Wave equations, examples and qualitative properties

Eduard Feireisl

Abstract

This is a short introduction to the theory of nonlinear wave equations. After a preliminary part devoted to the simplified $1D-$problem, we shortly discuss the blow-up phenomena for the quasilinear and semilinear wave equations. Then we develop an existence theory for a class of semilinear wave equations under suitable restrictions on the structural properties of the nonlinearities. In the final part, we discuss the problem of free vibrations for the semilinear wave equation in the $1D-$geometry.

Contents

1 Introduction

There is a large number of real world phenomena that fits in the category of wave motion. We start with a simple example of *transport equation*

$$(\partial_t + c\partial_x)[u] = 0. \tag{1.1}$$

Obviously, the solutions of (1.1) also satisfy

$$\partial_{t,t}u(t,x) - c^2\partial_{x,x}u(t,x) = (\partial_t - c\partial_x) \circ (\partial_t + c\partial_x)[u] = 0. \tag{1.2}$$

Equation (1.2) is a simple example of *wave equation*; it may be used as a model of an infinite elastic string, propagation of sound waves in a linear medium, among other numerous applications. We shall discuss the basic properties of solutions to the wave equation (1.2), as well as its multidimensional and non-linear variants. To begin, we remark that (1.2) falls in the category of *hyperbolic equations*, in accordance with the form of its principal part in the frequency (Fourier) variables

$$\partial_{t,t}u(t,x) - c^2\partial_{x,x}u = \mathcal{F}^{-1}_{(\xi_0,\xi_1)\to(t,x)}\left[(\xi_0^2 - \xi_1^2)\mathcal{F}_{(t,x)\to(\xi_0,\xi_1)}[u]\right],$$

where $\mathcal{F}$ denotes the standard Fourier transform.

Equation (1.2) can be written in several rather different form. Setting $\partial_t u = v$ we may rewrite (1.2) as a system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}, \text{ with } \mathcal{A} = \begin{bmatrix} 0 & \mathrm{Id} \\ c^2\partial^2_{x,x} & 0 \end{bmatrix}.$$

There is another way how to write (1.2), this time as a first order system

$$\partial_t u + \partial_x v = 0, \ \partial_t v + c^2\partial_x u = 0. \tag{1.3}$$

Equation (1.3) has a *non-linear* variant

$$\partial_t u + \partial_x v = 0, \ \partial_t v + \partial_x \sigma(u) = 0$$

that can be interpreted in two different ways, namely as

$$\partial^2_{t,t}u - \partial^2_{x,x}\sigma(u) = 0,$$

or, introducing a scalar potential Φ,

$$\partial_x\Phi = u, \ \partial_t\Phi = -v,$$

we obtain

$$\partial^2_{t,t}\Phi - \partial_x\left(\sigma(\partial_x\Phi)\right) = 0.$$

Finally, going back to (1.3), we easily deduce

$$\partial_t R + c\partial_x R = 0,\ \partial_t S - c\partial_x S = 0,\ R = v + cu,\ S = v - cu. \tag{1.4}$$

Note that the the systems decouples in (1.4) and can be easily solved by the method of characteristic lines.

1.1 Exercises

1.1.1 Duhamel's formula

Suppose that $X(t) = T(t)[X_0]$ is a solution operator for the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = A[X(t)],\ X(0) = X_0,$$

where A is a *linear* operator. Show (formally) that

$$Y(t) = \int_0^t T(t-s)[f(s)]\ \mathrm{d}s$$

is a solution of the non-homogeneous problem

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = A[Y(t)] + f(t),\ Y(0) = 0.$$

1.2 Bibliography

There several basic texts concerning the theory of wave motion and wave equations. The reader interested in physical aspects may consult the monographs by Billingham and King [2], Debnath [5], Lighthill [10]. Mathematical aspects are nicely exposed by John [7], Leis [8], Strauss [15] , Vaĭnberg [17]. For more recent and advanced treatment, we refer to Tao [16]. More references will be mentioned in relevant parts of the text.

2 1-D linear wave equation

Writing

$$\partial_{t,t}u(t,x) - c^2\partial_{x,x}u(t,x)$$
$$= (\partial_t - c\partial_x) \circ (\partial_t + c\partial_x)[u] = (\partial_t + c\partial_x) \circ (\partial_t - c\partial_x)[u],$$

we easily observe that solutions of (1.2) can be written in the form

$$u(t,x) = v(x+ct) + w(x-ct),\ t \in R,\ x \in R. \tag{2.1}$$

The general formula (2.1) yields solutions of (1.2) defined for both positive and *negative* values of the time t. The processes described by means of the wave equations like (1.2) are perfectly time reversible.

2.1 Uniqueness, finite speed of propagation

Multiplying the operator in (1.2) by u we obtain

$$\partial_t \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big) - c^2\partial_x\left(\partial_x u \partial_t u\right) = 0, \tag{2.2}$$

where the quantity

$$E = \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)$$

represents the *energy*. Given an interval $[a,b] \subset R^1$ we may integrate (2.2) over the cone

$$C_{a,b,\tau} = \Big\{ t \in (0,\tau),\ x \in R^1 \ \Big|\ 0 < t < \tau,\ x \in (a+ct, b-ct) \Big\},$$

and use the Gauss-Green theorem to obtain

$$\int_{a+\tau c}^{b-\tau c} \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)\ \mathrm{d}x = \int_a^b \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)\ \mathrm{d}x \tag{2.3}$$
$$-\frac{1}{\sqrt{1+c^2}}\int_0^\tau \frac{c}{2}\Big(|\partial_t u(t,a+ct)|^2 + c^2|\partial_x u(t,a+ct)|^2\Big) + c^2\left(\partial_x u \partial_t u\right)(t,a+ct)\ \mathrm{d}t$$
$$-\frac{1}{\sqrt{1+c^2}}\int_0^\tau \frac{c}{2}\Big(|\partial_t u(t,b-ct)|^2 + c^2|\partial_x u(t,b-ct)|^2\Big) - c^2\left(\partial_x u \partial_t u\right)(t,b-ct)\ \mathrm{d}t$$
$$\leq \int_a^b \frac{1}{2}\Big(|\partial_t u|^2 + c^2|\partial_x u|^2\Big)\ \mathrm{d}x.$$

Thus the values of the solution in the wave cone $C_{a,b,\tau}$ are uniquely determined by the value of the "initial data" in terms of $\partial_t u$ and $\partial_x u$ at the initial time $t = 0$. The solutions of the wave equation (1.2) admit a finite speed of propagation $c > 0$. This is a characteristic feature of all hyperbolic problems, meaning the solutions propagate along characteristic curves (lines).

2.2 D'Alembert solution operator

As we have observed in the previous discussion, the solutions of the wave equation (1.2) are

- given by the formula (2.1),
- uniquely determined by u and $\partial_t u$ at the initial time $t = 0$.

Consequently, in terms of the functions v, w introduced in (2.1),

$$u(0,x) = u_0(x) = v(x) + w(x),$$

$$\partial_t u(0,x) = u_1(x) = cv'(x) - cw'(x);$$

whence, going back to (2.1), we deduce the so-called *D'Alembert solution formula*:

$$u(t,x) = \frac{1}{2}\Big[u_0(x+ct) + u_0(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)\,\mathrm{d}s. \tag{2.4}$$

Jean le Rond dAlembert
(wikipedia)

French mathematician, philosopher, and encyclopedist
[1717 (Paris) - 1783 (Paris)]

- D'Alembert rule
- D'Alembert theorem (Gauss - D'Alembert):

  A polynomial of degree N with complex coefficients possesses exactly N complex roots
- D'Alembert martingale

It is easy to check that the function u given through (2.3) **(i)** solves the homogeneous wave equation (1.2) and **(ii)** satisfies the initial conditions

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ x \in R \tag{2.5}$$

as long as u_0, u_1 are twice continuously differentiable in R.

We note immediately that solutions of the wave equation obtained from (2.4) inherit the regularity of the initial data (2.5). What is more, formula (2.4) could be used to provide a kind of "generalized" solution to the initial-value problem (1.2), (2.5) provided the data u_0, u_1 are not smooth enough. Indeed, for non-smooth data, say

$$u_0 \in L^1_{\rm loc}(R^1),\ u_1 \in L^1_{\rm loc}(R^1),$$

we can find a sequence of smooth functions $u_{0,\varepsilon} \in C^\infty_c(R)$, $u_{1,\varepsilon} \in C^\infty_c(R)$ such that

$$u_{0,\varepsilon} \to u_0,\ u_{1,\varepsilon} \to u_1 \text{ in } L^1(K) \text{ for any compact set } K \in R^1$$

and use (2.4) to conclude that the corresponding (unique) solutions u_ε of (1.2), (2.5) converge in $L^1_{\rm loc}([0,T] \times R^1)$ to a (unique) function u that may be viewed as a "weak" solution of the same problem with the initial data u_0, u_1.

2.3 Dispersion and local energy decay

Before starting our study of more complicated and even nonlinear analogues of the wave equation (1.2), we take advantage of the simplicity of D'Alembert's formula (2.4) to illustrate other characteristic features of wave propagation. We have seen in Section 2.1 that the *total energy*

$$\int_{R^1} E(t,x)\ {\rm d}x = \frac{1}{2}\int_{R^1} \left(|\partial_t u(t,\cdot)|^2 + |\partial_x u(t,\cdot)|^2 \right)\ {\rm d}x$$

is a constant of motion, meaning *independent* of time for any solution of (1.2). Of course, we need the above integral to be finite at least at one time instant t t_0. This can be easily seen for *compactly supported* initial data u_0, u_1 by means of formula (2.3) and then extended via density argument to general u_0, u_1.

Consider the *local energy*

$$\mathcal{E}_{a,b}(t) = \int_a^b E(t,x)\ {\rm d}x \text{ for } -\infty < a < b < \infty.$$

Going back to D'Alembert's formula (2.4) we may compute

$$\int_{-T}^{T} \mathcal{E}_{a,b}(t)\ {\rm d}t = \frac{1}{2}\int_{-T}^{T}\int_a^b \left(|\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2 \right)\ {\rm d}x\ {\rm d}t$$

$$\leq 2(c+1)\int_{-T}^{T}\int_a^b \left(|\partial_x u_0(x+ct)|^2 + |\partial_x u_0(x-ct)|^2 \right) {\rm d}x\ {\rm d}t$$

$$+2(c+1)\int_{-T}^{T}\int_a^b \left(|\partial_x u_1(x+ct)|^2 + |\partial_x u_1(x-ct)|^2 \right)\ {\rm d}x\ {\rm d}t$$

$$\leq 4(b-a)\int_{R^1} \left(|\partial_x u_0(x)|^2 + |\partial_x u_1(x)|^2 \right)\ {\rm d}x$$

for any $T > 0$.

Letting $T \to \infty$ we may therefore infer that

$$\int_{-\infty}^{\infty}\left[\int_a^b E(t,x)\right] {\rm d}x\ {\rm d}t \leq 4(b-a)\int_{R^1} E(0,x)\ {\rm d}x, \tag{2.6}$$

which may be interpreted as *local energy decay.* In accordance with (2.1), waves - solutions of (1.2) emanating from spatially localized initial data - decay locally to zero in the integral sense (2.6). We may also observe *uniform* time decay, meaning

$$\left[\int_a^b E(t,x)\right] \to 0 \text{ as } t \to \infty$$

but only for *compactly supported* initial data. These phenomena are conditioned by unboundedness of the physical space R^1, where the waves have enough space to disperse. As we shall see later, the situation is completely different on *bounded* intervals, where the waves are reflected by the boundary.

Finally, we note that the local L^2-norm of a solution

$$\int_a^b |u(t,x)|^2 \, \mathrm{d}x$$

may remain bounded below away from zero as $t \to \infty$ for *certain* data as a direct consequence of D'Alembert's formula.

2.4 Wave equation on bounded intervals

Consider the 1-D wave equation

$$\partial^2_{t,t} u(t,x) - c^2 \partial^2_{x,x} u(t,x) = 0 \tag{2.7}$$

for x belonging to a *bounded* interval, say $(0,\pi)$, supplemented with the boundary conditions

$$u(t,0) = u(t,\pi) = 0, \ t > 0, \tag{2.8}$$

and the initial conditions,

$$u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \ x \in (0,\pi). \tag{2.9}$$

Our goal is to observe that D'Alembert's formula (2.4) can be adapted to the initial-boundary value problem (2.7 - 2.9). To this end, we first suppose that u_0, u_1 are spatially periodic with the period 2π. In such a case, it is easy to check that D'Alembert's formula yields a solution u with the same property, meaning periodic in x. In addition, assuming that both u_0 and u_1 are odd functions,

$$u_0(-x) = -u_0(x), \ u_1(-x) = -u_1(x), \ x \in R^1$$

we easily observe that u given by (2.4) is also odd. In particular, as a byproduct, we recover the boundary conditions (2.8).

We conclude that D'Alembert's formula yields a solution for the problem (2.7 - 2.9) provided the initial data u_0, u_1 were extended as odd, $2\pi-$periodic functions in R^1. Similarly, replacing *odd* by *even* we may deduce a solution formula for the problem with the so-called Neumann boundary condition

$$\partial_x u(t,0) = \partial_x u(t,\pi) = 0. \tag{2.10}$$

Carl Neumann
(wikipedia)

German mathematician, physicist, and mineralogist
[1832 (Koenigsberg) - 1925 (Leipzig)]

- works on Dirichlet principles
- Neumann series

At this point, it is important to notice that *extending* the function u_0 to be odd requires certain restrictions on $\partial^2_{x,x} u_0$ at the boundary points $x = 0, \pi$ provided we want C^2-solutions, namely,

$$\partial^2_{x,x} u_0(0) = \partial^2_{x,x} u_0(\pi) = 0. \tag{2.11}$$

Relations (2.11) are called *compatibility conditions*, and their meaning is the the initial datum u_0 satisfies the equation (2.7) at $t = 0$ provided we set, in accordance with (2.8),

$$\partial^2_{t,t} u(0,\cdot) = 0.$$

Similarly, we derive that the compatibility conditions for the Neumann problem (2.10) read simply

$$\partial_x u_0(0) = \partial_x u_0(\pi) = 0.$$

Next, we introduce the total energy

$$\frac{1}{2}\int_0^\pi \left(|\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2 \right) \, \mathrm{d}x,$$

exactly as in Section (2.1). However, unlike in the case of spatially localized solutions defined on the whole real line R^1, the total energy, though evaluated over a compact interval, *does not* decay to zero as $t \to \infty$. It can be easily seen that the total energy is actually *conserved*, meaning constant in time, as a consequence of our choice of the boundary conditions.

We conclude this part by a simple but rather interesting observation that *all* solutions to the initial-boundary value problem (2.7 - 2.9) are also $\frac{2\pi}{c}$-*time periodic*, a property that can be easily deduced from (2.4).

2.5 Riemann invariants, observability

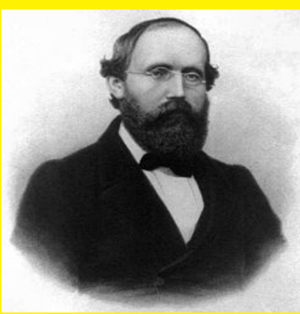
There are several ways how to write the wave equation (1.2). One possibility is to introduce the so-called *Riemann invariants*

$$R = \partial_t u + c\partial_x u,\ S = \partial_t u - c\partial_x u \tag{2.12}$$

and rewrite (1.2) as a *system*

$$\partial_t R(t,x) - c\partial_x R(t,x) = 0,\ \partial_t S(t,x) + c\partial_x S(t,x) = 0 \tag{2.13}$$

of two independent transport equations. Accordingly, the quantity R is constant along the lines $t \mapsto [t, x - ct]$, while S is constant on $t \mapsto [t, x + ct]$ for $x \in R$.



Georg Friedrich Bernhard Riemann
(wikipedia)

German mathematician
[1826 (Breselenz) - 1866 (Verbania, Italy)]

- Riemann problem
- zeta function
- Riemann hypothesis

Now, we exploit the relatively simple form of (2.13) to show *boundary observability* property for (1.2). To be more specific, we consider the situation described by the initial-boundary value problem (2.8), where the solutions u and, consequently, $\partial_t u$ vanish on the boundary $x = 0, \pi$. Since (2.13) is a system of two independent transport equations, we easily deduce that

$$\int_0^\pi R^2\left(\frac{2\pi}{c}, x\right)\ \mathrm{d}x \leq c\int_0^{2\pi/c} R^2(t,0)\ \mathrm{d}t = c^3\int_0^{2\pi/c} |\partial_x u(t,0)|^2\ \mathrm{d}t, \tag{2.14}$$

and, similarly,

$$\int_0^\pi S^2\left(\frac{2\pi}{c}, x\right)\ \mathrm{d}x \leq c\int_0^{2\pi/c} S^2(t,0)\ \mathrm{d}t = c^3\int_0^{2\pi/c} |\partial_x u(t,0)|^2\ \mathrm{d}t. \tag{2.15}$$

We recall the convention that u may be viewed as a $2\pi-$spatially periodic odd function, while $\partial_x u$ is $2\pi-$spatially periodic even.

Consequently, relations (2.14), (2.15) give rise to

$$\int_0^\pi E(2\pi/c, x)\ \mathrm{d}x = \frac{1}{2}\int_0^\pi \left(|\partial_t u|^2 + |\partial_x u|^2\right)(2\pi/c, x)\ \mathrm{d}x$$

$$\leq \mathrm{const}(c)\int_0^{2\pi/c} |\partial_x u(0,t)|^2\ \mathrm{d}t.$$

However, the total energy is a constant of motion and we deduce the *observability inequality*:

$$\int_0^\pi E_0\ \mathrm{d}x = \frac{1}{2}\int_0^\pi \left(|\partial_x u_0|^2 + |u_1|^2\right)\ \mathrm{d}x \leq \mathrm{const}(c)\int_0^{2\pi/c} |\partial_x u(t,0)|^2\ \mathrm{d}t. \tag{2.16}$$

The message hidden in (2.16) reads that the behavior of solutions to the boundary value problem (2.7 - 2.9) is entirely controlled (determined) by the boundary values of $\partial_x u$ on the time interval of the length at least $2\pi/c$.

Repeating the same arguments we can show a more general inequality

$$\int_{-\pi}^\pi E_0\ \mathrm{d}x \leq \mathrm{const}(c)\int_0^{2\pi/c} \left(|\partial_t u(t,\xi)|^2 + |\partial_x u(t,\xi)|^2\right)\mathrm{d}t \text{ for any } \xi \in [-\pi, \pi] \tag{2.17}$$

that holds for any $2\pi-$spatially periodic solution u, in particular for any solution of the initial-boundary value problems (2.7), (2.8), (2.9) and (2.7), (2.9), (2.10).

2.6 Uniqueness and data dependence

As we have seen in Section 2.1, *smooth* solutions are uniquely determined by their initial values on any wave cone. We have used the Gauss-Green formula, and, in particular, the existence of suitably defined traces. Here, we show that solutions of the wave equation (1.2) are still uniquely determined by the initial data even if we suppose much less regularity. To begin, we extend the class of solutions saying that u is a *weak solution* of the wave equation (1.2) on the space-time cylinder $(0,T)\times B$ if the integral identity

$$\int_0^T \int_B u(t,x)\Big(\partial^2_{t,t}\varphi(t,x) - c^2\partial^2_{x,x}\varphi(t,x)\Big)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{2.18}$$

holds for any test function $\varphi \in C^\infty_c((0,T)\times B)$.

First, we observe that the "initial values" of u and $\partial_t u$ at the time $t = 0$ can be well defined. To this end, we take a special test function $\varphi(t,x) = \psi(t)\phi(x)$ in (2.18). We easily check that the mapping

$$t \mapsto \int_B u(t,x)\phi(x)\ \mathrm{d}x\ \mathrm{d}t$$

has two derivatives with respect to the t variable integrable in $[0,T]$ provided $u \in L^1((0,T)\times B)$. In particular,

$$t \mapsto \int_B u(t,x)\phi(x)\ \mathrm{d}x\ \mathrm{d}t,\ \partial_t\left(t \mapsto \int_B u(t,x)\phi(x)\right)$$

may be viewed as *continuous* functions of $t \in [0,T]$. In particular, it makes sense to speal about the values of u and $\partial_t u$ at any time $t \in [0,T]$.

Assuming the solution u is more regular, say,

$$\partial_t u,\ \partial_x u \in L^1((0,T)\times B)$$

we deduce from (2.18) that

$$\left\{ \begin{array}{l} \int_0^T \int_B (\partial_t u + c\partial_x u)\,(\partial_t\varphi - c\partial_x\varphi)\ \mathrm{d}x\ \mathrm{d}t = 0, \\ \\ \int_0^T \int_B (\partial_t u - c\partial_x u)\,(\partial_t\varphi + c\partial_x\varphi)\ \mathrm{d}x\ \mathrm{d}t = 0. \end{array} \right\} \tag{2.19}$$

Thus the Riemann invariants R, S introduced in (2.12), being now solely integrable functions in $(0,T)\times B$, are still constant on the characteristic lines $t \mapsto [t, x-ct]$, $t \mapsto [t, x+ct]$, $x \in B$, respectively. In particular, the solution u is uniquely determined in the wave cone

$$C_{B,T} = \{t \in (0,T),\ y \in B \mid t \in (0,T),\ y = x + ct \text{ or } y = x - ct \text{ for a certain } x \in B\}$$

by the initial values u and $\partial_t u$ at $t = 0$, $x \in B$.

2.7 Exercises

2.7.1 Non-homogeneous wave equation

Using Duhamel's formula (see Section 1.1.1) show that

$$u(t,x) = \frac{1}{2}\Big[u_0(x+t) + u_0(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\ \mathrm{d}s$$

$$+\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \Big[f(s,z)\Big]\mathrm{d}z\ \mathrm{d}s$$

is a solution of the non-homogeneous problem

$$\partial^2_{t,t} u(t,x) - \partial^2_{x,x} u(t,x) = f(t,x),\ t>0,\ x\in R^1,$$

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ x\in R^1.$$

3 1-D nonlinear wave equation

We discuss briefly the situation when the speed of propagation dependends on $\partial_x u$, specifically,

$$\partial^2_{t,t} u(t,x) - \partial_x \sigma\Big(\partial_x u(t,x)\Big) = 0. \tag{3.1}$$

Our main goal is to show that, in general, equation (3.1) does not admit smooth solutions no matter how regular and small the initial data are.

3.1 Riemann invariants

Similarly to Section 2.5, we rewrite equation (3.1) in terms of the Riemann invariants. Writing

$$U = \partial_t u,\ V = \partial_x u$$

we obtain

$$\partial_t V - \partial_x U = 0,\ \partial_t U - \partial_x \sigma(V) = 0. \tag{3.2}$$

Furthermore, introducing

$$h(Z) = \int_0^Z \sqrt{\sigma'(s)}\ \mathrm{d}s$$

we get

$$\partial_t h(V) - \sqrt{\sigma'(V)}\partial_x U = 0,\ \partial_t U - \sqrt{\sigma'(V)}\partial_x h(V) = 0;$$

whence

$$\partial_t \left[U + h(V)\right] - \sqrt{\sigma'(V)}\partial_x \left[U + h(V)\right] = 0, \tag{3.3}$$

and

$$\partial_t \left[U - h(V)\right] + \sqrt{\sigma'(V)}\partial_x \left[U - h(V)\right] = 0. \tag{3.4}$$

By analogy with Section 2.5, the quantities

$$R = U + h(V),\ S = U - h(V)$$

are termed *Riemann invariants*

3.2 Shock waves

It is easy to deduce from (3.3), (3.4) that the nonlinear equation (3.1) does not admit, in general, global in time smooth solutions. Indeed we can take that initial data so that $U = h(V)$, meaning $S = 0$. In accordance with (3.4), this property is preserved at any positive time as S is constant along characterisic curves

$$\mathbf{X}' = \sqrt{\sigma'(V(t,\mathbf{X})},\ \mathbf{X}(0) = \mathbf{X}_0.$$

In particular, equation (3.3) reads

$$\partial_t U - \frac{1}{2}\sqrt{\sigma'(h^{-1}(U))}\partial_x U = 0, \tag{3.5}$$

which is nothing other than a quasilinear transport equation discuss. In particular, solutions of (3.5) may develop discontinuities (shock waves) in a finite time even if the initial data are taken smooth and small.

3.3 Exercises

3.3.1 Shock waves for transport equations

Using the method of characteristics show that solutions of the $1D-$transport equation

$$\partial_t u + \partial_x \sigma(u) = 0$$

develops singularities in a finite time provided σ is a *non-linear* function.

3.4 Bibliography

A classical introduction to the theory of shock waves is the monograph by Smoller [14]. A more recent exposition of the theory of nonlinear conservation laws can be found in Dafermos [4] or Benzoni-Gavage and Serre [1], [13].

4 Semilinear equations

We finish our study of the wave equations by the semilinear problem

$$\partial^2_{t,t} u(t,x) - \partial^2_{x,x} u(t,x) + f(u(t,x)) = 0, \tag{4.1}$$

together with its multidimensional analogue

$$\partial^2_{t,t} u(t,x) - \Delta_x u(t,x) + f(u(t,x)) = 0, \tag{4.2}$$

supplemented with suitable boundary as well as initial conditions.

In contrast with the example of a *quasilinear* equation examined in the previous section, the equations (4.1), (4.2) are nonlinear only on the lower order terms. Thus we expect, at least under certain hypotheses imposed on f, that the solutions will inherit regularity of the initial data.

4.1 Finite time blow-up

Solutions of *non-linear* equations may not exist an arbitrary long time intervals. We have seen an example of a singular behavior in Section 3.2, where solutions of a quasilinear equation developed singularities in the form of shock waves in a finite time. For *semi-linear* equations like (4.1), (4.2), solutions may develop a *blow-up*, where the amplitude becomes infinite in a finite time. In contrast with the shock waves, where usually the solutions my be “continued” in some form, the blow up behavior may lead to the ultimate state with hypothetial “infinite” energy. Here, we employ the method based on *convexity* of the nonlinear response function developed by Levine [9].

Consider regular solutions of the semilinear equation (4.1), supplemented, for definiteness, with the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0. \tag{4.3}$$

Johann Peter Gustav Lejeune Dirichlet
(wikipedia)

German mathematician
[1805 (Dueren) - 1859 (Goettingen)]

- number theory
- Dirichlet principle
- mathematical analysis, function theory

Multiplying the equation by $\partial_t u$, integrating by parts and making use of the boundary conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\pi \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\partial_x u|^2 + F(u)\right)(t,\cdot)\ \mathrm{d}x = 0,$$

where we have set

$$F(u) = -\int_0^u f(z)\ \mathrm{d}z.$$

The quantity

$$E(t) = \int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\partial_x u|^2 + F(u)\right)(t,x)\ \mathrm{d}x$$

plays the role of *energy* for the semilinear wave equation (4.2), and, as we have just observed, it is a constant of motion. Note however that "energy" defined in such a way may be negative.

Introducing

$$I(t) = \frac{1}{2}\int_\Omega |u(t,x)|^2\ \mathrm{d}x$$

we easily compute

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = \int_\Omega u(t,x)\partial_t u(t,x)\ \mathrm{d}x,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) = \int_\Omega \left(|\partial_t u(t,x)|^2 + u(t,x)\partial^2_{t,t}u(t,x)\right)\ \mathrm{d}x,$$

where, by virtue of (4.2),

$$\int_\Omega u(t,x)\partial^2_{t,t}u(t,x)\ \mathrm{d}x = -\int_\Omega \Big(|\nabla_x u(t,x)|^2 + f(u(t,x))u(t,x)\Big)\ \mathrm{d}x.$$

Thus, combining the previous two identities, we arrive at

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) = (2+2\lambda)\int_\Omega |\partial_t u(t,x)|^2\ \mathrm{d}x + 2\lambda\int_\Omega |\nabla_x u(t,x)|^2\ \mathrm{d}x \\ + \int_\Omega \Big((2+4\lambda)F(u) - uf(u)\Big)(t,x)\ \mathrm{d}x - (2+4\lambda)E\end{aligned} \tag{4.4}$$

for any $\lambda \geq 0$.

Suppose that

- there exists $\varepsilon > 0$ such that

$$(2+\varepsilon)F(u) \geq uf(u) \text{ for all } u \in R; \tag{4.5}$$

- the energy $E(t) = E < 0$ is negative.

Consequently, we can take $\lambda = \varepsilon/4$ and compute

$$\begin{aligned}I(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) \geq (1+\lambda)\int_\Omega |u(t,x)|^2\ \mathrm{d}x \int_\Omega |\partial_t u(t,x)|^2\ \mathrm{d}x \\ \geq (1+\lambda)\left|\frac{\mathrm{d}}{\mathrm{d}t}I(t)\right|^2,\end{aligned} \tag{4.6}$$

where we have used the Cauchy-Schwartz inequality. Moreover, it follows from (4.5) that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}I(t) \geq -(2+4\lambda)E > 0,$$

in particular, I is strictly convex and there is $\tau > 0$ such that

$$I(\tau) > 0,\ I'(\tau) > 0.$$

Thus, finally, dividing (4.6) on II', we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{\mathrm{d}}{\mathrm{d}t}I(t)\right) \geq \frac{\mathrm{d}}{\mathrm{d}t}\log\Big(I^{1+\lambda}(t)\Big) \text{ for all } t \geq \tau,$$

from which we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \geq \mu I^{1+\lambda}(t), \text{ with } \mu = \frac{\mathrm{d}}{\mathrm{d}t} I(\tau) I^{-1-\lambda}(\tau) > 0. \tag{4.7}$$

Relation (4.7) yields the existence of a *finite* number T such that

$$\lim_{t \to T-} I(t) = \infty. \tag{4.8}$$

We conclude that solutions of the problem (4.2), (4.3) with negative total energy E must blow-up in a finite time as soon as that nonlinearity f satisfy the convexity hypothesis (4.5). It is easy to check that the latter holds, for instance, if

$$f(u) = mu - |u|^{p-1}u, \ m \geq 0, \ p > 1.$$

Moreover, for such an f, we can find a couple of (smooth) functions $u_0 = u_0(x)$, $u_1 = u_1(x)$ satisfying

$$E_0 = \int_\Omega \left(\frac{1}{2} |u_1(x)|^2 + \frac{1}{2} |\partial_x u_0(x)|^2 + F(u_0(x)) \right) \, \mathrm{d}x < 0.$$

Accordingly, any classical solution u of the nonlinear wave equation (4.2) emanating from the initial data

$$u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x), \ x \in \Omega \tag{4.9}$$

must blow-up in a finite time T, specifically,

$$\int_\Omega |u(t, x)|^2 \, \mathrm{d}x \to \infty \text{ as } t \to T.$$

It is worth noting that the arguments used in the above discussion were based mainly on the structural properties of the non-linearity f. Accordingly, similar examples may be constructed for other types of boundary conditions and also on a large class of (unbounded) spatial domains, in particular for $\Omega = R^3$.

4.2 Soliton solutions, breathers

The example discussed in the previous section showed that the semilinear wave equation need not to possess a global-in-time solution for a certain class of convex nonlinearities. Here, we consider a seemingly similar problem, namely the so-called *Sine-Gordon equation*

$$\partial_{t,t}^2 u(t,x) - \partial_{x,x}^2 u(t,x) + \sin(u(t,x)) = 0,\ x \in R, \tag{4.10}$$

where the solutions are defined on the whole real line and decay for large x,

$$\lim_{|x|\to\infty} u(t,x) = 0. \tag{4.11}$$

Equation (4.10) possesses an explicit solution, namely

$$u(t,x) = 4\arctan\left(\frac{\sqrt{1-\omega^2}\cos(\omega t)}{\omega\cosh\left(\sqrt{1-\omega^2}x\right)}\right) \tag{4.12}$$

for $0 < \omega < 1$.

The solution given through (4.12) is called *breather*; it is time-periodic and spatially localized. Breathers belong to the class of solutions to nonlinear evolutionary equations termed *solitons*. Solitons are stable objects and may interact. There is a vast literature devoted to solitons and their basic properties. The evolutionary equations possessing soliton solutions are typically *completely integrable*, meaning, possessing and infinite family of conserved quantities. Here, we restrict ourselves to claiming that the Sine-Gordon equation (4.10) possesses this kind of spatially localized solutions.

4.3 A priori bounds

Unlike the equations with convex nonlinearities discussed in Section 4.1, the solutions of the Sine-Gordon equation (4.10) remain bounded on compact time intervals. Indeed we may write

$$\partial_{t,t}^2 u(t,x) - \partial_{x,x}^2 u(t,x) + \sin(u(t,x))$$

$$= \partial_{t,t}^2 u(t,x) - \partial_{x,x}^2 u(t,x) + u(t,x) + \sin(u(t,x)) - u(t,x);$$

whence, multiplying (4.10) on $\partial_t u$ and integrating by parts, we may infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_R \left(|\partial_t u|^2 + |\partial_x u|^2 + |u|^2\right)(t,x)\ \mathrm{d}t \tag{4.13}$$

$$\leq \int_R \left(\sin(u(t,x)) - u(t,x)\right)\partial_t u(t,x)\ \mathrm{d}t \leq 2\int_\Omega |u(t,x)||\partial_t u(t,x)|\ \mathrm{d}x$$

$$\leq \int_\Omega |u(t,x)|^2\ \mathrm{d}x + \int_\Omega |\partial_t u(t,x)|^2\ \mathrm{d}x.$$

Thus, by virtue of Gronwall's lemma,

$$\int_R \left(|\partial_t u|^2 + |\partial_x u|^2 + |u|^2 \right)(t,x)\ \mathrm{d}t \tag{4.14}$$
$$\leq \exp(2t) \int_R \left(|\partial_t u|^2 + |\partial_x u|^2 + |u|^2 \right)(0,x)\ \mathrm{d}t.$$

Thomas Hakon Gronwall
(wikipedia)

Swedish mathematician
[1877 (Dylta bruk) - 1932 (New York)]

- Gronwall area theorem
- Gronwall inequality

The relation (4.14) yields bounds, uniform with respect to compact time intervals, on the L^2-norms of $\partial_t u$, $\partial_x u$, and u in terms of the initial data. Specifically, denoting

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x)$$

we get

$$\sup_{t\in[0,T]} \left(\|\partial_t u(t,\cdot)\|_{L^2(R)} + \|u(t,\cdot)\|_{W^{1,2}(R)} \right) \tag{4.15}$$
$$\leq c(T) \left(\|u_1\|_{L^2(R)} + \|u_0\|_{W^{1,2}(R)} \right).$$

We may wish to deduce similar bounds on higher order derivatives. To this end, we take the time derivative of the equation (4.10), and, denoting $\partial_t u = v$ we obtain

$$\partial^2_{t,t} v - \partial^2_{x,x} v + \cos(u) v = 0.$$

Since $|\cos(u)| \leq 1$, we can repeat the arguments leading to (4.15) to obtain

$$\sup_{t\in[0,T]} \left(\|\partial^2_{t,t} u(t,\cdot)\|_{L^2(R)} + \|\partial_t u(t,\cdot)\|_{W^{1,2}(R)} \right) \tag{4.16}$$

$$\leq c(T) \left(\|u_0\|_{W^{2,2}(R)} + \|u_1\|_{W^{1,2}(R)} \right),$$

where we have used (4.10) to express

$$\partial^2_{t,t} u(0,\cdot) = \partial^2_{x,x} u_0 - \sin(u_0).$$

Moreover, as

$$|\sin(u)| \leq |u|,$$

we may use once more the equation (4.10) to include $\partial^2_{x,x}$ in the left-hand side of (4.16):

$$\sup_{t\in[0,T]} \left(\|\partial^2_{t,t} u(t,\cdot)\|_{L^2(R)} + \|\partial_t u(t,\cdot)\|_{W^{1,2}(R)} + \|u(t,\cdot)\|_{W^{2,2}(R)} \right) \tag{4.17}$$

$$\leq c(T) \left(\|u_0\|_{W^{2,2}(R)} + \|u_1\|_{W^{1,2}(R)} \right).$$

Thanks to the standard embedding relations

$$W^{1,2}(R) \hookrightarrow BC(R), \tag{4.18}$$

the estimate (4.17) yields uniform bounds on u and its first derivatives in the space of bounded and continuous function on R. In particular, we may continue the above procedure be differentiating (4.10) in t and x to obtain uniform bounds on the solutions in the Sobolev space $W^{k,2}(R)$ of an arbitrary order $k = 0, 1, \ldots$. We note that, by virtue of (4.18), that solutions are classical, meaning twice continuously differentiable, if $k \geq 3$. Unlike the situation treated in Section 4.1, where the norm of solutions blows-up in a finite time, the solutions of the Sine-Gordon equation (4.10) are controlled by the initial data. This is obviously due to the specific properties of the nonlinearity, here represented by a uniformly Lipschitz function $\sin(u)$.

Sergei Lvovich Sobolev
(wikipedia)

Russian mathematician
[1908 (Saint Petersburg) - 1989 (Moscow)]

- Sobolev spaces
- Sobolev equation
- Sobolev inequality

The estimates (4.15 - 4.17) are *formal*. They have been derived under the principal hypothesis that a sufficiently smooth solution u *exists*. Such a type of bounds is usually called *a priori estimates* in the literature. Intuitively, the available *a priori bounds* determine the function spaces framework suitable for a given *nonlinear* problem. From this point of view, the scale of Sobolev spaces $W^{k,2}$ resulting from the "energy" estimates (4.15 - 4.17) is more convenient for second-order problems like (4.10) rather than the classical framework of continuous functions.

4.4 Exercises

4.4.1 Global in time solutions for the blow-up nonlinearities

Consider the semilinear wave equation in the form

$$\partial^2_{t,t} u - \partial^2_{x,x} u + u = |u|^{r-1} u, \ r > 1,$$

supplemented with the homogeneous Dirichlet boundary conditions

$$u(t, 0) = u(t, \pi) = 0$$

Show that for “small” initial data the solutions remain bounded for $t \to \infty$.

4.5 Bibliography

Here, we follow the presentation in the spirit of the modern geometric theory of evolutionary equations, see Cazenave and Haraux [3] or, in the context of parabolic problems, Quittner and Souplet [12].

5 Well-posedness for semilinear wave equations

We focus on a semilinear wave equation in the form

$$\partial^2_{t,t} u(t, x) - \Delta_x u(t, x) + f(u(t, x)) = 0, \ t > 0, \ x \in \Omega, \tag{5.1}$$

where $\Omega \subset R^N$ is a bounded domain with a regular boundary on which we prescribe the homogeneous Dirichlet condition

$$u(t, \cdot)|_{\partial\Omega} = 0. \tag{5.2}$$

Given the *initial data*

$$u(0, \cdot) = u_0, \ \partial_t u(0, \cdot) = u_1 \text{ in } \Omega, \tag{5.3}$$

our main goal will be to show that the resulting *initial-boundary value problem* (5.1 - 5.3) possesses a (possibly unique) solution on a given time interval $(0, T)$.

We follow the nowadays standard scheme based on

- *a priori* bounds;
- *compactness* or (weak) sequential stability;
- *approximate scheme* and convergence.

Such a way of a *constructive* proof of existence is easily adaptable when solving the real world problems, where the chosen approximate scheme coincides with the expected numerical implementation. Although it may seem at the first glance that *a priori* bounds as well as the property of *compactness* of the (hypothetical) family of solutions are superfluous in the proof of existence, they represent the natural preliminary steps in identifying the suitable function spaces framework as well as the approximate scheme.

In general, given a nonlinear problem, we first try to identify as many *a priori* bounds as possible in order to guarantee *compactness* or sequential stability of a hypothetical class of solutions. Sequential stability means that any sequence of smooth solutions bounded in terms of *a priori estimates* possesses at least a subsequence that converges to another solution of the same problem. Having clarified these two rather crucial issues, we may try to *construct* solutions by means of a suitable approximation scheme, the convergence of which can be established by the tools developed in the preceding two steps.

5.1 A priori bounds

Basically all *a priori* bounds available for solutions of the problem (5.1 - 5.3) follow from the so called *energy method*.

5.1.1 Basic energy estimates

We adopt the procedure introduced in Section 4.3. Multiplying the equation (5.1) on $\partial_t u$, integrating the resulting expression over Ω, and using the Gauss-Green theorem together with the boundary condition (5.2) to eliminate the boundary terms, we obtain the standard *energy balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = 0,\ E(t) = \int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + F(u) \right)(t,x)\ \mathrm{d}x, \tag{5.4}$$

where we have denoted

$$F(u) = \int_0^u f(z)\ \mathrm{d}z.$$

As we have seen in Section 4.1, the relation (5.4) itself is not strong enough to yield *uniform* bounds unless we impose certain structural restrictions on f. Our aim is that the energy E represents a kind of "norm" in a suitable space. Since Ω is a bounded and regular domain, say of the class C^2, we have the *Poincaré inequality*:

$$\int_\Omega |\nabla_x v|^2 \, \mathrm{d}x \geq \Lambda^2 \int_\Omega |v|^2 \, \mathrm{d}x \text{ for any } v \in W_0^{1,2}(\Omega), \tag{5.5}$$

where $\Lambda > 0$ is the first (minimal) eigenvalue of the Dirichlet Laplacean in Ω,

$$-\Delta w = \Lambda w \text{ in } \Omega, \ w|_{\partial\Omega} = 0.$$

Henri Poincaré
(wikipedia)

French mathematician, physicists, and philosopher
[1854 (Nancy) - 1912 (Paris)]

- Poincaré's conjecture
- Poincaré - Bendixson theorem
- Poincaré's mapping

Inequality (5.5) motivates the following *hypothesis* imposed on f:

$$f'(u) \geq -c \text{ for all } u \in R^1, \tag{5.6}$$

where c is a certain (positive) constant. Accordingly, we may write

$$f(u) = g(u) + h(u), \ \text{with } g(u) = f(u) + cu - f(0), \ h(u) = f(0) - cu,$$

where

$$g'(u) \geq 0, \ g(0) = 0.$$

In particular,

$$G(u) = \int_0^u g(z)\ \mathrm{d}z \geq 0 \text{ for all } u \in R^1. \tag{5.7}$$

Here and hereafter, the symbol c or c_i denotes a generic positive real constant that specific value of which may vary from line to line.

With (5.7) in mind, we rewrite the energy balance (5.4) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + G(u) \right)(t,x)\ \mathrm{d}x = \int_\Omega (cu - f(0))\partial_t u\ \mathrm{d}x,$$

where

$$\left| \int_\Omega (cu - f(0))\partial_t u\ \mathrm{d}x \right| \leq c\left(1 + \|u\|^2_{L^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)}\right).$$

Thus, seeing that

$$\int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + G(u) \right)(t,x)\ \mathrm{d}x \geq \frac{1}{4}\int_\Omega \left(|\partial_t u|^2 + |\nabla_x u|^2 + \Lambda |u|^2 \right)(t,x)\ \mathrm{d}x$$

we may apply Gronwall's lemma to deduce

$$\sup_{t\in[0,T]} \left(\|\partial_t u(t,\cdot)\|^2_{L^2(\Omega)} + \|u(t,\cdot)\|^2_{W^{1,2}(\Omega)} \right) \leq c_1(T, \tilde{E}_0), \tag{5.8}$$

where

$$\tilde{E}_0 = \int_\Omega \left(\frac{1}{2}|u_1|^2 + \frac{1}{2}|\nabla_x u_0|^2 + G(u_0) \right)(x)\ \mathrm{d}x.$$

5.1.2 Higher order energy bounds

In order to derive estimates on higher order derivatives, we multiply the equation (5.1) on $-\Delta_x u$ and integrate by parts to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}\left(|\Delta_x u|^2 + |\partial_t \nabla_x u|^2 \right)(t,x)\ \mathrm{d}x = \int_\Omega f(u(t,x))\partial_t \Delta_x u(t,x)\ \mathrm{d}x. \tag{5.9}$$

The integral on the right-hand side needs extra treatment. We write

$$\int_\Omega f(u(t,x))\partial_t \Delta_x u(t,x)\ \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f(u(t,x))\Delta_x u(t,x)\ \mathrm{d}x - \int_\Omega f'(u(t,x))\partial_t u(t,x)\Delta_x u(t,x)\ \mathrm{d}x,$$

where, by virtue of Hölder's inequality,

$$\left|\int_\Omega f(u(t,x))\Delta_x u(t,x)\ \mathrm{d}x\right| \leq \|f(u(t,\cdot))\|_{L^2(\Omega)}\|\Delta_x u(t,\cdot)\|_{L^2(\Omega)}, \tag{5.10}$$

and

$$\left|\int_\Omega f'(u(t,x))\partial_t u(t,x)\Delta_x u(t,x)\ \mathrm{d}x\right| \tag{5.11}$$
$$\leq \|f'(u(t,\cdot)\|_{L^p(\Omega)}\|\partial_t u(t,x)\|_{L^q(\Omega)}\|\Delta_x u(t,\cdot)\|_{L^2(\Omega)},\ \text{with}\ \frac{1}{p}+\frac{1}{q}=\frac{1}{2}.$$

Keeping in mind that we control $\partial_t \nabla_x u$ by the expression on the left-hand side of (5.9), we use the imbedding

$$W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega),\ \begin{cases} q=\infty \text{ for } N=1, \\ q<\infty \text{ arbitrary fimite if } N=2, \\ q\leq \frac{2N}{N-2} \text{ for } N>2. \end{cases} \tag{5.12}$$

Consequently, under the growth restriction

$$|f'(u)| \leq c\left(1+|u|^{r-1}\right),\ r \text{ arbitrary finite for } N=1,2, r=\frac{N}{N-2}, \tag{5.13}$$

the uniform energy bound (5.8), together with the embedding (5.2), imply that

$$\sup_{t\in[0,T]} \|f'(u(t,\cdot))\|_{L^p(\Omega)} \leq c(T,E_0), \begin{cases} p<\infty \text{ arbitrary finite if } N=1,2, \\ p=N \text{ for } N\geq 3; \end{cases} \tag{5.14}$$

which, together with (5.9), (5.11) and the standard Gronwall argument, give rise to *a priori bounds*

$$\sup_{t\in[0,T]}\left(\|\Delta_x u(t,\cdot)\|_{L^2(\Omega)} + \|\partial_t u(t,\cdot)\|_{W^{1,2}(\Omega)}\right)$$
$$\leq c\left(T,E_0,\|\Delta_x u_0\|_{L^2(\Omega)},\|u_1\|^2_{W^{1,2}(\Omega)}\right).$$

Finally, we may use the standard elliptic estimates for the operator Δ_x to conclude that

$$\sup_{t\in[0,T]}\left(\|u(t,\cdot)\|_{W^{2,2}(\Omega)} + \|\partial_t u(t,\cdot)\|_{W^{1,2}(\Omega)}\right) \tag{5.15}$$
$$\leq c\left(T,E_0,\|\Delta_x u_0\|_{L^2(\Omega)},\|u_1\|^2_{W^{1,2}(\Omega)}\right).$$

If f is sufficiently smooth and $\partial\Omega$ regular, we may continue differentiating the equation in time to deduce *a priori* bounds on higher order derivatives in terms of the initial data u_0, u_1.

5.2 Compactness - weak sequential stability

Similarly to the derivation of the *a priori* bounds performed in the preceding section, the following step is purely formal but very illustrative. Assuming we are given a sequence of (smooth) solutions $\{u_n\}_{n=1}^{\infty}$ of the initial-boundary value problem (5.1 - 5.3), bounded only via the energy bounds (5.8), our goal is to show that, at least for a suitable subsequence,

$$u_n \to u \text{ in a certain sense,}$$

where u is a (possibly weak) solution of the same problem.

To be more specific, we therefore suppose that

$$\sup_{t\in[0,T]} \Big(\|\partial_t u_n(t,\cdot)\|^2_{L^2(\Omega)} + \|u_n(t,\cdot)\|^2_{W^{1,2}(\Omega)} \Big) \leq c, \tag{5.16}$$

where the constant is independent of $n = 1, 2, \ldots$. Accordingly, at least for a suitable subsequence (not relabeled) we have

$$u_n \to u \text{ weakly-(*) in } L^\infty(0,T; W^{1,2}_0(\Omega)), \tag{5.17}$$

and

$$\partial_t u_n \to \partial_t u \text{ weakly-(*) in } L^\infty(0,T; L^2(\Omega)). \tag{5.18}$$

Moreover, we have

$$\|u_n\|_{W^{1,2}((0,T)\times\Omega)} \leq c;$$

whence, by virtue of the compact embedding

$$W^{1,2}(Q) \hookrightarrow\hookrightarrow L^2(Q),\ Q\text{bounded in } R^{N+1},$$

we may infer that

$$u_n \to u \text{ in } L^2((0,T)\times\Omega),$$

and, passing again to a subsequence as the case may be

$$f(u_n) \to f(u) \text{ a.a. in } (0,T)\times\Omega, \tag{5.19}$$

cf. Section 5.5.

Next, we have to make sure that the sequence $\{f(u_n)\}_{n=1}^\infty$ does not admit concentration points, meaning it is *equi-integrable* in $(0,T)\times\Omega$. To this end, we use the bounds (5.16), with embedding relation (5.12). Supposing

$$|f(u)| \le c\,(1+|u|^r)\,,\ r \text{ arbitrary finite for } N=1,2, r<\frac{2N}{N-2}, \tag{5.20}$$

which is obviously weaker than (5.13), we get

$$\{f(u_n)\}_{n=1}^\infty \text{ bounded in } L^q((0,T)\times\Omega), \begin{cases} q \text{ arbitrary for } N=1,2, \\ \\ q=\frac{1}{r}\frac{2N}{N-2}>1 \text{ if } N\ge 3, \end{cases},$$

which, together with (5.19), gives rise to the desired conclusion

$$f(u_n)\to f(u) \text{ weakly in } L^q((0,T)\times\Omega). \tag{5.21}$$

The above relation clearly allow us to pass to the limit in the equation (5.1) in the sense of distributions. However, it seems more convenient to introduce a concise *weak formulation* of the whole initial-boundary value problem (5.1 - 5.3), where the equation (5.1) with the initial conditions (5.3) are replaced by a family of integral identities:

$$\begin{aligned}\int_0^T\int_\Omega \left[u\partial^2_{t,t}\varphi+\nabla_x u\cdot\nabla_x\varphi+f(u)\varphi\right](t,\cdot)\ \mathrm{d}x\ \mathrm{d}t \\ =\int_\Omega u_0\partial_t\varphi(0,\cdot)\ \mathrm{d}x-\int_\Omega u_1\varphi(0,\cdot)\ \mathrm{d}x\end{aligned} \tag{5.22}$$

for any *test function* $\varphi\in C_c^\infty([0,T)\times\Omega)$. Note that the relation (5.22) includes the distributional formulation of the equation (5.1), together with the satisfaction of the initial conditions (5.2). Since our solutions satisfy

$$u\in L^\infty(0,T;W_0^{1,2}(\Omega)),$$

the homogeneous Dirichlet boundary conditions (5.2) are satisfied in the sense of traces.

Since the sequence of solutions $\{u_n\}_{n=1}^\infty$ converges in the sense specified in (5.17), (5.18), (5.21), it is easy to pass to the limit for $n\to\infty$ in the weak formulation (5.22) to conclude that the limit u is another solution of the same problem. As for the initial data

$$u_n(0,\cdot)=u_{0,n},\ \partial_t u_n(0,\cdot)=u_{1,n},$$

our hypotheses imply that

$$u_{0,n}\to u_0 \text{ weakly in } W_0^{1,2}(\Omega),\ u_{1,n}\to u_1 \text{ weakly in } L^2(\Omega)$$

at least for suitable subsequences, which is compatible with (5.22).

5.3 Approximate solutions, convergence

The final and the only constructive step of the existence theory consists in finding a suitable family of *approximate problems*. To this end, we consider a finite family of (sufficiently) smooth functions $\{w_n\}_{n=1}^N$ in Ω, satisfying the homogeneous boundary condition (5.2). We introduce a finite dimensional space X_N,

$$X_N = \mathrm{span}\{w_n \mid n = 1, 2, \dots, n\}$$

endowed with the Hilbert structure induced by the Lebesgue space $L^2(\Omega)$. Without loss of generality, we may therefore assume that w_n is taken to be a basis of X_N. Our aim is to construct the approximate solutions u_N in the form

$$u_N(t, x) = \sum_{n=1}^{N} a_n(t) w_n(x), \tag{5.23}$$

where a_n will be solutions of a certain system of (non-linear) ordinary differential equations.

Taking the L^2-scalar product of the equation (5.1) with w_n and integrating by parts we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle u(t); w_n \rangle = \langle \nabla_x u; \nabla_x w_n \rangle - \langle f(u); w_n \rangle, \tag{5.24}$$

where we have denoted

$$\langle v; w \rangle = \int_\Omega vw \ \mathrm{d}x$$

the (real) scalar product in $L^2(\Omega)$.

Using the ansatz (5.23) in (5.24) we obtain a system of second order equations

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} a_n(t) = \langle \nabla_x u_N; \nabla_x w_n \rangle - \langle f(u_N); w_n \rangle = G(a_1(t), \dots, a_N(t))_n, \tag{5.25}$$

$n = 1, \dots, N$, supplemented with the initial data

$$a_n(0) = \langle u_0; w_n \rangle, \ \frac{\mathrm{d}}{\mathrm{d}t} a_n(0) = \langle u_1; w_n \rangle. \tag{5.26}$$

The system (5.25), (5.26) is solvable, at least locally on a certain time interval $[0, T_{\max}]$. In order to see that one can take $T_{\max}$ and arbitrary positive number, we need *uniform bounds* independent of n, similar to the energy *a priori* bounds obtained in (5.8). To this end, mimicking the procedure leading to (5.8), we multiply (5.25) by $\partial_t a_n$ and take the sum over n to obtain a discrete version of the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}|u_N|^2 + \frac{1}{2}|\nabla_x u_N|^2 + F(u_N)\right)(t,\cdot)\ \mathrm{d}x = 0. \tag{5.27}$$

Consequently, exactly as in Section 5.1.1, we derive the uniform bounds

$$\sup_{t\in[0,T]}\left(\|\partial_t u_N(t,\cdot)\|^2_{L^2(\Omega)} + \|u_N(t,\cdot)\|^2_{W^{1,2}(\Omega)}\right) \le c_1(T+E_0)\exp(c_2 T), \tag{5.28}$$

for any $T < T_{\max}$, uniformly in $\varepsilon \to 0$.In particular, the existence interval of the approximate solutions can be extended to an arbitrary positive number T, and u_N satisfy the integral identity:

$$\int_0^\tau \int_\Omega \left[u_N \partial^2_{t,t}\varphi + \nabla_x u_N \cdot \nabla_x \varphi + f(u_N)\varphi\right](t,\cdot)\ \mathrm{d}x\ \mathrm{d}t \tag{5.29}$$

$$= \int_\Omega u_0 \partial_t \varphi(0,\cdot)\ \mathrm{d}x - \int_\Omega u_1 \varphi(0,\cdot)\ \mathrm{d}x$$

for any test function φ belonging to the class

$$\varphi \in C^1([0,T]; X_N). \tag{5.30}$$

5.3.1 Higher order estimates

One may wonder, if the higher order estimates analogous to (5.15) can be also derived at the approximate level. To this end, we have only to realize, that all we need is *invariance* of the space X_N with respect to the Dirichlet Laplacean. In other words, the basis $\{w_n\}_{n=1}^N$ must be taken as the first N eigenfunctions of the Laplace operator Δ_x in Ω, endowed with the Dirichlet boundary conditions. Accordingly, multiplication by $\partial_t \Delta_x u_N$ can be performed even at the level of approximate equation (5.25).

Pierre-Simon de Laplace
(wikipedia)

French mathematician, physicists, and astronomer
[1749 (Beamount-en-Auge) - 1827 (Paris)]

- Laplace operator
- Laplace transform
- Central Limit Theorem

5.3.2 Limit $n \to \infty$

With the uniform bounds (5.28) at hand, it is now a routine matter to let $N \to \infty$ in (5.29), where we can follow step by step the arguments of Section 5.2. In such a way, we obtain a weak solution u,

$$u \in L^\infty(0,T;W_0^{1,2}(\Omega)),\ \partial_t u \in L^\infty(0,T;L^2(\Omega))$$

satisfying the integral identity (5.22) for any

$$\varphi \in \cup_{N=1}^\infty C^1([0,T];X_N).$$

Finally, using the density of the functions w_n in, say, $W_0^{1,2}(\Omega)$, we conclude. We have proved the following result:

Theorem 5.1 *Let $\Omega \subset R^N$ be a bounded domain of the class $C^{2+\nu}$. Suppose that the nonlinearity f is a continuously differentiable function satisfying*

$$f'(u) \geq -c,$$

$$|f(u)| \leq c(1 + |u|^r), \begin{cases} r \text{ arbitrary finite for } N = 1, 2 \\ \\ r < \frac{2N}{N-2} \text{ if } N \geq 3 \end{cases}$$

for all $u \in R^1$. Let the initial data u_0, u_1 belong to the class

$$u_0 \in W_0^{1,2}(\Omega), \ u_1 \in L^2(\Omega).$$

Then the wave equation (5.1), with the boundary condition (5.2), and the initial condition (5.3) possesses a weak solution u in the sense specified in (5.22),

$$u \in L^\infty(0, T; W_0^{1,2}(\Omega)), \ \partial_t u \in L^\infty(0, T; L^2(\Omega)).$$

If, in addition, f is continuously differentiable,

$$|f'(u)| \leq c\left(1 + |u|^{r-1}\right), \ r \text{ arbitrary finite for } N = 1, 2, r = \frac{N}{N-2} \tag{5.31}$$

and

$$u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \ u_1 \in W_0^{1,2}(\Omega),$$

then the weak solution belongs to the class

$$u \in L^\infty(0, T; W^{2,2}(\Omega)), \ \partial_t u \in L^\infty(0, T; W^{1,2}(\Omega)).$$

5.4 Energy equality, uniqueness

We conclude the chapter by addressing the problem of *uniqueness* of the weak solutions obtained in Theorem 5.1. To this end, we restrict ourselves to the class of nonlinearities f satisfying the growth condition (5.31). It is easy to check that

$$f(u) \in L^\infty(0, T; L^2(\Omega))$$

as soon as u is a "finite energy" solution of (5.22), in particular, $u \in L^\infty(0, T; W_0^{1,2}(\Omega))$. Indeed this follows directly from the standard embedding (5.12). Thus u can be viewed as a (weak) solution of the *linear* equation

$$\partial^2_{t,t} u(t, x) - \Delta_x u(t, x) = g(t, x) = -f(u(t, x)) \in L^\infty(0, T; L^2(\Omega)). \tag{5.32}$$

We therefore start our discussion by a short excursion in the linear theory.

5.4.1 Linear equation and the Fourier method

The initial-boundary value problem associated to the linear equation (5.32) may be solved by the *Fourier method*. To this end, we consider the eigenvalue problem for the Dirichlet Laplacean:

$$-\Delta_x w = \lambda w \text{ in } \Omega, \ w|_{\partial\Omega} = 0. \tag{5.33}$$

As is well-known, the problem (5.33) possesses a family of solutions $\{w_n\}_{n=1}^\infty$, together with the associated (positive) real eigenvalues $\{\lambda_n\}_{n=1}^\infty$, where $\{w_n\}_{n=1}^\infty$ can be taken as an orthogonal basis of the Hilbert space $L^2(\Omega)$. Moreover, the functions w_n are smooth (twice continuously differentiable) as soon as $\partial\Omega$ is smooth (of class $C^{2+\nu}$).

Joseph Fourier
(wikipedia)

French mathematician, and physicist
[1768 (Auxerre) - 1830 (Paris)]

- Fourier series
- Fourier transform

Revoking the functional calculus associated to the self-adjoint operator $-\Delta_D$ - the Dirichlet Laplacean - we may define

$$H(-\Delta_D)[v] = \sum_{n=1}^{\infty} H(a_n[v]) w_n,$$

where

$$a_n[v] = \langle v; w_n \rangle \equiv \int_\Omega v w_n \ \mathrm{d}x \text{ are the Fourier coefficients.}$$

Accordingly, the function

$$u(t,\cdot) = \frac{1}{2}\left[\exp\left(\mathrm{i}\sqrt{-\Delta_D} t\right)[u_0] + \exp\left(-\mathrm{i}\sqrt{-\Delta_D} t\right)[u_0]\right] \tag{5.34}$$

$$+\frac{1}{2\sqrt{-\Delta}}\left[\exp\left(\mathrm{i}\sqrt{-\Delta_D} t\right)[u_1] - \exp\left(-\mathrm{i}\sqrt{-\Delta_D} t\right)[u_1]\right]$$

$$+\int_0^t \frac{1}{2\sqrt{-\Delta}}\left[\exp\left(\mathrm{i}\sqrt{-\Delta_D}(t-s)\right)[g(s,\cdot)] - \exp\left(-\mathrm{i}\sqrt{-\Delta_D}(t-s)\right)[g(s,\cdot)]\right] \ \mathrm{d}s$$

represents a weak solution of the linear equation (4.9), supplemented with the boundary conditions (5.2) and the initial conditions (5.3). More specifically, u satisfies the integral identity

$$\int_0^T \int_\Omega \left[u \partial^2_{t,t}\varphi - u \cdot \Delta_x \varphi - g\varphi\right](t,\cdot) \ \mathrm{d}x \ \mathrm{d}t \tag{5.35}$$

$$= \int_\Omega u_0 \partial_t \varphi(0,\cdot) \ \mathrm{d}x - \int_\Omega u_1 \varphi(0,\cdot) \ \mathrm{d}x$$

for any test function φ belonging to the class $\varphi \in C^\infty_c([0,T) \times \Omega)$ provided, for instance

$$u_0 \in W^{1,2}_0(\Omega), \ u_1 \in L^2(\Omega), \ g \in L^1(0,T; L^2(\Omega)).$$

Indeed relation (5.35) can be verified first for the approximate data

$$u_{0,N} = \sum_{n=1}^N a_n[u_0] w_n, \ u_{1,N} = \sum_{n=1}^N a_n[u_1] w_n, \ g_N(t,\cdot) = \sum_{n=1}^N a_n[g(t,\cdot)] w_n,$$

for which the equation (5.32) holds in the standard sense, and then we let $N \to \infty$. Moreover, it is easy check, by a direct inspection of (5.34), that

$$u \in C([0,T]; W^{1,2}_0(\Omega)), \ \partial_t u \in C([0,T]; L^2(\Omega)),$$

and that u satisfies the energy balance

$$\int_\Omega \frac{1}{2}\left(|\partial_t u|^2 + |\nabla_x u|^2\right)(t,\cdot) \ \mathrm{d}x = \int_\Omega \frac{1}{2}\left(|u_1|^2 + |\nabla_x u_0|^2\right) \ \mathrm{d}x \tag{5.36}$$

$$+\int_0^t g(s,\cdot)\partial_t u(s,\cdot)\ \mathrm{d}s.$$

Finally, we observe that the weak solutions satisfying (5.35) are uniquely determined by the data u_0, u_1, and g. Indeed the difference v of two integrable solutions satisfies

$$\int_0^T \int_\Omega v\left(\partial^2_{t,t}\varphi - \Delta_x\varphi\right)\ \mathrm{d}x\ \mathrm{d}t = 0 \text{ for any } \varphi \in C_c^\infty([0,T)\times\Omega); \tag{5.37}$$

whence, by a simple density argument, the identity (5.37) holds for any φ,

$$\varphi \in C^2([0,T]\times\overline{\Omega}),\ \varphi|_{\partial\Omega} = 0,\ \varphi(T,\cdot) = \partial_t\varphi(T,\cdot) = 0. \tag{5.38}$$

On the other hand, the image of the space of test functions (5.38) under the wave operator $\partial^2_{t,t} - \Delta_x$ is dense in, say, $L^2((0,T)\times\Omega)$; whence (5.37) yields immediately $v \equiv 0$.

We have proved the following result:

Theorem 5.2 *Let $\Omega \subset R^N$ be a bounded domain of class $C^{2+\nu}$. Assume that the data belong to the regularity class*

$$u_0 \in W_0^{1,2}(\Omega),\ u_1 \in L^2(\Omega) \text{ and } g \in L^1(0,T;L^2(\Omega)).$$

Then the initial-boundary value problem (5.32), (5.2), (5.3) admits a weak solution u,

$$u \in C([0,T];W_0^{1,2}(\Omega)),\ \partial_t u \in C([0,T];L^2(\Omega)),$$

unique in the space of integrable functions $L^1((0,T)\times\Omega)$. Moreover, u is given by formula (5.34) and satisfies the energy balance (5.36).

5.4.2 Energy equality and uniqueness for the nonlinear problem

As already observed in (5.32), the solution u of the semilinear equation (5.1) constructed in Theorem 5.1 can be seen as a solution of the linear equation (5.32) with $g = -f(u)$, where $g \in L^\infty(0,T;L^2(\Omega))$ as soon as the nonlinearity f obeys the hypothesis (5.31). Consequently, we may apply the conclusion of Theorem 5.2 to deduce that

$$u \in C([0,T];W_0^{1,2}(\Omega)),\ \partial_t u \in C([0,T];L^2(\Omega)), \tag{5.39}$$

satisfies the energy balance (5.36); whence

$$\int_\Omega \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + F(u)\right)(t,\cdot)\ \mathrm{d}x \tag{5.40}$$
$$= \int_\Omega \left(\frac{1}{2}|u_1|^2 + \frac{1}{2}|\nabla_x u_0|^2 + F(u_0)\right)\ \mathrm{d}x.$$

Suppose now that u^1, u^2 are two solutions emanating from the same initial data. Taking $v = u^1 - u^2$ and revoking (5.36) again, we obtain

$$\frac{1}{2}\int_\Omega \left(|\partial_t v|^2 + |\nabla_x v|^2\right)(t,\cdot)\ \mathrm{d}x = \int_0^t \int_\Omega \left(f(u^2) - f(u^1)\right)\partial_t v\ \mathrm{d}x\ \mathrm{d}s \tag{5.41}$$
$$\leq \int_0^t \int_\Omega \left|f'(\xi u^1 + (1-\xi)u^2)\right| |v||\partial_t v|\ \mathrm{d}x\ \mathrm{d}s,$$

where, since u^1, u^2 belong to the regularity class (5.39) and f obeys (5.31),

$$\left|f'(\xi u^1 + (1-\xi)u^2)\right| \in L^\infty(0,T;L^r(\Omega)),\ r = \begin{cases} \infty \text{ if } N = 1 \\ \text{arbitrary finite for } N = 2 \\ N \text{ for } N \geq 3. \end{cases}$$

Consequently, by means of Hölder's inequality and the embedding (5.12), we may infer that

$$\frac{1}{2}\int_\Omega \left(|\partial_t v|^2 + |\nabla_x v|^2\right)(t,\cdot)\ \mathrm{d}x \leq c\int_0^t \|\nabla_x v\|_{L^2(\Omega)}\|\partial_t\|_{L^2(\Omega)};$$

whence a straightforward application of Gronwall's lemma yields $v = u^1 - u^2 \equiv 0$.

Summarizing the previous discussion, we obtain:

Theorem 5.3 *Let $\Omega \subset R^N$ be a bounded domain of class $C^{2+\nu}$. Assume that the initial data belong to the regularity class*

$$u_0 \in W_0^{1,2}(\Omega),\ u_1 \in L^2(\Omega),$$

and that f is a continuously differentiable function on R^1 satisfying

$$f'(u) \geq -c,$$

$$|f'(u)| \leq c\left(1 + |u|^{r-1}\right),\ r \text{ arbitrary finite for } N = 1, 2, r = \frac{N}{N-2},$$

for all $u \in R^1$.

Then the initial-boundary value problem (5.1), (5.2), (5.3) admits a weak solution u,

$$u \in C([0,T]; W_0^{1,2}(\Omega)),\ \partial_t u \in C([0,T]; L^2(\Omega)),$$

unique in the space

$$u \in L^\infty(0,T; W_0^{1,2}(\Omega)),\ \partial_t u \in L^\infty(0,T; L^2(\Omega)).$$

Moreover, u satisfies the energy equality (5.40).

5.5 Exercises

(i) Show that

$$w_n \to w \text{ weakly in } L^q(Q) \text{ for a certain } q > 1,$$

and

$$w_n \to \tilde{w} \text{ a.a. in } Q$$

implies $w = \tilde{w}$ a.a. in Q.

(ii) Find a sequence of functions such that

$$v_n \to 0 \text{ a.a. in } Q,$$

where $Q \subset R^1$ is an open interval, such that $\{v_n\}$ *does not* converge to zero weakly in $L^1(Q)$.

5.6 Bibliography

The basic strategy of this chapter is taken over from the classical monograph of J.-L.Lions [11].

6 Free vibrations

We consider a simple 1D analogue of the semilinear wave equation introduced in the previous section:

$$\partial^2_{t,t} u(t,x) - \partial^2_{x,x} u(t,x) + f(u(t,x)) = 0, \ x \in (0,\pi), \ t \in R^1, \tag{6.1}$$

supplemented with the homogeneous Dirichlet boundary conditions

$$u(t,0) = u(t,\pi) = 0. \tag{6.2}$$

Equation (6.1), with (6.2), may be viewed as a simple model of a (nonlinear) vibrating string. Of course, the length π of the string can be replaced by an arbitrary positive number. As we have seen in Theorem 5.1 above, the problem (6.1), (6.2), together with the initial data

$$u(0,\cdot) = u_0, \ \partial_t u(0,\cdot) = u_1 \tag{6.3}$$

is *well-posed* for any data

$$u_0 \in W^{1,2}_0(0,\pi), \ u_1 \in L^2(0,\pi)$$

and any continuously differentiable function f obeying the growth restrictions specified in Theorem 5.1. In addition, we suppose that $f(0) = 0$ so that $u \equiv 0$ is a trivial solution of (6.1), (6.2).

6.1 Wave equation as a Hamiltonian system

The wave equation (6.1) can be written in the form

$$\partial_t u = v, \ \partial_t v = \Delta_x u - f(u). \tag{6.4}$$

Thus defining the functional

$$\mathcal{H}(u,v) = \int_\Omega \frac{1}{2}|v|^2 + \frac{1}{2}|\partial_x u|^2 + F(u) \ \mathrm{d}x$$

we may be viewed as

$$\partial_t u = D_v \mathcal{H}(u,v), \ \partial_t v = -D_u \mathcal{H}(u,v), \tag{6.5}$$

where D_v, D_u are understood as Fréchet (variational) derivatives. The functional $\mathcal{H}$ plays a role of the *Hamiltonian* associated to (6.4), and, in accordance with our discussion in Section 4.1, it is constant along solution trajectories, meaning

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}\Big(u(t,\cdot), v(t,\cdot)\Big) = 0.$$

Thus equation (6.1) can be interpreted as an infinite-dimensional Hamiltonian system.

6.1.1 Periodic solutions and free vibrations

As we have observed above, *any* solution to the problem (6.1 - 6.3) with $f \equiv 0$ is *time-periodic*. A natural question arises, namely if the same can be said about the solutions to the *nonlinear* problem. Such a statement, however, seems quite ambitious and may not even be true in general. Instead we pose a simpler question: Does the problem (6.1), (6.2), with $f(0) = 0$, possess a non-trivial (non-zero) periodic solution? The time-periodic solutions of (6.1), (6.2) are associated with the *free vibrations* in the string model. Note that there is no dissipative mechanism incorporated in the model (6.1), (6.2) as the total energy represented by the Hamiltonian $\mathcal{H}$ is constant in time.

We are therefore interested in the solutions of (6.1), (6.2) defined on the whole real line R^1 and satisfying, in addition,

$$u(t + 2\pi, x) = u(t, x) \text{ for all } t \in R^1,\ x \in (0, \pi). \tag{6.6}$$

The length of the period - 2π - is obviously related to the length of the underlying space interval. As we shall see, it is possible to construct time periodic solutions with a period which is a *rational* multiple of π. The existence for other (irrational) periods is still an outstanding open problem.

6.2 Variational formulation

We start by introducing the concept of *weak solution* to the problem (6.1), (6.2), (6.6). To this end, we consider the class of test functions φ that are **(i)** twice continuously differentiable in $R^1 \times [0, 1]$, **(ii)** satisfy the homogeneous Dirichlet boundary conditions (6.2), and **(iii)** are time periodic with the period 2π, meaning they satisfy (6.6). Denote

$$Q = \{(t, x) \mid t \in (0, 2\pi),\ x \in (0, \pi)\}.$$

Definition 6.1 *We shall say that a function u is a* weak solution *to the problem (6.1), (6.2), and (6.6) if*

$$u \in L^1(Q),\ f(u) \in L^1(Q),$$

and the integral identity

$$\int_0^{2\pi} \int_0^{\pi} \Big[u(t,x) \Big(\partial^2_{t,t} \varphi(t,x) - \partial^2_{x,x} \varphi(t,x) \Big) + f\left(u(t,x)\right) \varphi(t,x) \Big] \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{6.7}$$

holds for any test function $\varphi \in C^2(R^2 \times [0,\pi]$ such that

$$\varphi(t,0) = \varphi(t,\pi) = 0,\ \varphi(t+2\pi,x) = \varphi(t,x) \ \textit{for any}\ t \in R^1,\ x \in (0,\pi).$$

Note that such a concept of weak solution is, in fact, weaker than that one introduced in Section 5.2 for the initial-boundary value problem. Indeed, in contrast with (5.22), we do not assume *any* integrability properties of the (generalized) derivatives $\partial_t u$, $\partial_x u$. The reason is that they are simply not available (in a direct manner) for the component of u belonging to the kernel of the differential operator $\partial^2_{t,t} - \partial^2_{x,x}$, supplemented with the boundary conditions (6.2), (6.6).

In the remaining part of this section, we show that the problem (6.1), (6.2), (6.6) possesses *infinitely many* (weak) solutions for a suitable class of nonlinearities f. As a matter of fact, it is enough to show existence of at least one solution u such that $u \neq 0$, and u effectively depending on the time variable. Then u is time-periodic with some minimal period $2\pi/N$ and we may repeat the same procedure for the periodic solutions with the period $2\pi/(N+1)$.

To conclude this section, we introduce the functions

$$e_{j,k}(t,x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \sin(jt)\sin(kx) \ \text{for}\ j>0, k>0, \\ \\ \frac{1}{\pi}\sin(kx),\ j=0 \\ \\ \frac{1}{\sqrt{2\pi}} \cos(jt)\sin(kx) \ \text{for}\ j<0,\ k>0. \end{cases}$$

The functions $\{e_{j,k}\}_{j \in Z, k>0}$ form a basis of the Hilbert space $L^2(Q)$, and we may introduce the Fourier coefficients

$$a_{j,k}[v] = \int_0^{2\pi} \int_0^{\pi} v e_{j,k} \ \mathrm{d}x \ \mathrm{d}t$$

for any $v \in L^1(Q)$.

Now, it is easy to see that the weak formulation (6.7) is, in fact, equivalent to the identity

$$(j^2 - k^2)a_{j,k}[u] = a_{j,k}\left[f(u)\right] \text{ for all } j \in Z,\ k = 1, 2, \dots \tag{6.8}$$

6.3 Direct method for finding time-periodic solutions

We use a *direct method* to find solutions of the problem (6.1), (6.2), (6.6), more specifically, we reformulate the problem as finding *critical points* of the so-called action functional:

$$\mathcal{A}[v] = \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{2}|\partial_x v|^2 - \frac{1}{2}|\partial_t v|^2 + F(v(t,x))\right)\mathrm{d}x\ \mathrm{d}t,\ F(v) = \int_0^v f(z)\ \mathrm{d}z.$$

Indeed computing formally $\partial\mathcal{A}(u)$ we get

$$\langle\partial\mathcal{A}[u];\varphi\rangle = \int_0^{2\pi}\int_0^{\pi}\Big[\partial_x u\partial_x\varphi - \partial_t u\partial_t\varphi + f(u)\varphi\Big]\mathrm{d}x\ \mathrm{d}t;$$

whence we recover (6.7) by setting $\partial\mathcal{A}(u) = 0$.

6.4 Formal computations, a priori bounds

Suppose, for a moment, that all quantities in question are smooth. Moreover, assume that we have found a sequence $\{u_n\}_{n=1}^{\infty}$ of *critical points* of the functional $\mathcal{A}$ such that

$$\mathcal{A}[u_n] \to F_0 < 0 \text{ as } n \to \infty. \tag{6.9}$$

Since u_n are critical points of $\mathcal{A}$, meaning, in particular, that

$$\langle\partial\mathcal{A}[u_n];u_n\rangle = 0,$$

we get

$$\mathcal{A}[u_n] - \frac{1}{2}\langle\partial\mathcal{A}[u_n];u_n\rangle \tag{6.10}$$

$$= \int_0^{2\pi}\int_0^{\pi}\Big(F(u_n) - \frac{1}{2}f(u_n)u_n\Big)\ \mathrm{d}x\ \mathrm{d}t \to F_0 < 0 \text{ as } n \to \infty.$$

In order to exploit (6.10), we restrict ourselves to a very particular class of *superlinear* nonlinearities, namely

$$f(u) = |u|^{p-1}u,\ p > 1, \tag{6.11}$$

where

$$F(u) = \frac{1}{p+1}|u|^{p+1}.$$

Since $p > 1$ we deduce from (6.10) that

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_0^{2\pi} \int_0^{\pi} |u_n|^{p+1} \ \mathrm{d}x \ \mathrm{d}t \to -F_0 > 0. \tag{6.12}$$

In particular, we may infer that

$$u_n \to u \text{ weakly in } L^{p+1}(Q), \tag{6.13}$$

and

$$f(u_n) \to \overline{f(u)} \text{ weakly in } L^q(Q), \ q = \frac{p+1}{p}, \tag{6.14}$$

passing to a subsequence as the case may be.

With the estimates at hand, we can let $n \to 0$ in (6.8) to conclude that

$$(j^2 - k^2) a_{j,k}[u] = a_{j,k} \left[\overline{f(u)}\right] \text{ for all } j \in Z, \ k = 1, 2, \dots \tag{6.15}$$

Consequently, our ultimate goal is to show that

$$\overline{f(u)} = f(u).$$

To this end, we use the theory of monotone operators. It is convenient to decompose

$$L^2(Q) = \mathcal{K} \oplus \mathcal{R},$$

where $\mathcal{K}$ is the kernel of the periodic D'Alambert operator $\partial^2_{t,t} - \partial^2_{x,x}$,

$$\mathcal{K} = \left\{ v \in L^2(Q) \ \middle| \ a_{k,j}[v] = 0 \text{ whenever } j^2 - k^2 \neq 0 \right\},$$

$$\mathcal{R} = \left\{ v \in L^2(Q) \ \middle| \ a_{k,j}[v] = 0 \text{ for } j^2 - k^2 = 0 \right\}.$$

Accordingly, we write

$$u_n = P[u_n] + P^{\perp}[u_n],$$

where P denotes the orthogonal projection onto the space $\mathcal{R}$. Finally, we introduce a scale of norms on the space $\mathcal{R}$,

$$\|v\|^2_{\beta} = \sum_{j,k,j^2 \neq k^2} \left| j^2 - k^2 \right|^{\beta} a^2_{k,j}[v], \ \beta \geq 0.$$

Now, we claim the following auxilliary result.

Lemma 6.1

$$\sum_{j,k,j^2\neq k^2} \frac{1}{|j^2-k^2|^\alpha} < \infty$$

for any $\alpha > 1$.

Proof:
We write

$$\sum_{j,k,j^2\neq k^2} \frac{1}{|j^2-k^2|^\alpha} < \infty = \sum_{k>0} \frac{1}{k^{2\alpha}} + 2 \sum_{j>0,k>0,j\neq k} \frac{1}{|j+k|^\alpha} \frac{1}{|j-k|^\alpha},$$

where

$$\sum_{j>0,k>0,j\neq k} \frac{1}{|j+k|^\alpha} \frac{1}{|j-k|^\alpha} = \sum_{k>0,m\neq 0,m>-k} \frac{1}{|2k+m|^\alpha} \frac{1}{m^\alpha}$$
$$\leq \sum_{k\neq 0,m\neq 0} \frac{1}{|k|^\alpha} \frac{1}{|m|^\alpha},$$

where the last series converges as soon as $\alpha > 1$.

Q.E.D.

Using Lemma 6.1 we easily observe that

$$\|P[v]\|_{L^\infty(Q)} \leq c(\beta)\|P[v]\|_\beta \text{ for all } \beta > 1, \tag{6.16}$$

with $c(\beta) \to \infty$ for $\beta \nearrow 1$. Indeed

$$\|P[v]\|_{L^\infty(Q)} \leq c \sum_{j^2\neq k^2} |a_{j,k}[v]|,$$

where

$$\sum_{j^2\neq k^2} |a_{j,k}[v]| = \sum_{j^2\neq k^2} |j^2-k^2|^{-\beta/2}|j^2-k^2|^{\beta/2} |a_{j,k}[v]|$$

$$\leq \left[\sum_{j^2 \neq k^2} \frac{1}{|j^2 - k^2|^\beta} \right]^{1/2} \|P[v]\|_\beta \, ;$$

whence the desired conclusion follows directly from Lemma 6.1.

Seeing that, obviously,

$$\|P[v]\|_{L^2(Q)} \leq c \|P[v]\|_{\beta=0} \tag{6.17}$$

we may interpolate (6.16), (6.17) to obtain the following conclusion:

For any $2 \leq q < \infty$, there is $\beta(q) < 1$ such that

$$\|P[v]\|_{L^q(Q)} \leq c(p) \, \|P[v]\|_{\beta(q)} \, , \tag{6.18}$$

in particular

$$\|P[v]\|_{L^q(Q)} \leq c(q) \, \|P[v]\|_{\beta=1} \, . \tag{6.19}$$

Going back to (6.8) we deduce that

$$\|P[u_n]\|_{\beta=1}^2 = \sum_{j,k,j^2 \neq k^2} |j^2 - k^2| a_{j,k}^2 [u_n]$$

$$= \sum_{j,k} a_{j,k} f(u_n) \mathrm{sgn}(j^2 - k^2) a_{j,k}[P[u_n]] = \int_0^{2\pi} \int_0^\pi f(u_n) P[\tilde{u}_n] \; \mathrm{d}x \; \mathrm{d}t,$$

where we have set

$$\tilde{u}_n = \sum_{j,k} \mathrm{sgn}(j^2 - k^2) a_{j,k}[u_n] e_{j,k}.$$

In accordance with (6.19), we get

$$\left| \int_0^{2\pi} \int_0^\pi f(u_n) P[\tilde{u}_n] \; \mathrm{d}x \; \mathrm{d}t \right| \leq c_1 \|f(u_n)\|_{L^{(p+1)/p}(Q)} \|P[\tilde{u}_n\|_{L^{p+1}(Q)}$$

$$\leq c_2 \|P[\tilde{u}_n\|_{\beta=1} = c_2 \|P[u_n\|_{\beta=1},$$

where we have used the relations (6.14) and (6.19). Consequently, we may infer that

$$\|P[u_n]\|_{\beta=1} \leq c \tag{6.20}$$

uniformly for $n \to \infty$.

It follows from (6.20) that

$$\|P[u_n] - P[u]\|_\beta \to 0 \text{ for } n \to \infty \text{ as soon as } 0 \leq \beta < 1,$$

we can use the embedding (6.20) to conclude that

$$P[u_n] \to P[u] \text{ (strongly) in } L^q(Q) \text{ for any } 1 \le q < \infty. \tag{6.21}$$

It remains to show strong convergence of the component $P^\perp[u_n]$. To this end, we use monotonicity of the functions f. Returning once more to the relation (6.8) we deduce that

$$\int_0^{2\pi} \int_0^\pi f(u_n) P^\perp[u_n] \, \mathrm{d}x \, \mathrm{d}t = 0,$$

while (6.15) yields

$$\int_0^{2\pi} \int_0^\pi \overline{f(u)} P^\perp[u] \, \mathrm{d}x \, \mathrm{d}t = 0,$$

in particular,

$$\int_0^{2\pi} \int_0^\pi f\left(u_n\right) P^\perp[u_n] \, \mathrm{d}x \, \mathrm{d}t \to \int_0^{2\pi} \int_0^\pi \overline{f(u)} P^\perp[u] \, \mathrm{d}x \, \mathrm{d}t \tag{6.22}$$

as $n \to \infty$.

On the other hand, thanks to the specific form of the nonlinearity f,

$$\|u_n - u\|^{p+1}_{L^{p+1}(Q)} \le c \int_0^{2\pi} \int_0^\pi \left(f(u_n) - f(u)\right) \left(u_n - u\right) \, \mathrm{d}x \, \mathrm{d}t$$

$$= c \left[\int_0^{2\pi} \int_0^\pi \left(f(u_n) - f(u)\right) \left(P[u_n] - P[u]\right) \, \mathrm{d}x \, \mathrm{d}t \right.$$

$$\left. + \int_0^{2\pi} \int_0^\pi \left(f(u_n) - f(u)\right) \left(P^\perp[u_n] - P^\perp[u]\right) \, \mathrm{d}x \, \mathrm{d}t \right],$$

where, in agreement with (6.21), (6.22), the two integrals on the right-hand side tend to zero for $n \to \infty$.

Thus we may infer that

$$u_n \to u \text{ (strongly) in } L^{p+1}(Q), \tag{6.23}$$

in particular,

$$\overline{f(u)} = f(u), \tag{6.24}$$

therefore u is a weak solution of the problem (6.1), (6.2), (6.6). Finally, we let $n \to \infty$ in (6.12) to conclude that

$$\left(\frac{1}{2} - \frac{1}{p+1} \right) \int_0^{2\pi} \int_0^\pi |u|^{p+1} = -F_0 > 0, \tag{6.25}$$

meaning $u \neq 0$.

6.5 Finite-dimensional approximation

In this section, we *construct* a sequence of approximate solutions to the problem (6.1), (6.2), (6.6) by means of a Galerkin scheme. We take a family of finite-dimensional spaces

$$X_n = \left\{ v \in L^2(Q) \;\middle|\; a_{j,k}[v] = 0 \text{ for all } k > n, |j| > n \right\},$$

together with the action functional $\mathcal{A}_{n,M}$,

$$\mathcal{A}_{n,M}[v] = \int_0^{2\pi} \int_0^{\pi} \left(\frac{M^2}{2} |\partial_x v|^2 - \frac{M^2}{2} |\partial_t v|^2 + F(v(t,x)) \right) \mathrm{d}x \ \mathrm{d}t,$$

$$F(v) = \int_0^v f(z) \ \mathrm{d}z, \ v \in X_n,$$

where

$$\langle \partial \mathcal{A}_n[u]; \varphi \rangle = \int_0^{2\pi} \int_0^{\pi} \Big[M^2 \left(\partial_x u \partial_x \varphi - \partial_t u \partial_t \varphi \right) + f(u)\varphi \Big] \mathrm{d}x \ \mathrm{d}t$$

for $u, \varphi \in X_n$ and M a positive integer.

We observe that, formally for the time being, the critical points of $\mathcal{A}_{n,M}$ for $n \to \infty$ will yield weak solutions of the problem

$$\partial^2_{t,t} \tilde{u} - \partial^2_{x,x} \tilde{u} + \frac{1}{M^2} f(\tilde{u}) = 0,$$

which, after the simple change of variables

$$u(t,x) = \tilde{u}(Mt, Mx)$$

give rise to solutions of the original problem (6.1), (6.2), (6.6), provided $\tilde{u}$ has been extended as a 2π-periodic odd function in the x-variable.

Our next goal will be to show that $\mathcal{A}_{n,M}$ possesses a family of critical points $u_n \in X_n$ satisfying

$$-c_1 < \mathcal{A}_{n,M}[u_n] < -c_2 < 0 \tag{6.26}$$

for M sufficiently large, where the constants c_1, c_2 are independent of $n \to \infty$.

6.5.1 Critical points on X_n

We use the following result, see [6] :

Lemma 6.2 *Let X be a finite-dimensional (Hilbert) space and $J \in C^1(X,R)$ such that*

$$\lim_{\|v\|_X \to \infty} J(v) = \infty, \tag{6.27}$$

and X admits a decomposition

$$X = V_1 \oplus V_2 \oplus V_3,$$

$$J \leq b \text{ on } \mathcal{S} \cap (V_1 \oplus V_2),\ \mathcal{S} = \{v \in X \,|\|v\|_X = r > 0\}, \tag{6.28}$$

$$J > b \text{ on } V_3 \tag{6.29}$$

and

$$J > a \text{ on } V_2 \oplus V_3 \tag{6.30}$$

for certain real numbers a,b.

Then there exists a critical point v_c of J in X such that

$$\partial J(v_0) = 0,\ J(v) \in [a,b].$$

Our goal is to apply Lemma 6.2 in the situation

$$X = X_n,\ J = \mathcal{A}_{n,M},$$

where X_n is endowed with the norm

$$\|v\|^2_{X_n} = \|P[v]\|^2_{\beta=1} + \|P^{\perp}[v]\|^2_{L^2(Q)}.$$

Moreover, we take

$$V_1 = \operatorname{span}\left\{e_{j,k}\ |k^2 - j^2 < -3\right\} \cap X_n,\ V_2 = \left\{e_{j,k}\ |k^2 - j^2 = -3\right\} \cap X_n,$$

$$V_3 = \left\{e_{j,k}\ |k^2 - j^2 \geq 0\right\} \cap X_n.$$

Step 1:

It is easy to check that $X_n = V_1 \oplus V_2 \oplus V_3$ and that

$$\mathcal{A}_{n,M} = \sum_{|j|,k \leq n} M^2(k^2 - j^2)a^2_{j,k}[v] + \int_Q F(v)\ \mathrm{d}x \text{ for } v \in X_n.$$

In particular, we have

$$\mathcal{A}_{n,M} \geq 0 \text{ on } V_3, \tag{6.31}$$

and $\mathcal{A}_{n,M}$ satisfies the coercivity hypothesis (6.27) in X_n for any fixed $n = 1, 2, ...$

Step 2:

We consider the values of $\mathcal{A}_{n,M}$ on the sphere $\{\|v\|_{X_n} = 1\}$ intersected with the space $V_1 \cap V_2$. Keeping in mind the embedding relation (6.19), we get

$$\mathcal{A}_{n,m}[v] \leq -\sum_{|j|,k\leq n} M^2|k^2 - j^2|a^2_{j,k}[v] + \frac{1}{p}\|v\|^{p+1}_{L^{p+1}(Q)} \tag{6.32}$$

$$\leq -M^2\|v\|^2_{X_n} + c\|v\|^{p+1}_{X_n} \leq -M^2 + c < 0 \text{ for any } v \in \mathcal{S} \cap (V_1 \oplus V_2)$$

provided M is taken large enough. It is important to notice that the estimate (6.32) is independent of M.

Step 3:

Finally, fixing M so that (6.32) holds and taking $v \in V_2 \oplus V_3$ we get

$$\mathcal{M}_{n,M} \geq -3M^2 \sum_{j^2-k^2=-3} a^2_{j,k}[v] + \frac{1}{p+1}\int_Q |v|^{p+1}\,\mathrm{d}x$$

$$\geq -3M^2\|v\|^2_{L^2(Q)} + \frac{1}{p+1}\|v\|^{p+1}_{L^{p+1}(Q)}$$

$$\geq -3M^2\|v\|^2_{L^2(Q)} + \frac{c}{p+1}\|v\|^{p+1}_{L^2(Q)} \geq -c(M),$$

where, similarly to the previous step, the bound on the right-hand side is independent of n.

We infer that there exists a positive integer $M > 0$ such that all hypotheses of Lemma 6.2 are satisfied with the constant $a < b < 0$ independent of M. Thus we obtain the existence of critical points $u_n \in X_n$ of the functional $\mathcal{A}_{n,M}$ satisfying (6.26). Exactly as in Section 6.4 we can pass to the limit for $u_n \to u$, where u is a non-zero weak solution of the problem (6.1), (6.2), (6.6) in the sense specified in Definition 6.1.

We have proved the following result:

Theorem 6.1 *Let f be of the form*

$$f(u) = |u|^{p-1}u \text{ for all } u \in R^1, \text{ for a certain } p > 1.$$

The the problem (6.1), (6.2), (6.6) admits infinitely many non-zero weak solutions u, $u \in L^{p+1}(Q)$, in the sense of Definition 6.1.

6.6 Exercises

6.6.1 Coercivity of monotone functions

(i)

Show that for $p \geq 1$ there is positive constant $c(p)$ such that

$$\Big(|u|^{p-1}u - |v|^{p-1}v\Big)\Big(u - v\Big) \geq c(p)|u - v|^{p+1}.$$

(ii)

Let $f : [0, \infty) \to [0, \infty)$ be a convex function such that $f(0) = 0$. Show that

$$(f(u) - f(v))(u - v) \geq f(|u - v|)|u - v| \text{ for any } u, v \geq 0.$$

References

[1] S. Benzoni-Gavage and D. Serre. *Multidimensional hyperbolic partial differential equations, First order systems and applications*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007.

[2] J. Billingham and A. C. King. *Wave motion.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2000.

[3] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations.* Oxford University Press, Oxford, 1998.

[4] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics.* Springer-Verlag, Berlin, 2000.

[5] L. Debnath. *Nonlinear partial differential equations for scientists and engineers.* Birkhäuser/Springer, New York, third edition, 2012.

[6] E. Feireisl. Time periodic solutions to a semilinear beam equation. *Nonlinear Anal.*, **12**:279–290, 1988.

[7] F. John. *Plane waves and spherical means applied to partial differential equations.* Dover Publ., New York, 2004.

[8] R. Leis. *Initial-boundary value problems in mathematical physics.* B.G. Teubner, Stuttgart, 1986.

[9] H. A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.*, **192**:1–21, 1974.

[10] J. Lighthill. *Waves in Fluids.* Cambridge University Press, Cambridge, 1978.

[11] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Gautthier - Villars, Paris, 1969.

[12] P. Quittner and P. Souplet. *Superlinear parabolic problems.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2007. Blow-up, global existence and steady states.

[13] D. Serre. *Systems of conservations laws.* Cambridge university Press, Cambridge, 1999.

[14] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York, 1967.

[15] W. A. Strauss. *Nonlinear wave equations*, volume 73 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989.

[16] T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.

[17] B. R. Vaĭnberg. *Asymptotic methods in equations of mathematical physics*. Gordon & Breach Science Publishers, New York, 1989. Translated from the Russian by E. Primrose.