

Lemma 6. Let $T_1,\dots,T_n \in B(K)$ be a commuting n-tuple satisfying the conditions of Theorem 3. Let H be a separable infinite-dimensional Hilbert space. Then there exist operators $V_\alpha : H \to K \quad (\alpha \in \mathbb{Z}_+^n)$ satisfying

(i) $V_{0,\dots,0}$ is an isometry;
(ii) $T_j V_\alpha = 0 \quad (\alpha \in \mathbb{Z}_+^n, \alpha_j = 0)$;

(iii) $T_jV_\alpha = V_{\alpha-e_j} \quad (\alpha\in\mathbb{Z}_+^n, \alpha_j\ge 1)$;
(iv) $\|V_\alpha\|\le (2k^2)^{n|\alpha|}$.

Proof. Choose an isometry $V_{0,\dots,0}: H\to\bigcap_{i=1}^n N(T_i)$.

We construct the mappings V_α inductively by induction on $|\alpha|$. Let $\alpha\in\mathbb{Z}_+^n$, $|\alpha|\ge 1$ and suppose that the mappings $V_\beta : H\to K$ satisfying (i) – (iv) have already been constructed for all $\beta\in\mathbb{Z}_+^n$ satisfying $|\beta|<|\alpha|$. In particular, the mappings V_β are already constructed for all $\beta\le\alpha$, $\beta\ne\alpha$.

Let $G=\{j:\alpha_j\ne 0\}$ and let $m=\operatorname{card} G$. Consider the m-tuple of operators $T_j|\bigcap_{i\notin G}N(T_i)$, $j\in G \quad (j\in G)$. Note that these operators also satisfy the conditions of Theorem 3. For $F\subset G, F\ne G$ let $W_F : H\to\bigcap_{i\notin G}N(T_i)$ be defined by $W_F=V_{(\beta_1,\dots,\beta_n)}$, where $\beta_j=\alpha_j \quad (j\in F)$, $\beta_j=\alpha_j-1 \quad (j\in G\setminus F)$ and $\beta_j=0 \quad (j\notin G)$. Clearly the operators W_F satisfy the conditions of Lemma 5. So there exists an operator $V_\alpha : H\to\bigcap_{i\notin G}N(T_i)\subset K$ satisfying

$$T_jW_\alpha = W_{G\setminus\{j\}} = V_{\alpha-e_j} \qquad (j\in G),$$

$$T_jV_\alpha = 0 \qquad (j\notin G)$$

and

$$\|V_\alpha\|\le (2k^2)^{\operatorname{card} G}\cdot\max\{\|W_F\| : F\subset G, \operatorname{card} F=\operatorname{card} G-1\}\le (2k^2)^n\cdot\max\{\|V_\beta\| : |\beta|=|\alpha|-1\}.$$

Continuing in this way we construct the operators V_α with the required properties. □

Proof of Theorem 3. Let $T_1,\dots,T_n\in B(K)$ be a commuting n-tuple satisfying the conditions of Theorem 3.

Let H be a separable Hilbert space and $A_1,\dots,A_n\in B(H)$ a commuting n-tuple of operators. Let $c=\max\{\|A_j\| : 1\le j\le n\}$. Without loss of generality we may assume that c is sufficiently small (it will be clear from the proof the precise condition which c should satisfy).

Let $V_\alpha : H\to K \quad (\alpha\in\mathbb{Z}_+^n)$ be the operators constructed in Lemma 6. Define $V: H\to K$ by

$$Vh=\sum_{\alpha\in\mathbb{Z}_+^n}V_\alpha A^\alpha h \qquad (h\in H).$$

We have

$$\Big\|\sum_{\alpha\in\mathbb{Z}_+^n,\alpha\ne(0,\dots,0)}V_\alpha A^\alpha\Big\|\le\sum_{\alpha\in\mathbb{Z}_+^n,\alpha\ne(0,\dots,0)}(2k^2)^{n|\alpha|}c^{|\alpha|}$$

$$=\sum_{r=1}^\infty(2k^2)^{nr}c^r\cdot\operatorname{card}\{\alpha\in\mathbb{Z}_+^n : |\alpha|=r\}=\sum_{r=1}^\infty(2k^2)^{nr}c^r\binom{r+n-1}{n-1}$$

$$\le 2^{n-1}\sum_{r=1}^\infty 2^r(2k^2)^{nr}c^r<1$$

if c is sufficiently small. Then $\|V-V_{0,\dots,0}\|<1$, so V is a bounded operator. Since $V_{0,\dots 0}$ is an isometry, V is bounded below and its range VH is closed.

For all $j=1,\dots,n$ and $h\in H$ we have

$$VA_jh=\sum_{\alpha\in\mathbb{Z}_+^n}V_\alpha A^\alpha A_jh$$

and

$$T_jVh=\sum_{\alpha\in\mathbb{Z}_+^n}T_jV_\alpha A^\alpha h=\sum_{\alpha\in\mathbb{Z}_+^n,\alpha_j\ge 1}T_jV_{\alpha_1,\dots,\alpha_n}A^\alpha h=\sum_{\alpha\in\mathbb{Z}_+^n,\alpha_j\ge 1}V_{\alpha-e_j}A^\alpha h.$$

So $VA_j=T_jV\quad(j=1,\dots,n)$. Hence VH is a closed subspace of K invariant for all $T_j\quad(j=1,\dots,n)$ and V is the similarity between the restrictions $(T_1|VH,\dots,T_n|VH)$ and $(A_1,\dots,A_n)$. □

For $n=2$ we can formulate a simpler statement.

Lemma 7. Let $T_1,T_2\in B(K)$ be commuting surjective operators. The following statements are equivalent:

(i) $N(T_1T_2)=N(T_1)+N(T_2)$;
(ii) $T_1N(T_2)=N(T_2)$;
(iii) $T_2N(T_1)=N(T_1)$.

Proof. (i)⇒(ii): Clearly $T_1N(T_2)\subset N(T_2)$. Let $x\in N(T_2)$. Since T_1 is surjective, there exists $y\in K$ such that $T_1y=x$. Thus $T_1T_2y=0$ and by the assumption $y=y_1+y_2$ for some $y_1\in N(T_1)$ and $y_2\in N(T_2)$. Then $T_1(y-y_1)=T_1y=x$ and $T_2(y-y_1)=T_2y_2=0$. So $x\in T_1N(T_2)$.

(ii)⇒(i): The inclusion $N(T_1)+N(T_2)\subset N(T_1T_2)$ is always true.

Let $x\in N(T_1T_2)$. Then $T_1x\in N(T_2)$, and so there exists $y\in N(T_2)$ with $T_1y=T_1x$. Hence $x-y\in N(T_1)$ and $x=(x-y)+y\in N(T_1)+N(T_2)$.

The equivalence (i)⇔(iii) follows from the symmetry. □

Corollary 8. Let $T_1,T_2\in B(K)$ be commuting surjective operators satisfying

(i) $\dim N(T_1)\cap N(T_2)=\infty$;
(ii) $N(T_1T_2)=N(T_1)+N(T_2)$.

Then the pair (T_1,T_2) is universal for all commuting pairs.

Examples 9. (1) Let H be a separable infinite-dimensional Hilbert space. Consider the space $K=H^2(\mathbb{Z}_+^n,H)$ consisting of all functions $f:\mathbb{Z}_+^n\to H$ satisfying

$$\|f\|^2:=\sum_{\alpha\in\mathbb{Z}_+^n}\|f(\alpha)\|^2<\infty.$$

The operators $T_1,\dots,T_n\in B(K)$ are defined by

$$(T_jf)(\alpha)=f(\alpha+e_j)\qquad(\alpha\in\mathbb{Z}_+^n).$$

Clearly the operators $T_1,\dots,T_n$ may be interpreted as adjoints of the multiplication operators $M_{z_1},\dots,M_{z_n}$ by the variables $z_1,\dots,z_n$ in the vector-valued Hardy space $H^2(\mathbb{D}^n,H)$, where $\mathbb{D}^n$ is the unit polydisc in $\mathbb{C}^n$.

Clearly the n-tuple $T_1,\dots,T_n$ satisfies the conditions of Theorem 3, so it is universal for commuting tuples.

2. Instead of the Hardy space in the polydisc $\mathbb{D}^n$ it is possible to consider the Hardy space $H^2(B_n,H)$ where B_n is the unit ball in $\mathbb{C}^n$. Again, the adjoints of multiplication operators $M_{z_1},\dots,M_{z_n}$ in this space form an n-tuple universal for commuting tuples.

Both of these examples play an important role in the multivariable dilation theory — the first example in the theory of regular dilations, see e.g. [CV], and the second one in the dilation theory of spherical contractions, see e.g. [MV]. In fact both n-tuples are universal in a stronger sense; they contain a unitarily equivalent copy of any commuting n-tuple of operators on a separable Hilbert space with sufficiently small norms.

3. Non-commuting case

Definition 10. We say that an n-tuple $T_1, \dots, T_n \in B(K)$ of operators is universal for all n-tuples if it has the following property: for each n-tuple $A_1, \dots, A_n \in B(H)$ there exist a constant $c \neq 0$ and a subspace $M \subset K$ invariant for all $T_1, \dots, T_n$ such that the n-tuples $(cA_1, \dots, cA_n)$ and $(T_1|M, \dots, T_n|M)$ are similar.

Theorem 11. Let $T_1, \dots, T_n \in B(K)$ satisfy the following properties:

(i) $\dim \bigcap_{j=1}^n N(T_j) = \infty$;

(ii) $T_j\Big(\bigcap_{i\neq j} N(T_i)\Big) = K$ for each $j = 1, \dots, n$.

Then the n-tuple $(T_1, \dots, T_n)$ is universal.

Proof. For $j = 1, \dots, n$ let $\hat{T}_j : K \to \bigcap_{i,i\neq j} N(T_i)$ be a right inverse of the restriction of T_j to the subspace $\bigcap_{i,i\neq j} N(T_i)$. Let $k = \max\{\|\hat{T}_j\| : j = 1, \dots, n\}$.

For $r \geq 0$ let F_r be the set of all finite sequences $\alpha_r, \alpha_{r-1}, \dots, \alpha_1$ of length r with $\alpha_j \in \{1, \dots, n\}$. Clearly $\operatorname{card} F_r = n^r$. Let $\mathcal{F} = \bigcup_{r=0}^\infty F_r$. For $r = 0$ the only element of F_0 will be denoted by $\emptyset$.

Let H be a separable Hilbert space and let $A_1, \dots, A_n \in B(H)$. Write $c = \max\{\|A_1\| \dots, \|A_n\|\}$. Let $V_\emptyset : H \to \bigcap_{j=1}^n N(T_j)$ be an isometry.

For $(\alpha_r, \dots, \alpha_1) \in \mathcal{F}$ define the operator $V_{\alpha_r,\dots,\alpha_1} : H \to K$ by

$$V_{\alpha_r,\dots,\alpha_1} = \hat{T}_{\alpha_r} \cdots \hat{T}_{\alpha_1} V_\emptyset.$$

Then $T_j V_{\alpha_r,\dots,\alpha_1} = 0$ if $\alpha_r \neq j$. If $\alpha_r = j$ then $T_j V_{\alpha_r,\dots,\alpha_1} = V_{\alpha_{r-1},\dots,\alpha_1}$. Moreover, $\|V_{\alpha_r,\dots,\alpha_1}\| \leq k^r$.

Define $V : H \to K$ by

$$Vh = \sum_{(\alpha_r,\dots,\alpha_1)\in\mathcal{F}} V_{\alpha_r,\dots,\alpha_1} A_{\alpha_1} \cdots A_{\alpha_r} h \qquad (h \in H).$$

We have

$$\|V - V_\emptyset\| \leq \sum_{(\alpha_r,\dots,\alpha_1)\in\mathcal{F}\setminus F_0} \|V_{(\alpha_r,\dots,\alpha_1)}\| c^r \leq \sum_{r=1}^\infty k^r n^r c^r = \frac{cnk}{1 - cnk} < 1$$

if c is small enough. So V is a bounded linear operator. Since $V_\emptyset$ is an isometry, V is bounded below and its range $M := VH$ is a closed subspace of K.

For all j and $h \in H$ we have $VA_j h = \sum_{\alpha\in\mathcal{F}} V_{\alpha_r,\dots,\alpha_1} A_{\alpha_1} \cdots A_{\alpha_r} A_j h$ and

$$T_j V h = \sum_{(\alpha_r,\dots,\alpha_1)\in\mathcal{F}} T_j V_{\alpha_r,\dots,\alpha_1} A_{\alpha_1} \cdots A_{\alpha_r} h = \sum_{\alpha_{r-1},\dots\alpha_1} V_{\alpha_{r-1},\dots,\alpha_1} A_{\alpha_1} \cdot A_{\alpha_{r-1}} A_j h.$$

So $T_j V = V A_j$. Hence M is a subspace of K invariant for all $T_1, \dots, T_n$ and the n-tuples $(A_1, \dots, A_n)$ and $(T_1|M, \dots, T_n|M)$ are similar. □

Example 12. Let $\mathcal{F} = \bigcup_{k=0}^{\infty} F_k$ denote as above the set of all words $\alpha = (\alpha_k, \dots, \alpha_1)$ with $\alpha_j \in \{1, \dots, n\}$ for all j. Consider the space K of all functions $f : \mathcal{F} \to H$ with

$$\|f\|^2 := \sum_{\alpha \in \mathcal{F}} \|f(\alpha)\|^2 < \infty.$$

Define the operators $S_1, \dots, S_n \in B(K)$ by

$$(S_j f)(\alpha_r, \dots, \alpha_1) = f(j, \alpha_r, \dots, \alpha_1) \qquad ((\alpha_r \dots, \alpha_1) \in \mathcal{F}).$$

Then the n-tuple $(S_1, \dots, S_n) \in B(K)^n$ is universal.

Again, this example plays an important role in the dilation theory for non-commuting tuples of operators, see [P].

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