Scaling and singular limits in fluid mechanics

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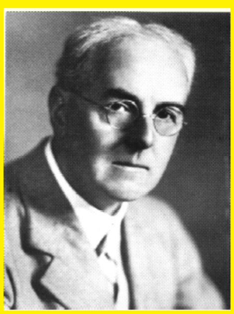


L.F. Richardson (1881-1953)

Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong. . . Some verbal statements have not this merit.

Contents

1 Introduction

We start by introducing the *Navier-Stokes-Fourier system* in the "entropy" form:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{1.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \ \sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \tag{1.3}$$

The system governs the evolution of a compressible, viscous, and heat conducting fluid described in terms of the *mass density* $\varrho = \varrho(t, x)$, the *absolute temperature* $\vartheta = \vartheta(t, x)$, and the *velocity field* $\mathbf{u} = \mathbf{u}(t, x)$, see Gallavotti [20]. Furthermore, the symbol $\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$ stands for the viscous stress, here given by the standard *Newton rheological law*

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.4}$$

and $\mathbf{q}(\vartheta, \nabla_x \vartheta)$ is the heat flux determined by the *Fourier law*

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta. \tag{1.5}$$

Finally, $p = p(\varrho, \vartheta)$ is the *pressure* and $s = s(\varrho, \vartheta)$ the *specific entropy* related to the *specific internal energy* $e = e(\varrho, \vartheta)$ by *Gibbs' equation*

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right). \tag{1.6}$$

In addition to (1.6) we impose the *thermodynamic stability hypothesis*

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \tag{1.7}$$

that will play a crucial role in the analysis (see Callen [7] for the physical background of (1.6), (1.7)).

Equations (1.1 - 1.3) represent our *primitive system* providing a complete description of a given fluid in motion. Given the enormous scope of applications of continuum fluid mechanics, solutions of the Navier-Stokes-Fourier system describe the motion of general gases and compressible liquids around or without presence of rigid bodies, the atmosphere and oceans in meteorology, and even the evolution of gaseous stars. Obviously, these phenomena may occur on very different time and spatial scales, where *simplified models* may provide equally good if not better picture of reality. Our goal is to show how these models can be *rigorously* derived as singular limits of a scaled version of (1.1 - 1.3), where certain characteristic numbers tend to zero or become excessively large.

1.1 Scaling and dimensionless equations

The method of *scaling* is well known and frequently used in engineering. Instead of considering the physical quantities in their original (typically S.I.) units, we replace a quantity X by $X/X_{\rm char}$, where $X_{\rm char}$ is the *characteristic* value of X. Applying this procedure to the system (1.1 - 1.3) and keeping the same symbols for physical quantities and their dimensionless counterparts, we arrive at the following *scaled* Navier-Stokes-Fourier system:

$$[{\rm Sr}]\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0, \tag{1.8}$$

$$[{\rm Sr}]\partial_t(\varrho \mathbf{u}) + {\rm div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \left[\frac{1}{{\rm Ma}^2}\right] \nabla_x p(\varrho, \vartheta) = \left[\frac{1}{\rm Re}\right] {\rm div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{1.9}$$

$$[{\rm Sr}]\partial_t(\varrho s(\varrho, \vartheta)) + {\rm div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \left[\frac{1}{\rm Pe}\right] {\rm div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \sigma = \frac{1}{\vartheta}\left(\left[\frac{{\rm Ma}^2}{\rm Re}\right] \mathbb{S} : \nabla_x \mathbf{u} - \left[\frac{1}{\rm Pe}\right] \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right), \tag{1.10}$$

with the *characteristic numbers*:

Čeněk Strouhal

Strouhal number:

$$[{\rm Sr}] = \frac{\rm length_{char}}{\rm time_{char} velocity_{char}}$$

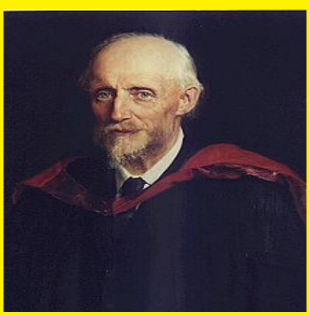
Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large.

Ernst Mach [1838-1916]

Mach number:

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}}/\mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes *incompressibility*.



Osborne Reynolds [1842-1912]

Reynolds number:

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to *turbulent* flows, where the viscosity of the fluid is negligible.

Jean Claude Eugène Péclet [1793-1857]

Péclet number:

$$[\mathrm{Pe}] = \frac{\mathrm{pressure}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{heat\ conductivity}_{\mathrm{char}}\mathrm{temperature}_{\mathrm{char}}}$$

Similarly to Reynolds number, high Péclet number corresponds to low heat conductivity of the fluid that may be attributed to turbulent flows.

The reader will have noticed that specific values of characteristic numbers may correspond to physically different systems. For instance, high Reynolds number may be associated to low viscosity of the fluid or to extremely large length scales. We refer to the survey of Klein et al. [27] for a thorough discussion of singular limits and the applications of scaling in numerical analysis.

1.2 Inviscid, incompressible limit

We focus on the situation when

$$\mathrm{Sr} = 1,\ \mathrm{Ma} = \varepsilon,\ \mathrm{Re} = \varepsilon^{-a},\ \mathrm{Pe} = \varepsilon^{-b},\ a, b > 0,$$

where $\varepsilon > 0$ is a small parameter. Our goal is to identify the limit system for $\varepsilon \to 0$, meaning the *inviscid, incompressible limit* of the scaled Navier-Stokes-Fourier system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.11}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{1.12}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \varepsilon^b \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \sigma = \frac{1}{\vartheta} \left(\varepsilon^{2+a} \mathbb{S} : \nabla_x \mathbf{u} - \varepsilon^b \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{1.13}$$

supplemented with the initial conditions:

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \tag{1.14}$$

where the reference values $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants.

1.3 Limit system

Formally, it is easy to identify the limit system of equations. Indeed the fact that the Mach number is small indicates incompressibility of the limit fluid flow; whence the limit system reads:

$$\mathrm{div}_x \mathbf{v} = 0 \tag{1.15}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0, \tag{1.16}$$

$$\partial_t \mathcal{T} + \mathbf{v} \cdot \nabla_x \mathcal{T} = 0, \tag{1.17}$$

which is nothing other than the *incompressible Euler system*, supplemented with the transport equation for the temperature deviation $\mathcal{T}$.

1.4 Boundary conditions

Real fluid systems are typically confined to a physical space - a domain $\Omega \subset R^3$. Accordingly, the boundary behavior of certain quantities must be specified. In order to avoid the so far unsurmountable problem of the boundary layer in the inviscid limit, we restrict ourselves to the *Navier slip boundary condition*

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \varepsilon^c [\mathbb{S}(\vartheta, \nabla_x \mathbf{u})\mathbf{n}]_{\tan} + \beta(\vartheta)\mathbf{u}|_{\partial\Omega} = 0,\ c, \beta > 0. \tag{1.18}$$

In addition, we impose the no-flux condition for the total energy, specifically, in terms of the heat flux $\mathbf{q}$,

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = -\beta \varepsilon^d |\mathbf{u}|^2|_{\partial\Omega},\ d = 2 + a - c - b. \tag{1.19}$$

Condition (1.19) implies, in particular, that the total energy of the system is a conserved quantity:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(\varepsilon^2 \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \Big) \ \mathrm{d}x = 0. \tag{1.20}$$

1.5 Singular limit

Our main goal is to discuss the singular limit of the scaled Navier-Stokes-Fourier system (1.11 - 1.14), supplemented with the boundary conditions (1.18), (1.19), for $\varepsilon \to 0$. Our work plan reads as follows:

- In Section 2, we introduce the relative entropy inequality together with the concept of dissipative solutions to the (primitive) Navier-Stokes-Fourier system.
- We use the relative entropy inequality to derive stability estimates for the solutions of the scaled system, see Section 3.
- In Section 4, we analyze the asymptotic behavior of *acoustic waves* and show the relevant dispersive estimates.
- Section 5 contains final comments and concluding remarks.

2 Weak and dissipative solutions

Solutions of the system (1.11 - 1.13), (1.18), (1.19) satisfy, together with the *total energy balance* (1.20), the total entropy production relation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x = \int_\Omega \sigma \ \mathrm{d}x + \varepsilon^{2+a-c} \int_{\partial\Omega} \frac{\beta}{\vartheta} |\mathbf{u}|^2 \ \mathrm{dS}_x. \tag{2.1}$$

Thus, adding (1.20), (2.1) together, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \Big(\varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta) \Big) \right] \ \mathrm{d}x + \frac{\Theta}{\varepsilon^2} \int_\Omega \sigma \ \mathrm{d}x + \Theta \varepsilon^{a-c} \int_{\partial\Omega} \frac{\beta}{\vartheta} |\mathbf{u}|^2 \ \mathrm{dS}_x = 0 \tag{2.2}$$

for any positive *constant* Θ. Relation (2.2) is usually termed *total dissipation inequality.* The functional

$$(\varrho, \vartheta, \mathbf{u}) \mapsto \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \Big(\varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta) \Big) \right] \ \mathrm{d}x$$

is a *Lyapunov function* for the Navier-Stokes-Fourier system.

2.1 Ballistic free energy

The functional

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big) \tag{2.3}$$

is called *ballistic free energy*, see Ericksen [13]. We compute

$$\frac{\partial^2 H_\Theta(\varrho,\Theta)}{\partial\varrho^2} = \frac{1}{\varrho}\frac{\partial p(\varrho,\Theta)}{\partial\varrho} \ \text{and} \ \frac{\partial H_\Theta(\varrho,\theta)}{\partial\vartheta} = \varrho\frac{\partial s(\varrho,\vartheta)}{\partial\vartheta}(\vartheta-\Theta).$$

Using the hypothesis of thermodynamic stability (1.7) we therefore conclude that

$$\varrho \mapsto H_\Theta(\varrho,\Theta) \text{ is strictly convex,} \tag{2.4}$$

and

$$\vartheta \mapsto H_\Theta(\varrho,\vartheta) \text{ is decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta \text{ for any fixed } \varrho. \tag{2.5}$$

2.2 Relative entropy

Motivated by the discussion in the preceding section, we introduce the *relative entropy functional* in the form

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right) \tag{2.6}$$
$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta)\right)\ \mathrm{d}x.$$

In the light of the coercivity properties (2.4), (2.5) it is easy to check that the relative entropy represents a kind of distance between the trio $(\varrho,\vartheta,\mathbf{u})$ and $(r,\Theta,\mathbf{U})$. Going back to the total dissipation inequality (2.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\Big|\overline{\varrho},\overline{\vartheta},0\right) + \frac{\overline{\vartheta}}{\varepsilon^2}\int_\Omega \sigma\ \mathrm{d}x + \overline{\vartheta}\varepsilon^{a-c}\int_{\partial\Omega}\frac{\beta}{\vartheta}|\mathbf{u}|^2\ \mathrm{dS}_x = 0, \tag{2.7}$$

where we have set

$$\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right) \tag{2.8}$$
$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + \frac{1}{\varepsilon^2}\left(H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta)\right)\right]\ \mathrm{d}x,$$

and where $\overline{\varrho}$, $\overline{\vartheta}$ are the positive constants appearing in the initial conditions (1.14) and chosen in such a way that

$$\int_{\Omega} (\varrho - \overline{\varrho}) \ \mathrm{d}x = 0 \tag{2.9}$$

If Ω is a bounded domain, the satisfaction of (2.9) is guaranteed if the perturbation $\varrho^{(1)}_{0,\varepsilon}$ is taken of zero integral mean as the total mass of the fluid

$$M_0 = \int_{\Omega} \varrho(t, \cdot) \ \mathrm{d}x$$

is a constant of motion. In general, the constants $\overline{\varrho}$, $\overline{\vartheta}$ will be always chosen in such a way that (2.8) holds. The trio $(\overline{\varrho}, \overline{\vartheta}, 0)$ is trivially a solution to the Navier-Stokes-Fourier system (1.1 - 1.3) that is called a *static state*. In view of the coercivity properties of the relative entropy established in (2.4), (2.5), relation (2.8) yields stability of the "static" states with respect to perturbations.

Our next goal is to derive a relation (inequality) similar to (2.7) provided $(\varrho, \vartheta, \mathbf{u})$ is a *weak* solution of the Navier-Stokes-Fourier system, and $(r, \Theta, \mathbf{U})$ is an arbitrary trio of "test functions" satisfying natural boundary conditions. To this end, a short excursion in the theory of weak solutions to the Navier-Stokes-Fourier system is needed.

2.3 Weak solutions

Following [17, Chapter 3] we introduce a concept of *weak solution* to the Navier-Stokes-Fourier system (1.1 - 1.3), with the boundary conditions (1.18), (1.19), and the initial condition

$$\varrho(0, \cdot) = \varrho_0, \ \vartheta(0, \cdot) = \vartheta_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{2.10}$$

To simplify presentation, we suppose that $\Omega \subset R^3$ is a *bounded* domain with smooth boundary.

2.3.1 Constitutive relations

Besides the existing restrictions imposed on the thermodynamic functions p, e, and s through Gibbs' equation (1.6) and the thermodynamic stability hypothesis (1.7), we introduce rather technical but still physically grounded assumptions required by the existence theory developed in [17]. More specifically, we suppose that the pressure p is given in the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \ P(0) = 0. \tag{2.11}$$

Here, the term proportional to ϑ^4 is attributed to the *radiation pressure* while the specific form

$$\vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right)$$

can be derived from the Gibbs' equation (1.6) as the universal formula for the *monoatomic gas* satisfying

$$p(\varrho, \vartheta) = \frac{2}{3} \varrho e(\varrho, \vartheta),$$

see [17, Chapter 1].

Accordingly, we take

$$e(\varrho, \vartheta) = \frac{3}{2} \vartheta \left(\frac{\vartheta^{3/2}}{\varrho} \right) P \left(\frac{\varrho}{\vartheta^{3/2}} \right) + \frac{a}{\varrho} \vartheta^4, \tag{2.12}$$

and

$$s(\varrho, \vartheta) = S \left(\frac{\varrho}{\vartheta^{3/2}} \right) + \frac{4a}{3} \frac{\vartheta^3}{\varrho}, \tag{2.13}$$

where

$$S'(Z) = -\frac{3}{2} \frac{\frac{5}{3} P(Z) - P'(Z) Z}{Z^2}$$

The thermodynamic stability hypothesis (1.7) stated in terms of the structural properties of the function P gives rise to:

$$P'(Z) > 0, \ 0 < \frac{\frac{5}{3} P(Z) - P'(Z) Z}{Z^2} < c \text{ for all } Z > 0, \ \lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.14}$$

Finally, we suppose the Third law of thermodynamics in the form

$$\lim_{Z \to \infty} S(Z) = 0. \tag{2.15}$$

As for the transport coefficients μ, λ, β and κ, we shall assume that they are continuously differentiable functions of the absolute temperature $\vartheta \in [0, \infty)$ satisfying:

$$\mu \in C^1[0, \infty) \text{ is globally Lipschitz continuous, } 0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta), \tag{2.16}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta), \tag{2.17}$$

and

$$\underline{\beta}(1 + \vartheta) \leq \beta(\vartheta) \leq \overline{\beta}(1 + \vartheta), \ \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3). \tag{2.18}$$

2.3.2 Variational formulation

We introduce a weak (variational) formulation of the Navier-Stokes-Fourier system (1.1 - 1.3), taking into account the boundary conditions (1.18), (1.19), together with the initial conditions (2.10).

We say that a trio $(\varrho, \vartheta, \mathbf{u})$ is a *weak solution* of the Navier-Stokes-Fourier system if

$$\varrho \geq 0,\ \varrho \in C_{\rm weak}([0,T]; L^{5/3}(\Omega)) \cap L^q((0,T)\times\Omega) \text{ for a certain } q > \frac{5}{3}, \tag{2.19}$$

$$\vartheta > 0 \text{ a.a. in } (0,T)\times\Omega,\ \vartheta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),\ \log(\vartheta) \in L^2(0,T;W^{1,2}(\Omega)), \tag{2.20}$$

$$\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \varrho\mathbf{u} \in C_{\rm weak}([0,T];L^{5/4}(\Omega)), \tag{2.21}$$

and the following integral identities are satisfied:

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\ {\rm d}x\right]_{t=0}^{\tau} = \int_0^\tau \int_\Omega \left(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right)\ {\rm d}x\ {\rm d}t \tag{2.22}$$

for any $\tau \in [0,T]$, and any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$;

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi(t,\cdot)\ {\rm d}x\right]_{t=0}^{\tau} = \int_0^\tau \int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\varphi + (\varrho\mathbf{u}\times\mathbf{u}) : \nabla_x\varphi + p(\varrho,\vartheta){\rm div}_x\varphi - \mathbb{S} : \nabla_x\varphi\right)\ {\rm d}x\ {\rm d}t \tag{2.23}$$

$$- \int_0^\tau \int_{\partial\Omega} \beta\mathbf{u}\cdot\varphi\ {\rm dS}_x$$

for any $\tau \in [0,T]$, and any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega};R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$;

$$\left[\int_\Omega \varrho s(\varrho,\vartheta)\varphi(t,\cdot)\ {\rm d}x\right]_{t=0}^{\tau} \geq \int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi - \mathbb{S}\right)\ {\rm d}x\ {\rm d}t \tag{2.24}$$

$$+ \int_0^\tau \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi\ {\rm d}x\ {\rm d}t + \int_0^\tau \int_{\partial\Omega} \beta|\mathbf{u}|^2\varphi\ {\rm dS}_x\ {\rm d}t$$

for a.a. $\tau \in [0,T]$, and any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$, $\varphi \geq 0$.

Since the weak formulation is formulated for the unscaled system, we have taken $\varepsilon = 1$ in the boundary conditions (1.18), (1.19). Note that the initial conditions are "hidden" in the quantities on the left-hand side of the above integral formulas.

While the integral identities (2.22), (2.23) represent the standard weak formulation of the equations (1.1), (1.2), the reader will have noticed that the entropy balance (1.3) has been replaced by *inequality* (2.24) corresponding to the entropy production rate

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right).$$

In order to compensate for this obvious lack of information, the variational formulation will be augmented, similarly to [17, Chapter 3], by the *total energy balance*

$$\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ {\rm d}x\right]_{t=0}^{\tau} = 0 \text{ for a.a. } \tau \in [0,T]. \tag{2.25}$$

Although apparently quite general, the resulting concept of weak solution is mathematically tractable. In particular, we report the following *global existence* result in the class of weak solutions, see [17, Theorems 3.1,3.2].

Theorem 2.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the thermodynamic functions p, e, s, and the transport coefficients μ, η, β, κ comply with the structural restrictions introduced in Section 2.3.1. Finally, let the initial data be taken such that*

$$\varrho_0 > 0, \ \vartheta_0 > 0 \ a.a. \ in \ \Omega, \ E_0 = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right) \ \mathrm{d}x < \infty.$$

Then the Navier-Stokes-Fourier possesses a weak solution in $(0,T) \times \Omega$ for any $T > 0$ in the sense specified through (2.19 - 2.25).

Possible generalizations with respect to the structural properties of p, e, and s as well as relaxation of the growth conditions (2.16), (2.17) are discussed at length in [17, Chapter 3]. We also remark that the initial density ϱ_0 may be taken only non-negative in Ω, however, such a generalization seems to be at odds with the standard derivation of the Navier-Stokes system as a model of *non-dilute* fluids.

2.4 Dissipative solutions

The dissipative solutions of the Navier-Stokes-Fourier system will be characterized by *relative entropy inequality* we are going to derive. After a bit tedious but absolutely routine manipulation we obtain

$$\begin{aligned} & \left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau} \qquad (2.26) \\ & + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_{\partial\Omega} \frac{\Theta \beta}{\vartheta} |\mathbf{u}|^2 \ \mathrm{dS}_x \ \mathrm{d}t \\ & \leq \int_0^\tau \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_{\partial\Omega} \beta \mathbf{u} \cdot \mathbf{U} \ \mathrm{dS}_x \ \mathrm{d}t \\ & + \int_0^\tau \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x \mathrm{d}t \end{aligned}$$

$$-\int_0^\tau \int_\Omega \left(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \frac{r-\varrho}{r}\Big(\partial_t p(r,\Theta) + \mathbf{U}\cdot\nabla_x p(r,\Theta)\Big) \, \mathrm{d}x \, \mathrm{d}t$$

for any (smooth) solution $(\varrho, \vartheta, \mathbf{u})$ of the Navier-Stokes-Fourier system and any trio of smooth "test" functions $(r, \Theta, \mathbf{U})$ satisfying

$$r > 0, \;\; \Theta > 0, \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{2.27}$$

Relation (2.26) is called *relative entropy inequality*. Our next observation is that it can be extended to the class of *weak* solutions. Indeed we may write

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right) = \sum_{i=1}^{6} I_i,$$

where

$$I_1 = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right) \, \mathrm{d}x,$$

$$I_2 = -\int_\Omega \varrho\mathbf{u}\cdot\mathbf{U} \, \mathrm{d}x, \;\; I_3 = \int_\Omega \frac{1}{2}\varrho|\mathbf{U}|^2 \, \mathrm{d}x,$$

$$I_4 = -\int_\Omega \varrho s(\varrho,\vartheta)\Theta \, \mathrm{d}x, \;\; I_5 = -\int_\Omega \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}\varrho \, \mathrm{d}x,$$

and

$$I_6 = \int_\Omega \left(\frac{\partial H_\Theta(r,\Theta)}{\partial\varrho} r - H(r,\Theta)\right) \, \mathrm{d}x.$$

Since the functions $(r, \Theta, \mathbf{U})$ are smooth and $\mathbf{U}$ satisfies the relevant boundary conditions, all quantities $[I_i]_{t=0}^{\tau}$ can be expressed by means of the *weak formulation* (2.22 - 2.25).

Motivated by a similar definition introduced by DiPerna and Lions [30] in the context of inviscid fluids, we say that $(\varrho, \vartheta, \mathbf{u})$ *is a* dissipative solution *to the Navier-Stokes-Fourier system if the relative entropy inequality (2.26) holds for all smooth test functions satisfying (2.27).*

As we have just observed, the *weak* solutions of the Navier-Stokes-Fourier system in a *bounded* regular domain Ω are dissipative solutions. The relative entropy inequality is a powerful tool that has been successfully applied to study

- the unconditional stability of the static states and attractors for the full Navier-Stokes-Fourier system, see [19];

- the problem of *weak-strong uniqueness*, see [16];
- the singular limits for low Mach and high Reynolds and Péclet numbers, see [15].

Here, we focus on the last issue performing the limit $\varepsilon \to 0$ in the scaled system.

2.4.1 Possible extensions

The concept of dissipative solution can easily be extended to problems on general *unbounded* domain. In such a situation, the constants $\overline{\varrho}$, $\overline{\vartheta}$ are taken to characterize the far field behavior, specifically,

$$\varrho \to \overline{\varrho},\ \vartheta \to \overline{\vartheta} \text{ as } |x| \to \infty. \tag{2.28}$$

Moreover, we shall always assume that the velocity vanishes for large x,

$$\mathbf{u} \to 0 \text{ as } |x| \to \infty. \tag{2.29}$$

Now, the relative entropy inequality remains formally the same as (2.26), where, in addition to (2.27), the test functions r, Θ, $\mathbf{U}$ admit suitable "far field" behavior. We may assume that

$$r - \overline{\varrho},\ \vartheta - \overline{\vartheta},\ \mathbf{U} \in C_c^\infty([0,T] \times \overline{\Omega}), \tag{2.30}$$

or that they decay rapidly to their asymptotic limits depending on the integrability of the weak solutions.

As the relative entropy inequality contains a complete piece of information we need to perform the singular limit we are interested in, we focus in the future only on *dissipative solutions*. Note that the *global-in-time existence* of dissipative solutions occupying a general unbounded physical space can be easily shown be the method of *invading domains*, where we construct weak (dissipative) solutions on a family of bounded domains

$$\Omega_R = \Omega \cap \{|x| < R\}$$

and let $R \to \infty$, see Jesslé, Jin, Novotný [21].

Since in the future we will deal exclusively with the scaled system (1.11 - 1.13), let us reformulate (2.26) in the $\varepsilon-$framework:

$$
\begin{aligned}
&\left[\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle|r,\Theta,\mathbf{U}\right)\right]_{t=0}^{\tau} \qquad (2.31)\\
&+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\varepsilon^a\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\varepsilon^{b-2}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^{a-c}\int_0^\tau\int_{\partial\Omega}\frac{\Theta\beta}{\vartheta}|\mathbf{u}|^2\,\mathrm{dS}_x\,\mathrm{d}t\\
&\le\int_0^\tau\int_\Omega\Big(\varrho\Big(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})+\varepsilon^a\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t+\varepsilon^{a-c}\int_0^\tau\int_{\partial\Omega}\beta\mathbf{u}\cdot\mathbf{U}\,\mathrm{dS}_x\,\mathrm{d}t\\
&+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big[\Big(p(r,\Theta)-p(\varrho,\vartheta)\Big)\mathrm{div}\mathbf{U}+\frac{\varrho}{r}(\mathbf{U}-\mathbf{u})\cdot\nabla_x p(r,\Theta)\Big]\,\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\left(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta+\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta+\varepsilon^b\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\right)\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\frac{r-\varrho}{r}\Big(\partial_t p(r,\Theta)+\mathbf{U}\cdot\nabla_x p(r,\Theta)\Big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for all test functions

$$
r>0,\ \Theta>0,\ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega}=0,\ \varrho-r,\ \vartheta-\Theta\in C_c^\infty([0,T]\times\overline{\Omega}),\ \mathbf{U}\in C_c^\infty([0,T]\times\overline{\Omega};R^3). \qquad (2.32)
$$

As we shall see in the next section, the integrability properties of the dissipative solutions on unbounded domains differ from those on bounded ones. As a matter of fact, they follow directly from (2.31).

3 Stability

Anticipating global-in-time existence of dissipative solutions $(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon)$ satisfying the relative entropy inequality (2.31), we derive uniform bounds *independent of* $\varepsilon\to 0$. To this end, it is convenient to introduce the following notation:

$$
h=h_{\rm ess}+h_{\rm res},\ h_{\rm ess}=\Psi(\varrho_\varepsilon,\vartheta_\varepsilon)h,\ h_{\rm res}=h-h_{\rm ess},
$$

$$
\Psi\in C_c^\infty(0,\infty)^2,\ 0\le\Psi\le 1,\ \Psi=1 \text{ on an open neighborhood of the point } (\overline{\varrho},\overline{\vartheta})
$$

for any measurable function h. The idea behind this notation is the it is the *essential* component $h_{\rm ess}$ that bears all the relevant information while the *residual* part $h_{\rm res}$ disappears in the asymptotic limit.

3.1 Coercivity of the relative entropy and uniform bounds

Let $K \subset \overline{K} \subset (0,\infty)^2$ be an open set containing (r,Θ). It follows from relations (2.4), (2.5), and the structural restrictions imposed of the functions e, s in Section 2.3.1 that

$$H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta) \geq c(K)\left(|\varrho - r|^2 + |\vartheta - \Theta|^2\right) \ \text{for all } (\varrho,\vartheta) \in K, \tag{3.1}$$

$$H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta) \geq c(K)\left(1 + \varrho e(\varrho,\vartheta) + \varrho s(\varrho,\vartheta)\right) \ \text{whenever } (\varrho,\vartheta) \in [0,\infty)^2 \setminus K. \tag{3.2}$$

3.1.1 First application of the relative entropy inequality

The desired uniform bounds follow immediately from the relative entropy inequality (2.31) evaluated at $r = \overline{\varrho}$, $\Theta = \overline{\vartheta}$, $\mathbf{U} = 0$ yielding

$$\left[\mathcal{E}_\varepsilon\left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \middle| \overline{\varrho}, \overline{\vartheta}, 0\right)\right]_{t=0}^{\tau} \tag{3.3}$$

$$+ \int_0^\tau \int_\Omega \frac{\overline{\vartheta}}{\vartheta}\left(\varepsilon^a \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon - \varepsilon^{b-2} \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right) \ \mathrm{d}x \ \mathrm{d}t + \varepsilon^{a-c} \int_0^\tau \int_{\partial\Omega} \frac{\overline{\vartheta}\beta(\vartheta_\varepsilon)}{\vartheta_\varepsilon} |\mathbf{u}_\varepsilon|^2 \ \mathrm{dS}_x \ \mathrm{d}t$$
$$\leq 0.$$

Observing that $\mathcal{E}_\varepsilon\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \middle| \overline{\varrho}, \overline{\vartheta}, 0\right)$ remains bounded for $\varepsilon \to 0$ as soon as

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega)} + \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2(\Omega;R^3)} \leq c, \tag{3.4}$$

we deduce the following list of estimates:

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;R^3)} \leq c, \tag{3.5}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}} \right\|_{L^2(\Omega)} \leq c, \tag{3.6}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\mathrm{ess}} \right\|_{L^2(\Omega)} \leq c, \tag{3.7}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left[\|1_{\mathrm{res}}\|_{L^1(\Omega)} + \|[\varrho_\varepsilon]_{\mathrm{res}}\|^{5/3}_{L^{5/3}(\Omega)} + \|[\vartheta_\varepsilon]_{\mathrm{res}}\|^4_{L^4(\Omega)}\right] \leq \varepsilon^2 c, \tag{3.8}$$

together with the "integral" bounds:

$$\varepsilon^a \int_0^T \int_\Omega \left|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\right|^2 \ \mathrm{d}x \ \mathrm{d}t + \varepsilon^{a-c} \int_0^T \int_{\partial\Omega} |\mathbf{u}_\varepsilon|^2 \ \mathrm{dS}_x \ \mathrm{d}t \leq c, \tag{3.9}$$

$$\varepsilon^{b-2} \int_0^T \int_\Omega |\nabla_x \vartheta_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c, \tag{3.10}$$

where all constants are independent of $\varepsilon \to 0$.

3.2 Convergence

The uniform bounds (3.5 - 3.10) are sufficient to pass to the limit in the family of solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)$ for $\varepsilon \to 0$. We obtain, in accordance with (3.5), (3.6), and (3.8),

$$\mathrm{ess} \sup_{t \in (0,T)} \|\varrho_\varepsilon(t, \cdot) - \overline{\varrho}\|_{L^2(\Omega) + L^{5/3}(\Omega)} \leq \varepsilon c \tag{3.11}$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \mathcal{T} \text{ weakly-(*) in } L^\infty(0,T; L^2(\Omega)), \tag{3.12}$$

and

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \tilde{\mathbf{u}} \text{ weakly-(*) in } L^\infty(0,T; L^2(\Omega; R^3)). \tag{3.13}$$

3.2.1 Another use of the relative entropy inequality

Of course, our goal is to show that $\tilde{\mathbf{u}} = \sqrt{\overline{\varrho}}\mathbf{v}$, where $\mathbf{v}$ is a solution of the limit Euler system (1.15), (1.16), and that $\mathcal{T}$ solves (1.17). To this end, we use again the relative entropy inequality (2.26), this time for the choice of "test functions" that corresponds to the first order $\varepsilon-$approximation. Specifically, we rewrite formally the system (1.11 - 1.13) as

$$\varepsilon \partial_t \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \tag{3.14}$$

$$\varepsilon \partial_t \left(\varrho_\varepsilon \mathbf{u}_\varepsilon\right) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \partial_\vartheta (\overline{\varrho}, \overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right) = \mathbf{F}_{1,\varepsilon}, \tag{3.15}$$

$$\partial_t \left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right) + \mathrm{div}_x \left[\left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right) \mathbf{u}_\varepsilon \right] = F_{2,\varepsilon}, \tag{3.16}$$

where, in view of the uniform bounds established in Section 3.1.1, the "forces" $\mathbf{F}_{1,\varepsilon}$, $F_{2,\varepsilon}$ tend to zero for $\varepsilon \to 0$.

Thus we have

$$\varrho_\varepsilon \approx \overline{\varrho} + \varepsilon R_\varepsilon, \ \vartheta_\varepsilon \approx \overline{\vartheta} + \varepsilon \mathcal{T}_\varepsilon, \ \mathbf{u}_\varepsilon \approx \mathbf{v} + \nabla_x \Phi_\varepsilon,$$

where $\mathbf{v}$ is a solution of the Euler system (1.15), (1.16), and the functions R_ε, $\mathcal{T}_\varepsilon$, Φ_ε satisfy the *acoustic equation*

$$\left\{ \begin{array}{c} \partial_t \left(\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon \right) + \omega \Delta \Phi_\varepsilon = 0, \\ \\ \partial_t \nabla_x \Phi_\varepsilon + \nabla_x \left(\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon \right) = 0, \\ \\ \nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{array} \right\} \tag{3.17}$$

together with the *transport equation*

$$\partial_t \left(\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon \right) + \mathrm{div}_x \left[\left(\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon \right) \left(\mathbf{v} + \nabla_x \Phi_\varepsilon \right) \right] = 0. \tag{3.18}$$

In accordance with (3.14 - 3.16),

$$\alpha = \frac{1}{\overline{\varrho}} \partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) > 0,\ \beta = \frac{1}{\overline{\varrho}} \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}),\ \delta = \overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) > 0,\ \omega = \overline{\varrho} \left(\alpha + \frac{\beta^2}{\delta} \right) > 0.$$

The initial values are determined by (1.14), more specifically,

$$R_\varepsilon(0, \cdot) = R_{0,\varepsilon,\delta} = \left[\varrho^{(1)}_{0,\varepsilon} \right]_\delta,\ \mathcal{T}_\varepsilon(0, \cdot) = \mathcal{T}_{0,\varepsilon,\delta} = \left[\vartheta^{(1)}_{0,\varepsilon} \right]_\delta,$$

while

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],\ \nabla_x \Phi_{0,\varepsilon} = \nabla_x \Phi_{0,\varepsilon\,\delta} = \left[\mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}] \right]_\delta,$$

where $\mathbf{H}$ denotes the standard *Helmholtz projection* onto the space of solenoidal functions in Ω, and where $[\cdot]_\delta$ are suitable *regularizing operators* specified below. The reason for regularizing the data is that we want to take

$$r = r_\varepsilon = R_\varepsilon,\ \Theta = \Theta_\varepsilon = \mathcal{T}_\varepsilon,\ \mathbf{U} = \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality (2.31).

Seeing that

$$\mathcal{E}_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \middle| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right)(0) \approx \int_\Omega \varrho_{0,\varepsilon} \left| \mathbf{H}[\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0] + \mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}] - \left[\mathbf{H}^\perp[\mathbf{u}_{0,\varepsilon}] \right]_\delta \right|^2 \ \mathrm{d}x$$

$$+ \int_\Omega \left(\left| \varrho^{(1)}_{0,\varepsilon} - \left[\varrho^{(1)}_{0,\varepsilon} \right]_\delta \right|^2 + \left| \vartheta^{(1)}_{0,\varepsilon} - \left[\vartheta^{(1)}_{0,\varepsilon} \right]_\delta \right|^2 \right) \ \mathrm{d}x$$

we suppose that

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega), \tag{3.19}$$

$$\vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega), \tag{3.20}$$

and

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3). \tag{3.21}$$

The leading idea of the proof of convergence towards the limit (target) system is to let first $\varepsilon \to 0$, then $\delta \to 0$, in the relative entropy inequality and to use a Gronwall type argument to "absorb" all terms in the remainder on the right-hand side of (2.31). To this end, we have to make sure that

- the Euler system (1.15), (1.16) possesses a *regular solution* on some time interval $[0, T_{\max})$ for the initial datum
$$\mathbf{v}(0, \cdot) = \mathbf{H}[\mathbf{u}_0];$$
- the acoustic waves described by the system (3.17) become "negligible" in the asymptotic limit $\varepsilon \to 0$.

3.3 Solvability of the Euler system

The Euler system (1.15), (1.16) is well known to possess local-in-time regular solutions provided the initial datum $\mathbf{v}_0$ is sufficiently smooth. Results of this type were obtained by many authors, see Beirao da Veiga [4], Kato [23], Kato and Lai [24], among others. Moreover, in the remarkable work, Beale, Kato, and Majda [3] identified a celebrated regularity criterion, namely, the local smooth solution $\mathbf{v}$ can be extended up to the critical time $T_{\max}$ provided

$$\int_0^{T_{\max}} \|\mathbf{curl}\ \mathbf{v}\|_{L^\infty}\ \mathrm{d}t < \infty.$$

Of course, these results depend also on the geometry of the underlying physical space Ω. Starting with the known local existence result of Kato and Lai [24] on *bounded domains*, we can construct local-in-time solutions on a general (unbounded) domain Ω by taking

$$\Omega_R = \Omega \cap \{|x| < R\}$$

and passing to the limit for $R \to \infty$. Such a method works provided

- we restrict ourselves to finite energy solutions decaying to zero for $|x| \to \infty$ in sufficiently high-order Sobolev spaces;
- we are interested only in local-in-time solutions.

Indeed the technique of Kato [23], Kato and Lai [24] is based on *energy estimates* obtain via multiplication of the equations by $\mathbf{v}$ and its derivatives and the resulting *existence time* can be taken *independent* of the size of the domain.

In what follows, we shall therefore assume that the initial velocity satisfies

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] \in W^{k,2}(\Omega; R^3) \text{ for a certain } k > \frac{5}{2}, \tag{3.22}$$

for which the Euler system (1.15), (1.16), supplemented with the boundary condition

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{3.23}$$

admits a unique solution on the time interval $[0, T_{\max})$ belonging to the class

$$\mathbf{v} \in C([0, T_{\max}), W^{k,2}(\Omega, R^3)),\ \partial_t \mathbf{v} \in C([0, T_{\max}); W^{k-1,2}(\Omega; R^3)). \tag{3.24}$$

Note that *global* existence of solutions to the Euler system, say in the class of weak solutions, is a delicate problem, where many surprising new facts emerged only recently in the work by DeLellis and Székelyhidi [11], [12], Wiedemann [35].

4 Acoustic waves

We study the decay properties of solutions to the *acoustic equation* (3.17) written in a more concise form as

$$\varepsilon \partial_t Z + \Delta \Phi = 0,\ \varepsilon \partial_t \Phi + Z = 0, \tag{4.1}$$

$$\nabla_x \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{4.2}$$

$$\Phi(0\cdot) = \Phi_0,\ Z(0, \cdot) = Z_0, \tag{4.3}$$

which is nothing other than a (scaled) linear wave equation for the *acoustic potential* Φ supplemented with the homogeneous Neumann boundary conditions.

4.1 Neumann Laplacean, Duhamel's formula

The (non-negative) Neumann Laplacean $-\Delta_N$ is a non-negative self-adjoint operator in the Hilbert space $L^2(\Omega)$ with a domain of definition

$$\mathcal{D}(-\Delta_N) = \left\{ w \in W^{1,2}(\Omega) \ \middle|\ \int_\Omega \nabla_x w \cdot \nabla_x \phi \ \mathrm{d}x = \int_\Omega g\phi \ \mathrm{d}x \text{ for a certain } g \in L^2(\Omega) \text{ and all } \phi \in C_c^\infty(\overline{\Omega}) \right\},$$

and we set

$$-\Delta_N w = g.$$

As a consequence of the standard elliptic theory, we have

$$\mathcal{D}(-\Delta_N) \in W^{2,2}_{\rm loc}(\Omega),$$

where the estimates can be extended up to the boundary $\partial\Omega$ provided the latter is smooth.

Solutions of the acoustic equation (4.1 - 4.3) can be written by means of *Duhamel's formula* as

$$\Phi(t,\cdot) = \frac{1}{2}\exp\left({\rm i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{{\rm i}}{\sqrt{-\Delta_N}}Z_0\right] + \frac{1}{2}\exp\left(-{\rm i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{{\rm i}}{\sqrt{-\Delta_N}}Z_0\right], \quad (4.4)$$

$$Z(t,\cdot) = \frac{1}{2}\exp\left({\rm i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[{\rm i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right] + \frac{1}{2}\exp\left(-{\rm i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[-{\rm i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]. \quad (4.5)$$

4.1.1 Decay and dispersive estimates

Our strategy is based on *eliminating* the effect of acoustic waves by means of dispersion. In other words, if Ω is "large", solutions of (4.1 - 4.3) will decay to zero locally in space as $t \to \infty$, therefore they will vanish as $\varepsilon \to 0$ for any positive time. A direct inspection of Duhamel's formula (4.4), (4.5) yields immediately that such a scenario is precluded by the presence of *trapped modes* - eigenvalues with corresponding eigenfunctions in $L^2(\Omega)$. In particular, all bounded domains must be excluded from future analysis.

On the other hand, the existence of eigenvalues of the Neumann Laplacean on a general *unbounded* domain is a delicate and highly unstable problem, see Davies and Parnovski [10]. Examples of domains, where Δ_N has void point spectrum are R^3, exterior domains in R^3, flat waveguides in R^3, see Lesky and Racke [29].

From now on, we shall therefore *assume* that the point spectrum of Δ_N defined in Ω is empty. In such a case, the celebrated RAGE theorem can be used to obtain *local decay estimates* for solutions of the acoustic equation, see Cycon et al. [9, Theorem 5.8]):

Theorem 4.1 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto the space of continuity H_c of A, specifically,*

$$H = H_c \oplus {\rm cl}_H\Big\{{\rm span}\{w \in H \mid w \textit{ an eigenvector of } A\}\Big\}.$$

Then

$$\left\| \frac{1}{\tau}\int_0^\tau \exp(-{\rm i}tA)CP_c\exp({\rm i}tA)\ {\rm d}t \right\|_{\mathcal{L}(H)} \to 0 \textit{ as } \tau \to \infty. \quad (4.6)$$

Taking $H = L^2(\Omega)$, $A = \sqrt{-\Delta_N}$, $C = \chi^2 G(-\Delta_N)$, with

$$\chi \in C_c^\infty(\overline{\Omega}),\ \chi \geq 0,\ G \in C_c^\infty(0,\infty),\ 0 \leq G \leq 1,$$

we may apply Theorem 4.1 for $\tau = 1/\varepsilon$ to obtain

$$\int_0^T \left\| \chi G(-\Delta_N) \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)[X] \right\|^2_{L^2(\Omega)} \mathrm{d}t \leq \omega(\varepsilon) \|X\|^2_{L^2(\Omega)} \text{ for any } X \in L^2(\Omega), \tag{4.7}$$

where

$$\omega(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Relation (4.7) is a kind of spatially and "frequency" localized estimates that are quite general and require only the absence of eigenvalues of the operator Δ_N in Ω and certain smoothness of $\partial\Omega$. The decay rate characterized through ω may be arbitrarily slow depending on the geometrical properties of $\partial\Omega$, see [14]. The "optimal" rate $\omega(\varepsilon) \approx \varepsilon$ can be achieved provided the operator Δ_N satisfies the *limiting absorption principle* (LAP), see Leis [28], Vaĭnberg[34] :

The cut-off resolvent operator

$$(1 + |x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm \mathrm{i}\delta]^{-1} \circ (1 + |x|^2)^{-s/2},\ \delta > 0,\ s > 1 \tag{4.8}$$

can be extended as a bounded linear operator on $L^2(\Omega)$ for $\delta \to 0$ and μ belonging to *compact* subintervals of $(0, \infty)$.

If Δ_N satisfies (LAP), the relevant alternative to the RAGE theorem is provided by a result of Kato [22] (see also Burq et al. [6]):

Theorem 4.2 [Reed and Simon [31, Theorem XIII.25 and Corollary]]

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin \mathbb{R}$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin \mathbb{R},\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty. \tag{4.9}$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|^2_X \ \mathrm{d}t \leq \Gamma^2.$$

If Δ_N satisfies (LAP), Theorem 4.2 yields (see [18] for details) the decay rate

$$\int_0^\infty \left\| \chi G(-\Delta_N) \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N} t\right)[X] \right\|^2_{L^2(\Omega)} \mathrm{d}t \leq c \|X\|^2_{L^2(\Omega)} \text{ for any } X \in L^2(\Omega), \tag{4.10}$$

which is equivalent to (4.8) with $\omega(\varepsilon) = \varepsilon$.

4.2 Smoothing operators

Motivated by the solution formulas (4.4), (4.5), we introduce the smoothing operators

$$\left\{ \begin{array}{c} [w]_\delta = G_\delta(\sqrt{-\Delta_N})[w],\ G_\delta \in C^\infty_c(R \setminus \{0\}),\ G_\delta(-Z) = G_\delta(Z),\ Z \in R \\ \\ 0 \le G_\delta \le 1,\ G_\delta(Z) \nearrow 1 \text{ as } \delta \to 0. \end{array} \right\} \tag{4.11}$$

First, as a consequence of the elliptic regularity,

$$\|[w]_\delta\|_{W^{k,2}(\Omega)} \le c(k,\delta)\|w\|_{L^2(\Omega)} \text{ for any } k = 0, 1, \ldots \tag{4.12}$$

as long as $\partial\Omega$ is smooth.

Next, we show that $[w]_\delta$ decays very fast as $|x| \to \infty$ for compactly supported w. Let us take

$$\varphi \in C^\infty_c(\Omega),\ \text{supp}[\varphi] \subset \{|x| < R\}.$$

Our goal is to estimate $[\varphi]_\delta$ outside the ball $\{|x| < 2R\}$. To this end, we introduce a weighted (pseudo-norm)

$$\|v\|^2_{s,(2R)^c} = \int_{\Omega\cap\{|x|>2R\}} |v|^2 |x|^{2s}\ \mathrm{d}x,$$

and write

$$G_\delta(\sqrt{-\Delta_N})[\varphi] = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{G}_\delta(t) \left(\exp(\mathrm{i}\sqrt{-\Delta_N} t) + \exp(-\mathrm{i}\sqrt{-\Delta_N} t) \right) [\varphi]\ \mathrm{d}t,$$

where $\tilde{G}_\delta$ denotes the Fourier transform of G_δ.

Next, we compute

$$\left\| G_\delta(\sqrt{-\Delta_N})[\varphi] \right\|_{s,(2R)^c} \le \frac{1}{2}\int_{-\infty}^{\infty} |\tilde{G}_\delta(t)| \left\| \left(\exp(\mathrm{i}\sqrt{-\Delta_N} t) + \exp(-\mathrm{i}\sqrt{-\Delta_N} t) \right) [\varphi] \right\|_{s,(2R)^c}\ \mathrm{d}t,$$

where

$$\left\| \left(\exp(\mathrm{i}\sqrt{-\Delta_N} t) + \exp(-\mathrm{i}\sqrt{-\Delta_N} t) \right) [\varphi] \right\|^2_{s,(2R)^c}$$

$$= \int_\Omega \text{sgn}^+(|x| - 2R)|x|^{2s} \left| \left(\exp(\mathrm{i}\sqrt{-\Delta_N} t) + \exp(-\mathrm{i}\sqrt{-\Delta_N} t) \right) [\varphi] \right|^2\ \mathrm{d}x.$$

However, because of the finite speed of propagation of the wave operator $\exp(\pm\mathrm{i}\sqrt{-\Delta_N} t)$, we may infer that

$$\int_\Omega \text{sgn}^+(|x| - 2R)|x|^{2s} \left| \left(\exp(\mathrm{i}\sqrt{-\Delta_N} t) + \exp(-\mathrm{i}\sqrt{-\Delta_N} t) \right) [\varphi] \right|^2\ \mathrm{d}x$$

$$\leq \mathrm{sgn}^+(|t|-R)(|t|+R)^{2s}\int_\Omega \left| \left(\exp(\mathrm{i}\sqrt{-\Delta_N}t) + \exp(-\mathrm{i}\sqrt{-\Delta_N}t) \right)[\varphi] \right|^2 \ \mathrm{d}x \leq c(s)|t|^{2s}\|\varphi\|_{L^2(\Omega)};$$

whence

$$\left\| G_\delta(\sqrt{-\Delta_N})[\varphi] \right\|^2_{s,(2R)^c} \leq c(s,\delta)\|\varphi\|_{L^2(\Omega)} \text{ provided } \mathrm{supp}[\varphi] \subset \{|x| < r\}.$$

Applying the same argument to $-\Delta^\alpha[\varphi]$ we deduce that

$$\sup_{x\in\Omega, |x|>2R} |x|^s |\partial_x^k [w]_\delta| \leq c(s,\delta,k)\|w\|_{L^2(\Omega)} \text{ for all } w \in L^2(\Omega),\ \mathrm{supp}[w] \subset \{|x| < R\}. \tag{4.13}$$

4.3 Dispersive estimates revisited

The local dispersive estimates (4.7), (4.10) are not strong enough to be used in the analysis of the inviscid limits. Some "global" version is needed, where the cut-off function $\chi \equiv 1$. Of course, this is not possible with the L^2-norm as the total energy of acoustic waves is conserved. On the other hand, if $\Omega = R^3$, solutions of the system (4.1 - 4.3) satisfy the *Strichartz estimates*:

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm \mathrm{i}\sqrt{-\Delta}t \right)[h] \right\|^p_{L^q(R^3)} \ \mathrm{d}t \leq \|h\|^p_{H^{1,2}(R^3)},\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ q < \infty, \tag{4.14}$$

where $H^{1,2}$ denotes the homogeneous Sobolev space of functions having first derivatives square integrable in R^3, see Keel and Tao [25], Strichartz [33].

In addition, the free Laplacean satisfies also the local energy decay in the form

$$\int_{-\infty}^{\infty} \left\| \chi \exp\left(\pm \mathrm{i}\sqrt{-\Delta}t \right)[h] \right\|^2_{H^{\alpha,2}(R^3)} \ \mathrm{d}t \leq c(\chi)\|h\|^2_{H^{\alpha,2}(R^3)},\ \alpha \leq \frac{3}{2},\ \chi \in C_c^\infty(R^3), \tag{4.15}$$

see Smith and Sogge [32, Lemma 2.2].

The estimates (4.14), (4.15) remain valid for the Neumann Laplacean on a "flat" space, for instance, on half-spaces in R^3.

4.3.1 Frequency localized Strichartz estimates

We assume that Ω is a "compact" perturbation of a larger domain on which the Neumann Laplacean satisfies the estimates (4.14), (4.15). For the sake of simplicity, we take the exterior domain $\Omega = R^3 \setminus K$, where K is a compact, not necessarily connected set. Applications to other domains like local perturbations of a half-space can be handled in a similar manner.

Our goal is to show

$$\int_{-\infty}^{\infty} \left\| G(-\Delta_N) \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N}t \right)[h] \right\|^p_{L^q(\Omega)} \leq c(G)\|h\|^p_{H^{1,2}(\Omega)},\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ q < \infty \tag{4.16}$$

for any $G \in C_c^\infty(0,\infty)$, adapting the method developed by Burq [5], Smith and Sogge [32].

We start by writing

$$U(t,\cdot) = G(-\Delta_N)\exp\left(\pm \mathrm{i}\sqrt{-\Delta_N}t\right)[h] = \exp\left(\pm \mathrm{i}\sqrt{-\Delta_N}t\right)G(-\Delta_N)[h]$$

as

$$U = v + w,\ v = \chi U,\ w = (1-\chi)U,$$

where

$$\chi \in C_c^\infty(R^3),\ 0 \le \chi \le 1,\ \chi \text{ radially symmetric},\ \chi(x) = 1 \text{ for } |x| \le R,$$

where R is so large that the ball $\{|x| < R\}$ contains K.

Accordingly,

$$w = w^1 + w^2,$$

where w^1 solves the homogeneous free wave equation

$$\partial^2_{t,t} w^1 - \Delta w^1 = 0 \text{ in } R^3,$$

supplemented with the initial conditions

$$w^1(0) = (1-\chi)G(-\Delta_N)[h],\ \partial_t w^1(0) = \pm \mathrm{i}(1-\chi)\sqrt{-\Delta_N}G(-\Delta_N)[h],$$

while

$$\partial^2_{t,t} w^2 - \Delta w^2 = F \text{ in } R^3,$$
$$w^2(0) = \partial_t w^2(0) = 0,$$

with

$$F = -\nabla_x \chi \nabla_x U - U\Delta\chi.$$

As a consequence of the standard Strichartz estimates (4.14), we get

$$\int_{-\infty}^{\infty} \left\| w^1 \right\|^p_{L^q(R^3)} \ \mathrm{d}t \le c(G) \|h\|^p_{H^{1,2}(R^3)},\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ q < \infty. \tag{4.17}$$

Furthermore, using Duhamel's formula, we obtain

$$w^2(\tau,\cdot) = \frac{1}{2\sqrt{-\Delta}}\left[\exp\left(\mathrm{i}\sqrt{-\Delta}\tau\right)\int_0^\tau \exp\left(-\mathrm{i}\sqrt{-\Delta}s\right)[\eta^2 F(s)]\ \mathrm{d}s\right]$$
$$-\frac{1}{2\sqrt{-\Delta}}\left[\exp\left(-\mathrm{i}\sqrt{-\Delta}\tau\right)\int_0^\tau \exp\left(\mathrm{i}\sqrt{-\Delta}s\right)[\eta^2 F(s)]\ \mathrm{d}s\right],$$

with

$$\eta \in C_c^\infty(R^3),\ 0 \le \eta \le 1,\ \eta \text{ radially symmetric},\ \eta = 1 \text{ on supp}[F].$$

Similarly to [5], we use the following result of of Christ and Kiselev [8]:

Lemma 4.1 *Let X and Y be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in the space of bounded linear operators from X to Y. Set*

$$\mathcal{T}[f](t) = \int_a^b K(t,s) f(s)\ \mathrm{d}s,\ \mathcal{W}[f](t) = \int_a^t K(t,s) f(s)\ \mathrm{d}s,$$

where

$$0 \leq a \leq b \leq \infty.$$

Suppose that

$$\|\mathcal{T}[f]\|_{L^p(a,b;Y)} \leq c_1 \|f\|_{L^r(a,b;X)}$$

for certain

$$1 \leq r < p \leq \infty.$$

Then

$$\|\mathcal{W}[f]\|_{L^p(a,b;Y)} \leq c_2 \|f\|_{L^r(a,b;X)},$$

where c_2 depends only on c_1, p, and r.

We aim to apply Lemma 4.1 in the situation

$$X = L^2(R^3),\ Y = L^q(R^3),\ q < \infty,\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ r = 2,$$

and

$$f = F,\ K(t,s)[F] = \frac{1}{\sqrt{-\Delta}} \exp\left(\pm \mathrm{i}\sqrt{-\Delta}(t-s)\right)[\eta^2 F].$$

Writing

$$\int_0^\infty K(t,s)F(s)\ \mathrm{d}s = \exp\left(\pm \mathrm{i}\sqrt{-\Delta} t\right) \frac{1}{\sqrt{-\Delta}} \int_0^\infty \exp\left(\mp \mathrm{i}\sqrt{-\Delta} s\right)[\chi^2 F(s)]\ \mathrm{d}s,$$

we have to show, keeping in mind the Strichartz estimates (4.14), that

$$\left\| \int_0^\infty \exp\left(\pm \mathrm{i}\sqrt{-\Delta} s\right)[\eta^2 F(s)]\ \mathrm{d}s \right\|_{L^2(R^3)} \leq c\|F\|_{L^2(0,\infty;L^2(R^3))}. \tag{4.18}$$

However,

$$\left\| \int_0^\infty \exp\left(\pm \mathrm{i}\sqrt{-\Delta} s\right)[\chi^2 F(s)]\ \mathrm{d}s \right\|_{L^2(R^3)}$$

$$= \sup_{\|v\|_{L^2(R^3)} \leq 1} \int_0^\infty \left\langle \exp\left(\pm \mathrm{i}\sqrt{-\Delta} s\right)[\chi^2 F(s)]; v \right\rangle\ \mathrm{d}s$$

$$= \sup_{\|v\|_{L^2(R^3)}\leq 1} \int_0^\infty \left\langle \chi F(s); \chi \exp\left(-\mathrm{i}\sqrt{-\Delta} s\right)[v] \right\rangle \ \mathrm{d}s;$$

whence the desired conclusion (4.18) follows from the local energy decay estimates stated in (4.15).

As the norm of F is bounded in view of the local estimates established in (4.15), we may infer that

$$\int_{-\infty}^\infty \left\| w^2 \right\|_{L^q(R^3)}^p \ \mathrm{d}t \leq c(G)\|h\|_{H^{1,2}(R^3)}^p, \ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty. \tag{4.19}$$

Finally, since $v = \chi U$ is compactly supported, we deduce from (4.15) combined with the standard elliptic regularity theory of Δ_N that

$$\int_0^\infty \|v\|_{L^q(\Omega)}^2 \ \mathrm{d}t \leq c(G)\|h\|_{H^{1,2}(\Omega)}^2; \tag{4.20}$$

while, by virtue of the energy estimates,

$$\sup_{t>0} \|v(t,\cdot)\|_{L^q(\Omega)} \leq c(G)\|h\|_{H^{1,2}(\Omega)}, \tag{4.21}$$

where $q < \infty$ is the same as in (4.16).

Interpolating (4.20), (4.21) and combining the result with the previous estimates, we get (4.16). As a matter of fact, our conclusion can be "strengthened" to

$$\int_{-\infty}^\infty \left\| G(-\Delta_N) \exp\left(\pm\mathrm{i}\sqrt{-\Delta_N} t\right)[h] \right\|_{L^q(\Omega)}^p \leq c(G)\|h\|_{L^2(\Omega)}^p, \ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty \tag{4.22}$$

for any $G \in C_c^\infty(0,\infty)$.

5 Concluding remarks

The uniform estimates established in Section 4, specifically (4.22), are sufficient to pass to the limit in the relative entropy inequality, first $\varepsilon \to 0$ then $\delta \to 0$. Thus we have shown (strong) convergence of the dissipative solutions of the scaled Navier-Stokes-Fourier system (1.11 - 1.14) to the (unique) solution of the target problem (1.15 - 1.17), endowed with the initial data

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0], \ \mathcal{T}(0,\cdot) = \overline{\varrho}\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta_0^{(1)} - \frac{1}{\overline{\varrho}}\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})\varrho_0^{(1)}. \tag{5.1}$$

The convergence takes place on any *compact* time interval $[0,T]$ provided $T < T_{\max}$, where $T_{\max} \leq \infty$ is the life span of the smooth solution to the target system. The details of the proof can be found in [15].

The result depends essentially on the geometric properties of the physical domain Ω. We have identified a class of *admissible domains* satisfying:

- Ω is an (unbounded) smooth domain in R^3, on which the Neumann Laplacean Δ_N satisfies the limiting absorption principle (4.8).

- There is $R > 0$ and a domain $D \subset R^3$ such that Δ_N satisfies the Strichartz and local decay estimates (4.14), (4.15) on D and $D \cap \{|x| > R\} = \Omega \cap \{|x| > R\}$.

Let us summarize our results (cf. [15, Theorem 3.1]):

Theorem 5.1 *Let $\Omega \subset R^3$ be an admissible domain in the sense specified above. Suppose that the thermodynamic functions p, e, s and the transport coefficients μ, λ, κ, and β comply with the structural restrictions introduced in Section 2.3.1, with*

$$b > 0,\ 0 < c < a < \frac{10}{3},$$

Furthermore, suppose that the initial data (1.14) satisfy

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(\Omega),\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega),$$

and

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),$$

where

$$\varrho^{(1)}_0,\ \vartheta^{(1)}_0 \in W^{1,2} \cap W^{1,\infty}(\Omega),\ \mathbf{H}[\mathbf{u}_0] = \mathbf{v}_0 \in W^{k,2}(\Omega; R^3) \text{ for a certain } k > \frac{5}{2}.$$

Let $T_{\max} \leq \infty$ be the maximal life-span of the regular solution $\mathbf{v}$ to the Euler system (1.15), (1.16, with the initial datum $\mathbf{v}(0,\cdot) = \mathbf{v}_0$. Finally, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ be a dissipative solution of the scaled Navier-Stokes-Fourier system (1.11 - 1.14) in $(0,T) \times R^3$, $T < T_{\max}$, supplemented with the boundary conditions (1.18), (1.19).

Then

$$\operatorname*{ess\,sup}_{t \in (0,T)} \|\ \varrho_\varepsilon(t,\cdot) - \overline{\varrho}\ \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c,$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\ \mathbf{v} \text{ in } L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\Omega; R^3)) \text{ and weakly-(*) in } L^\infty(0,T; L^2(\Omega; R^3)),$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \mathcal{T} \text{ in } L^\infty_{\rm loc}((0,T]; L^q_{\rm loc}(\Omega)),\ 1 \leq q < 2, \text{ and weakly-(*) in } L^\infty(0,T; L^2(\Omega)),$$

where $\mathbf{v}$, $\mathcal{T}$ is the unique solution of the Euler-Boussinesq system (1.15 - 1.17), with the initial data

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],\ \mathcal{T}_0 = \overline{\varrho}\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\vartheta^{(1)}_0 - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}\varrho^{(1)}_0.$$

5.1 Alternative techniques

An alternative approach to singular limits is based on strong solutions for *both* the primitive and the target system, see Klainerman and Majda [26]. Necessarily, the results are only *local-in-time* even if the target system happens to admit a global solution for a specific choice of the data. The initial data for the primitive system must be regular and their convergence to the limit values takes place in stronger topologies. We refer to Alazard [1], [2] for very interesting results concerning the full Navier-Stokes-Fourier system.

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