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Pointwise decay of stationary rotational viscous incompressible flows with nonzero velocity at infinity.

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Abstract

We consider a stationary viscous incompressible flow around a translating and rotating body. Optimal rates of decay are derived for the velocity and its gradient, on the basis of a representation formula involving a fundamental solution constructed by R. B. Guenther, E. A. Thomann, JMFM 8 (2006), 77-98, for a linearized system.

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Key words. Stationary incompressible Navier-Stokes system, rotating body, fundamental solution, representation formula, pointwise decay.

1. Introduction

We consider the following variant of the stationary incompressible Navier-Stokes system:

$$-\Delta u + \tau \cdot \partial_1 u + \tau \cdot (u \cdot \nabla)u - (\omega \times z) \cdot \nabla u + \omega \times u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad (1.1)$$
$$\text{for } \ z \in \mathbb{R}^3 \backslash \overline{\mathfrak{D}},$$

with zero flow at infinity:

$$u(x) \to 0 \ \text{ for } \ |x| \to \infty. \quad (1.2)$$

Problem (1.1), (1.2) together with some boundary condition on $\partial\mathfrak{D}$ constitutes a mathematical model describing stationary flow of a viscous incompressible fluid around a rigid body which moves at a constant velocity and rotates at a constant angular velocity. The open, bounded set $\mathfrak{D} \subset \mathbb{R}^3$ describes the rigid body, the given function $f : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ stands for an exterior force, and the unknowns $u : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ and $\pi : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}$ correspond respectively to the normalized velocity and pressure field of the fluid. In this model, the translational velocity of the rigid body is given by the vector $-\tau \cdot (1, 0, 0)$ and its angular velocity by $\omega := \varrho \cdot (1, 0, 0)$, for some $\tau \in (0, \infty)$ (Reynolds number) and $\varrho \in \mathbb{R} \backslash \{0\}$ (Taylor number). The velocity u was shifted in such a way that the limit velocity at

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infinity is zero (see (1.2)). For more extensive background information on (1.1), we refer to [26].

Suppose that $f \in L^{p_0}(\mathbb{R}^3)^3$ for some $p_0 \in (1,\infty)$ and f has compact support. Further suppose there is a pair of functions (u,π) with $u \in L^6(\overline{\mathfrak{D}}^c)^3$, $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ and $\pi \in L^2_{loc}(\overline{\mathfrak{D}}^c)$ satisfying (1.1) in the distributional sense ("Leray solution"). Such a solution exists under suitable assumptions on $\partial\mathfrak{D}$, $u|\partial\mathfrak{D}$ and p_0 ([28, Theorem XI.3.1]). Note that the condition $u \in L^6(\overline{\mathfrak{D}}^c)^3$, $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ means in particular that (1.2) holds in a weak sense; compare [24, Theorem II.5.1]. In this situation, it was shown by Galdi and Kyed [29] that

$$|\partial^\alpha u(x)| = O\big[\,\big(\,|x|\cdot\nu(x)\,\big)^{-1-|\alpha|/2}\,\big] \quad (|x|\to\infty), \tag{1.3}$$

where $\alpha \in \mathbb{N}_0^3$ with$|\alpha| := \alpha_1+\alpha_2+\alpha_3 \le 1$ (decay of u and ∇u). The term $\nu(x)$ in (1.3) is defined by

$$\nu(x) := 1+|x|-x_1 \quad (x \in \mathbb{R}^3). \tag{1.4}$$

Its presence in (1.3) may be considered as a mathematical manifestation of the wake extending downstream behind the rigid body. Even in the linear nonrotational case, that is, in the case of solutions to the Oseen system

$$-\Delta u + \tau\cdot\partial_1 u + \nabla\pi = f, \;\; \operatorname{div} u = 0, \tag{1.5}$$

the velocity cannot be expected to decay more rapidly than $\big(\,|x|\cdot\nu(x)\,\big)^{-1}$ for $|x|\to\infty$, nor its gradient more rapidly than $\big(\,|x|\cdot\nu(x)\,\big)^{-3/2}$ ([46]). Therefore the decay rate in (1.3) should be best possible in the present case, too.

The result we will prove in the work at hand may be stated as

Theorem 1.1 *Let $\tau \in (0,\infty)$, $\varrho \in \mathbb{R}\backslash\{0\}$, $\mathfrak{D} \subset \mathbb{R}^3$ open and bounded.*

Take $\gamma, S_1 \in (0,\infty)$, $p_0 \in (1,\infty)$, $A \in (2,\infty)$, $B \in [0,3/2]$, $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ measurable with $\overline{\mathfrak{D}} \subset B_{S_1}$, $A+\min\{B,1\} > 3$, $A+B \ge 7/2$, $f|B_{S_1} \in L^{p_0}(B_{S_1})^3$,

$$|f(y)| \le \gamma\cdot|y|^{-A}\cdot\nu(y)^{-B} \quad for\;\; y \in B^c_{S_1}. \tag{1.6}$$

Let $u \in L^6(\overline{\mathfrak{D}}^c)^3 \cap W^{1,1}_{loc}(\overline{\mathfrak{D}}^c)^3$, $\pi \in L^2_{loc}(\overline{\mathfrak{D}}^c)$ with $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$, $div\, u = 0$ and

$$\int_{\overline{\mathfrak{D}}}\big[\,\nabla u\cdot\nabla\varphi \;+\; \big(\tau\cdot\partial_1 u + \tau\cdot(u\cdot\nabla)u \\ -(\omega\times z)\cdot\nabla u + \omega\times u\big)\cdot\varphi - \pi\cdot div\,\varphi - f\cdot\varphi\,\big]\,dx = 0,$$

for $\varphi \in C_0^\infty(\overline{\mathfrak{D}}^c)^3$. Let $S \in (S_1,\infty)$. Then

$$|\partial^\alpha u(x)| \le D\cdot\big(\,|x|\cdot\nu(x)\,\big)^{-1-|\alpha|/2} \quad for\;\; x \in B^c_S,\;\; \alpha \in \mathbb{N}_0^3 \; with\; |\alpha| \le 1, \tag{1.7}$$

with the constant D depending on γ, S_1, p_0, A, B, $\|f|B_{S_1}\|_1$, u, π, S and on an arbitrary but fixed number $S_0 \in (0,S_1)$ with $\overline{\mathfrak{D}} \subset B_{S_0}$.

Note that we do not impose any conditions on the regularity of the boundary of $\mathfrak{D}$, and we need not prescribe any boundary conditions for u on $\partial\mathfrak{D}$. As an improvement with respect to the theory in [29], we do not require f to have compact support. Instead we only suppose that f decays as stated in (1.6). If f decays less fast, inequality (1.7) cannot be expected to hold. This may be seen by the very detailed study presented in [46] on the asymptotic behaviour of solutions to the Oseen system (1.5) in the whole space $\mathbb{R}^3$.

The work at hand, however, is not motivated by the more general assumptions on f we admit compared to the theory in [29]. Instead, our main purpose consists in introducing a new access to (1.7). In [29], this relation was reduced to estimates of solutions of the time-dependent Oseen system in the whole space $\mathbb{R}^3$. Here we will deduce (1.7) from a representation formula for u stated in Theorem 4.1 below and established in [6] and [10]. This formula involves a fundamental solution introduced by Guenther and Thomann [37] for the linearized system

$$-\Delta u + \tau \cdot \partial_1 u - (\omega \times z) \cdot \nabla u + \omega \times u + \nabla \pi = f, \;\; \operatorname{div} u = 0. \tag{1.8}$$

Deriving (1.7) from a representation formula is interesting because if $\omega = 0$, that is, in the case of a rigid body moving steadily but without rotation, the asymptotic behaviour of the flow field is usually deduced from such formulas (see [25, Section IX.6] for example). Therefore we hope the work at hand paves the way for carrying over other results besides (1.7) from the nonrotational to the rotational case. But our access is made difficult by the structure of the Guenther-Thomann fundamental solution. In fact, as was already pointed out in [20] for the case $\tau = 0$, a fundamental solution $\mathfrak{Z}(x, y)$ to (1.8) cannot be bounded by $c \cdot |x - y|^{-1}$ uniformly in $x, y \in \mathbb{R}^3$ with $|x|$ and $|y|$ large, contrary to what may be expected in view of the situation in the Stokes and Oseen case. Actually it seems that no uniform bound $c \cdot |x - y|^{-\epsilon}$ exists, for whatever $\epsilon \in (0, \infty)$. But due to Lemma 2.17 and Theorem 3.1 below, we are able to circumvent this difficulty. Both of these references pertain to convolutions of the Guenther-Thomann fundamental solution; the first provides a key element for obtaining pointwise bounds of these convolutions, whereas the second presents L^p-bounds. As a by-product of the latter result, we obtain the ensuing existence theorem for solutions to the linear system (1.8) in the whole space $\mathbb{R}^3$:

Theorem 1.2 *Let $p \in (1, 2)$. Then there is $u \in W^{2,p}_{loc}(\mathbb{R}^3)^3 \cap L^{(1/p-1/2)^{-1}}(\mathbb{R}^3)^3$, $\pi \in W^{1,p}_{loc}(\mathbb{R}^3) \cap L^{(1/p-1/3)^{-1}}(\mathbb{R}^3)$ such that*

$\partial_l u \in L^{(1/p-1/4)^{-1}}(\mathbb{R}^3)^3$, $\partial_l \partial_m u$, $\partial_1 u \in L^p(\mathbb{R}^3)^3$, $\partial_l \pi \in L^p(\mathbb{R}^3)$ for $1 \le l, m \le 3$, and such that (u, π) solves (1.8) in $\mathbb{R}^3$.

This theorem is not new; it was proved by Farwig, Hishida, Müller [20], [12]. These authors established it by adapting the proof of Lizorkin's multiplier theorem [52, Theorem 8] to problem (1.8), a procedure involving the heavy analysis of Littlewood-Paley theory. We are able to deduce our L^p-estimates of convolutions of the Guenther-Thomann fundamental solution – and thus Theorem 1.2 – directly from Lizorkin's multiplier theorem. This access to Theorem 1.2, although still rather technical, is considerably less involved than the one in [20].

In [6] – [9], we proved a representation formula, a decay estimates as in (1.7), and asymptotic expansions for weak solutions of the linear problem (1.8), as well as a representation formula for weak solutions of (1.1). In the context of these papers, a weak solution (u, π) of (1.8) or (1.1) is characterized by the assuptions that u is L^6-integrable outside a ball containing $\overline{\mathfrak{D}}$, and ∇u and π are L^2-integrable outside such a ball. The essential difference between this type of solution and a Leray solution as specified in Theorem 1.1 consists in the fact that the pressure part of a Leray solution needs to be L^2-integrable only locally, but not globally, in the complement of a ball around $\overline{\mathfrak{D}}$. It should be insisted that in the present context, the notions of weak solution and Leray solution do not refer to the behaviour of u or π near $\partial\mathfrak{D}$. In [10], we extended the results from [6] – [9] to Leray solutions of (1.8) and (1.1), respectively.

There is another type of solutions which is of interest in relation to (1.1) and (1.8), that is, physical reasonable (p. r.) solutions. They are characterized by the assumption that the velocity decays as $O(|x|^{-1})$ for $|x| \to \infty$. References [33] - [35] treat existence, uniqueness and the validity of (1.7) for these solutions. Note that the theory in [29] and in the work at hand means in particular that Leray solutions are physical reasonable.

Concerning further articles related to the work at hand, we mention [1], [13] – [19], [21] – [23], [27], [30], [31] [32], [36], [38]– [45], [47] – [51], [53] – [55]. We additionally remark that we proceed in a similar way as Farwig, Hishida [17], [18], who considered the linear equation (1.8) and the nonlinear one (1.1) without the Oseen term $\tau \cdot \partial_1 u$ (flow around a body that rotates but does not perform a translation). It turned out that a fundamental solution of (1.8) with $\tau = 0$ may be constructed in two steps: first a suitable rotational term is introduced into the fundamental solution of the time-dependent Stokes system; then the function obtained in this way is integrated with respect to time. With this fundamental solution as starting point, Farwig and Hishida succeeded in exhibiting detailed profiles of the flow in question, both in the linear ([18]) and in the nonlinear case ([17]). The profiles we obtained in [7] and [8] for the case $\tau \neq 0$ are less elaborated. This is due to the markedly more complicated structure of the Guenther-Thomann fundamental solution compared to the function constructed by Farwig and Hishida.

Concerning an approach to (1.8) in weighted spaces, Kračmar, Nečasová, Penel [43] – [45] worked in a L^2-framework with anisotropic weights, extending to (1.8) the theory established in [47], [48] for a simplified Oseen-type equation. In particular, a positive answer could be given to the question of existence of a wake, independently of [35], where the wake phenomenon is captured by the relation in (1.3). Another possibility to deal with (1.8) consists in working in an L^q-framework; then weighted multiplier and Littlewood-Paley theory are used, as well as the theory of one-sided Muckenhoupt weights corresponding to one-sided maximal functions. This approach was first introduced by Farwig, Hishida, Müller [20] (zero velocity at infinity) and Farwig [12], [13] (nonzero velocity at infinity) for the case that no weight is present, and then extended to the weighted case by Farwig, Krbec, Nečasová [21], [22] and Nečasová, Schumacher [55]. The case of singular data was studied in this framework in [42]. Stability estimates in the L^2-setting are proved in [34], and in the $L^{3,\infty}$-setting in [41].

Let us briefly indicate how we will proceed in the following. In Section 2, we will present various auxiliary results, most of them proved elsewhere, with the notable exception of Lemma 2.17. As indicated above, this lemma is an intermediate but crucial step in view of obtaining pointwise bounds for convolutions of the Guenther-Thomann fundamental solution. Section 3 deals with L^p-bounds of such convolutions. Establishing such bounds constitutes the main difficulty of our theory. We recall that as a by-product, these L^p-bounds yield existence of solutions to the linear system (1.8) in the whole space (Theorem 1.2). This point will also be discussed in Section 3.

At the beginning of Section 4, we will introduce a solution (u, π) of problem (1.1), (1.2) with properties as in Theorem 1.1. This solution will be kept fixed throughout the rest of this paper. Then, in Theorem 4.1, we will apply [10, Theorem 5], expressing u as a sum of certain integrals. This is the representation formula mentioned above; it will be used frequently. For this formula, additional assumptions on regularity of u and π near the boundary of the domain under consideration in that formula are needed. Thus, in order to avoid additional regularity assumptions near $\partial\mathfrak{D}$ for u and π, we will apply the representation formula in question not on $\overline{\mathfrak{D}}^c$, but on $B_{S_1}^c$, for some ball B_{S_1} containing $\overline{\mathfrak{D}}$. In this way, we will only need results on interior regularity of u and π, results which follow by the L^p-theory of the Stokes system, and thus need not be required as assumptions. In Section 4, we will exploit Theorem 4.1 in order to show that u belongs to $L^{12/5}(\overline{\mathfrak{D}}^c)^3$ and ∇u to $L^{12/7}(\overline{\mathfrak{D}}^c)^9$ (Theorem 4.4, Corollary 4.2). Besides Theorem 4.1, the proof of this result involves L^p-estimates from Section 3 and Banach's fixed point theorem. Being rather lengthy, this proof will be split into several steps, beginning with Lemma 4.2 and ending with Corollary 4.2. Less effort will be necessary to establish that ∇u belongs to L^4 outside a ball around $\overline{\mathfrak{D}}$ (Lemma 4.3).

Section 5 deals with pointwise estimates of u. A key role in this respect will be played by a function $\varphi(S)$ defined as the sup of $|u(x)|$ with respect to x from outside B_S, for suitable $S > 0$. Applying results from Section 2, in particular Lemma 2.17, as well as Corollary 4.2, we will show that $\varphi(S) \to 0$ for $S \to \infty$, and $\varphi(S) \leq \mathfrak{C} \cdot (S^{-1} + \varphi(S/2)^{7/6})$ if S is sufficiently large (Theorem 5.1). At this point, a result by Babenko [2] implies that $|u(x)|$ decays as $|x|^{-1}$ for $|x| \to \infty$ (Theorem 5.2). Then, by an argument based on [46, Theorem 3.2], we will obtain an estimate of u corresponding to (1.7) with $\alpha = 0$ (Theorem 5.3).

The last section (Section 6) treats the decay of ∇u. In a first step, we will exploit the preceding estimate of u and L^p-estimates of ∇u from Section 4 in order to show that ∇u is bounded outside a ball around $\overline{\mathfrak{D}}$ (Theorem 6.1). Once this result is available, the rest is canonical: we will again refer to [46, Theorem 3.2], applying an iteration argument that starts with Theorem 6.1 in order to establish (1.7) for $\alpha \in \mathbb{N}^3$ with $|\alpha| = 1$ (Theorem 6.2).

2. Notation. Auxiliary results.

If $A \subset \mathbb{R}^3$, we write A^c for the complement $\mathbb{R}^3 \backslash A$ of A. The symbol $|\ |$ denotes the Euclidean norm of $\mathbb{R}^3$ and also the length of a multiindex from $\mathbb{N}_0^3$, defined as in the line following (1.3). The open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. Put $e_l := (\delta_{jl})_{1 \leq j \leq 3}$ for $1 \leq l \leq 3$. Let $x \times y$ denote the usual vector product of $x, y \in \mathbb{R}^3$. Recall the function ν introduced in (1.4). We write C for numerical constants, and $C(\gamma_1, ..., \gamma_n)$ for constants depending on γ_1, ..., $\gamma_n \in (0, \infty)$, for some $n \in \mathbb{N}$.

If $p \in [1, \infty]$ and $A \subset \mathbb{R}^3$ measurable, the usual norm of the Lebesgue space $L^p(A)$ is denoted by $\| \ \|_p$. For $p \in [1, \infty)$, $k \in \mathbb{N}$ and open sets $A \subset \mathbb{R}^3$, we write $W^{k,p}(A)$ for the usual Sobolev space of order k and exponent p. If $B \subset \mathbb{R}^3$ is open, define $W^{k,p}_{loc}(B)$ as the set of all functions $g : B \mapsto \mathbb{R}$ such that $g|U \in W^{k,p}(U)$ for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset B$.

If $\mathfrak{H}$ is a normed space whose norm is denoted by $\| \ \|_{\mathfrak{H}}$, and if $n \in \mathbb{N}$, we equip the product space $\mathfrak{H}^n$ with a norm $\| \ \|^{(n)}_{\mathfrak{H}}$ defined by $\|v\|^{(n)}_{\mathfrak{H}} := \left(\sum_{j=1}^n \|v_j\|^2_{\mathfrak{H}} \right)^{1/2}$ for $v \in \mathfrak{H}^n$. But for simplicity, we will write $\| \ \|_{\mathfrak{H}}$ instead of $\| \ \|^{(n)}_{\mathfrak{H}}$.

Let $\tau \in (0, \infty)$ and $\varrho \in \mathbb{R}\backslash\{0\}$. These parameters are to be fixed for the rest of this article. As in Section 1, we define $\omega := \varrho \cdot e_1$. In addition, we put $\zeta := 2 \cdot \pi / |\varrho|$ (period of the rotation of the rigid body). Define the matrix $\Omega \in \mathbb{R}^{3\times3}$ by

$$\Omega := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

so that $\omega \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$. It will be convenient to introduce an abbreviation for the left-hand side of (1.8). To this end, we put

$$L(v)(z) := -\Delta v(z) + \tau \cdot \partial_1 v(z) - (\omega \times z) \cdot \nabla v(z) + \omega \times v(z)$$

for $v \in W^{2,1}_{loc}(B)$, $z \in B$, with $B \subset \mathbb{R}^3$ open. We further define a formal adjoint $\widetilde{L}$ of L by setting

$$\widetilde{L}(v)(z) := -\Delta v(z) - \tau \cdot \partial_1 v(z) + (\omega \times z) \cdot \nabla v(z) - \omega \times v(z)$$

for v, z as before. We note a technical detail that will be useful later on.

Lemma 2.1 *Let $k \in \mathbb{N}$, $z \in \mathbb{R}^3$, $s, r \in [0, \zeta]$. Then*

$$|z - \tau \cdot (k \cdot \zeta + s) \cdot e_1|^2 + k \cdot \zeta + s \geq C(\tau, \varrho) \cdot \left(|z - \tau \cdot (k \cdot \zeta + r) \cdot e_1|^2 + k \cdot \zeta + r \right). \quad (2.1)$$

Proof: First consider the case $|z - \tau \cdot k \cdot \zeta \cdot e_1| \leq 2 \cdot \tau \cdot \zeta$. Then

$$3 \cdot \tau \cdot \zeta \geq |z - \tau \cdot k \cdot \zeta \cdot e_1| + \tau \cdot \zeta \geq |z - \tau \cdot k \cdot \zeta \cdot e_1| + |\tau \cdot r \cdot e_1|$$
$$\geq |z - \tau \cdot (k \cdot \zeta + r) \cdot e_1|.$$

Since on the other hand, the left-hand side of (2.1) is bounded from below by $k \cdot \zeta$, and

$$k \cdot \zeta \geq \zeta/2 + k \cdot \zeta/2 = (3 \cdot \tau \cdot \zeta)^2 / (18 \cdot \tau^2 \cdot \zeta) + 2 \cdot k \cdot \zeta/4,$$

with $2 \cdot k \cdot \zeta \geq k \cdot \zeta + r$, inequality (2.1) follows in the case under consideration. Now assume that $|z - \tau \cdot k \cdot \zeta \cdot e_1| \geq 2 \cdot \tau \cdot \zeta$. Then we find

$$|z - \tau \cdot (k \cdot \zeta + s) \cdot e_1| \geq |z - \tau \cdot k \cdot \zeta \cdot e_1| - \tau \cdot s \geq |z - \tau \cdot k \cdot \zeta \cdot e_1|/2,$$

and

$$|z - \tau \cdot k \cdot \zeta \cdot e_1| \geq |z - \tau \cdot k \cdot \zeta \cdot e_1|/2 + \tau \cdot \zeta$$
$$\geq |z - \tau \cdot (k \cdot \zeta + r) \cdot e_1|/2 - \tau \cdot r/2 + \tau \cdot \zeta \geq |z - \tau \cdot (k \cdot \zeta + r) \cdot e_1|/2.$$

Observing that $k \cdot \zeta + s \geq k \cdot \zeta \geq k \cdot \zeta/2 + r/2$, we thus see that inequality (2.1) holds in the case $|z - \tau \cdot k \cdot \zeta \cdot e_1| \geq 2 \cdot \tau \cdot \zeta$ as well. □

We will further need the following Sobolev inequality:

Theorem 2.1 ([24, Theorem I.5.1]) *Let* $R > 0$, $v \in L^6(B_R^c) \cap W^{1,1}_{loc}(B_R^c)$ *with* $\nabla v \in L^2(B_R^c)^3 \cap L^{12/5}(B_R^c)^3$. *Then* $v \in L^{12}(B_R^c)$.

In order to estimate the Navier-Stokes nonlinearity, we will frequently refer to the following application of Hölder's inequality.

Lemma 2.2 *Let* $p, q \in (1, \infty)$ *with* $r := p \cdot q/(p+q) \geq 1$, $A \subset \mathbb{R}^3$ *measurable. Then there is* $C_0(p,q) > 0$ *with* $\|v^T \cdot W\|_r \leq C_0(p,q) \cdot \|v\|_p \cdot \|W\|_q$ *for* $v \in L^p(A)^3$, $W \in L^q(A)^{3\times 3}$.
In particular, $v^T \cdot W \in L^{3/2}(\mathbb{R}^3)^3$ *if* $p = 6$, $q = 2$.

Proof: Obviously $q/(p+q) < 1$, so $r < p$. Similarly we see that $r < q$. Thus the lemma follows from Hölder's inequality. □

Next we state a number of inequalities related to ν and Ω. Some of these inequalities are obvious, others require a proof that is quite involved (but well known).

Lemma 2.3 *The function* $e^{t\cdot\Omega}$ $(t \in \mathbb{R})$ *is periodic, with period* ζ. *Moreover,* $|e^{t\cdot\Omega} \cdot z| = |z|$ *for* $z \in \mathbb{R}^3$, $t \in \mathbb{R}$.

Proof: The eigenvalues of the matrix Ω are 0, $i \cdot |\varrho|$ and $-i \cdot |\varrho|$. □

Lemma 2.4 ([11, Lemma 4.3]) *Let* $b \in (1, \infty)$. *Then* $\int_{\partial B_r} \nu(x)^{-b}\, do_x \leq C(b) \cdot r$ *for* $r \in (0, \infty)$.

Corollary 2.1 *Let* $a \in (2, \infty)$, $b \in (1, \infty)$, $R \in (0, \infty)$. *Then* $\int_{B_R^c} |x|^{-a} \cdot \nu(x)^{-b}\, dx \leq C(a,b) \cdot R^{-a+2}$.

Lemma 2.5 *Let* $R \in (0, \infty)$. *Then* $|x| \geq C(R) \cdot \nu(x)$ *for* $x \in B_R^c$.

Proof: Obvious. □

Lemma 2.6 ([5, Lemma 2]) *Let* $M \in (0, \infty)$. *Then*
$|x - \tau \cdot t \cdot e_1|^2 + t \geq C(\tau, M) \cdot (|x|^2 + t)$ *for* $x \in B_M$, $t \in (0, \infty)$,
$|x - \tau \cdot t \cdot e_1|^2 + t \geq C(\tau, M) \cdot (|x| \cdot \nu(x) + t)$ *for* $x \in B_M^c$, $t \in (0, \infty)$.

Theorem 2.2 *Let* $a \in (1/2, \infty)$, $b \in [0, 3/2]$. *Then*

$$\begin{aligned}&\int_{\mathbb{R}^3} \big[\, (1 + |y - z|) \cdot \nu(y - z)\, \big]^{-3/2} \cdot (1 + |z|)^{-a} \cdot \nu(z)^{-b}\, dz \\ &\leq C(a,b) \cdot (1 + |y|)^{-c} \cdot \nu(y)^{-d} \quad \textit{for } y \in \mathbb{R}^3,\end{aligned}$$

where

$c := a - 1/2$, $d := b$ *if* $a \leq 2$, $a < b + 1$, $a + \min\{1, b\} \neq 3$, *and* $a \neq 1$ *or* $b \neq 0$*;*

$c := (a+b)/2$, $d := (a+b)/2 - 1/2$ *if* $b + 1 < a$ *and* $1 < a + b < 3$;

$c := d := 3/2$ *if* $a + b \geq 7/2$ *and* $a + \min\{1, b\} > 3$.

Proof: See the proof of [46, Theorem 3.2]. □

Theorem 2.3 *Let* $R, \delta \in (0, \infty)$, $a \in (1, \infty)$. *Then*

$$\int_0^\infty \big(|y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z|^2 + t \big)^{-a} \, dt \leq C(\tau, R, \delta, a) \cdot \big(|y| \cdot \nu(y) \big)^{-a+1/2}$$

for $y \in B^c_{(1+\delta)\cdot R}$, $z \in \overline{B_R}$, *and*

$$\int_0^\infty \big(|y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z|^2 + t \big)^{-a} \, dt \leq C(\tau, R, \delta, a) \cdot \big(|z| \cdot \nu(z) \big)^{-a+1/2}$$

for $z \in B^c_{(1+\delta)\cdot R}$, $y \in \overline{B_R}$.

Proof: For the first inequality, see [7, Theorem 2.19]. The second inequality holds according to the last paragraph of the proof of [7, Theorem 2.19]. This last paragraph further explains why the constants in the two inequalities in Theorem 2.3 do not depend on ϱ. □

Theorem 2.4 ([6, Theorem 3.1]) *Let* $k \in \{0, 1\}$, $R \in (0, \infty)$, $y, z \in B_R$ *with* $y \neq z$. *Then*

$$\int_0^\infty \big(|y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z|^2 + t \big)^{-3/2-k/2} \, dt \leq C(\tau, \varrho, R) \cdot |y - z|^{-1-k}.$$

Note there is an error in inequality [6, (3.7)]: instead of x, it should read $y + t \cdot U - e^{t\cdot\Omega} \cdot z$.

Let us introduce the fundamental solutions that will be needed in what follows. Put $\mathfrak{E}(x) := (4 \cdot \pi \cdot |x|)^{-1}$ for $x \in \mathbb{R}^3 \backslash \{0\}$ ("Newton potential"; fundamental solution of the Poisson equation); $E(x) := (4\cdot\pi\cdot|x|)^{-3}\cdot x$ for $x \in \mathbb{R}^3\backslash\{0\}$; $K(x, t) := (4\cdot\pi\cdot t)^{-3/2}\cdot e^{-|x|^2/(4\cdot t)}$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$ (fundamental solution of the heat equation);

$${}_1F_1(1, c, u) := \sum_{n=0}^\infty \big(\Gamma(c)/\Gamma(n + c) \big) \cdot u^n \quad \text{for } u \in \mathbb{R},\ c \in (0, \infty)$$

("Kummer function"), where the letter Γ denotes the usual Gamma function;

$$\mathfrak{H}_{jk}(x) := x_j \cdot x_k \cdot |x|^{-2},$$

$$\Lambda_{jk}(x, t) := K(x, t) \cdot \Big(\delta_{jk} - \mathfrak{H}_{jk}(x) - {}_1F_1\big(1, 5/2, |x|^2/(4 \cdot t)\big) \cdot \big(\delta_{jk}/3 - \mathfrak{H}_{jk}(x) \big) \Big)$$

for $x \in \mathbb{R}^3\backslash\{0\}$, $t \in (0, \infty)$, $j, k \in \{1, 2, 3\}$;

$$\Gamma(y, z, t) := \Lambda(y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z,\ t) \cdot e^{-t\cdot\Omega} \tag{2.2}$$

for $y, z \in \mathbb{R}^3$, $t \in (0, \infty)$ with $y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z \neq 0$;

$$\mathfrak{K}(y, z, t) := K(y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z,\ t) \quad \text{for } y, z \in \mathbb{R}^3,\ t \in (0, \infty);$$

$$\Psi(y, z, t) := (4 \cdot \pi)^{-1} \cdot \int_{\mathbb{R}^3} |y - x|^{-1} \cdot \mathfrak{K}(x, z, t) \, dx \quad \text{for } y, z, t \text{ as before;}$$

$$\mathfrak{Z}(y, z) := \int_0^\infty \Gamma(y, z, t) \, dt \quad \text{for } y, z \in \mathbb{R}^3 \text{ with } y \neq z.$$

Note that in the following, the letter Γ will not denote the standard Gamma function, but the function defined in (2.2), which constitutes the velocity part of the fundamental solution introduced by Guenther, Thomann [37] for the time-dependent variant of (1.8). The related pressure part is given by the function E introduced above. We further indicate that $\mathfrak{Z}$ is the velocity part of the Guenther-Thomann fundamental solution of (1.8) mentioned in Section 1, and E its pressure part.

Theorem 2.5 *For $h \in C_0^\infty(\mathbb{R}^3)$, $s \in \{1,2\}$, $x \in \mathbb{R}^3$, we have $\int_{\mathbb{R}^3} |x-y|^{-s} \cdot |h(y)|\, dy < \infty$, so we may define $\mathfrak{N}(h)(x) := \int_{\mathbb{R}^3} \mathfrak{E}(x-y) \cdot h(y)\, dy$. Let $g \in C_0^\infty(\mathbb{R}^3)$. Then $\mathfrak{N}(g) \in C^\infty(\mathbb{R}^3)$, $\partial^\alpha \mathfrak{N}(g) = \mathfrak{N}(\partial^\alpha g)$ for $\alpha \in \mathbb{N}_0^3$, and*

$$\partial_l \mathfrak{N}(g)(x) = \int_{\mathbb{R}^3} \partial_l \mathfrak{E}(x-y) \cdot h(y)\, dy \quad (1 \le l \le 3,\ x \in \mathbb{R}^3).$$

If $p \in (1,3)$, the inequality $\|\partial_l \mathfrak{N}(g)\|_{(1/p-1/3)^{-1}} \le C(p) \cdot \|g\|_p$ holds for $1 \le l \le 3$. In addition, $\Delta \mathfrak{N}(g) = -g$ and $\|\partial^\alpha \mathfrak{N}(g)\|_p \le C(p) \cdot \|g\|_p$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = 2$, $p \in (1,\infty)$.

For the convenience of the reader, we give some indications of the well-known

Proof: On writing $\mathfrak{N}(g)(x) = \int_{\mathbb{R}^3} \mathfrak{E}(z) \cdot g(x-z)\, dz$, and observing that $\mathfrak{E} \in L^1_{loc}(\mathbb{R}^3 \backslash \{0\})$, we may apply Lebesgue's theorem to obtain that $\mathfrak{N}(g) \in C^\infty(\mathbb{R}^3)$ and $\partial^\alpha \mathfrak{N}(g) = \mathfrak{N}(\partial^\alpha g)$ for $\alpha \in \mathbb{N}_0^3$. Starting from the equation $\partial_l \mathfrak{N}(g) = \mathfrak{N}(\partial_l g)$, then performing an integration by parts on $\mathbb{R}^3 \backslash B_\epsilon(x)$, for $\epsilon > 0$, and after that letting ϵ tend to zero, we obtain the formula for $\partial_l \mathfrak{N}(g)(x)$ stated in the theorem. This formula and the Hardy-Littlewood-Sobolev inequality ([57, p. 119, Theorem 1]) yield the estimate $\|\partial_l \mathfrak{N}(g)\|_{(1/p-1/3)^{-1}} \le C(p) \cdot \|g\|_p$ for $p \in (1,3)$, $1 \le l \le 3$. The last claim of the theorem may be obtained by twice integrating by parts in the expression $\mathfrak{N}(\partial_l \partial_m g)(x)$ $(1 \le l, m \le 3,\ x \in \mathbb{R}^3)$, and by applying Calderon-Zygmund's inequality ([57, p. 39, Theorem 3]). $\square$

Now we may prove the following variant of the Helmholtz decomposition.

Theorem 2.6 *Let $p, q \in (1,\infty)$, $f \in L^p(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$. Then there are sequences (ϕ_n), (φ_n), $(\widetilde{\varphi}_n)$ with $\phi_n \in C_0^\infty(\mathbb{R}^3)^3$, $div\, \phi_n = 0$, $\varphi_n, \widetilde{\varphi}_n \in C^\infty(\mathbb{R}^3)$ for $n \in \mathbb{N}$, and $\|\phi_n + \nabla \varphi_n - f\|_p \to 0$, $\|\phi_n + \nabla \widetilde{\varphi}_n - f\|_q \to 0$ $(n \to \infty)$.*

Proof: Choose a sequence (g_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|f - g_n\|_s \to 0$ $(n \to \infty)$ for $s = p$ and $s = q$. Let $n \in \mathbb{N}$, and abbreviate $U_n := \nabla \mathfrak{N}(\operatorname{div} g_n)$, with $\mathfrak{N}(\operatorname{div} g_n)$ defined in Theorem 2.5. By that theorem, we have $U_n \in C^\infty(\mathbb{R}^3)^3$ and $\operatorname{div}(g_n + U_n) = 0$. Moreover, again by Theorem 2.5, $U_{n,j} = \sum_{k=1}^3 \partial_j \partial_k \mathfrak{N}(g_k)$, $\partial_l U_{n,j} = \sum_{k=1}^3 \partial_j \partial_k \mathfrak{N}(\partial_l g_k)$ for $1 \le j, l \le 3$. Hence, once more by Theorem 2.5, $U_n \in W^{1,p}(\mathbb{R}^3)^3 \cap W^{1,q}(\mathbb{R}^3)^3$. Now we may conclude by [24, Theorem III.6.1] there is a sequence $(\phi_m^{(n)})_{m \ge 1}$ in $C_0^\infty(\mathbb{R}^3)^3$ with $\operatorname{div} \phi_m^{(n)} = 0$ for $m \in \mathbb{N}$, and $\|g_n + U_n - \phi_m^{(n)}\|_s \to 0$ $(m \to \infty)$ for $s = p$ and $s = q$. If $p > 3$, we further have $\mathfrak{N}(\operatorname{div} g) = \sum_{k=1}^3 \partial_k \mathfrak{N}(g_k) \in L^{(1/p-1/3)^{-1}}(\mathbb{R}^3)$ by Theorem 2.5. Thus we may refer to [24, Theorem II.6.2, II.5.1] to obtain a sequence $(\varphi_m^{(n)})_{m \ge 1}$ in $C_0^\infty(\mathbb{R}^3)$ with $\|\nabla \varphi_m^{(n)} + U_n\|_p \to 0$ $(m \to \infty)$. An analogous argument yields a sequence $(\widetilde{\varphi}_m^{(n)})_{m \ge 1}$ with $\|\nabla \widetilde{\varphi}_m^{(n)} + U_n\|_q \to 0$ $(m \to \infty)$. These observations imply Theorem 2.6. $\square$

Theorem 2.7 *Let $p \in (1,3)$, $g \in L^p(\mathbb{R}^3)^3$. Then, for a. e. $x \in \mathbb{R}^3$, the function $y \mapsto |x-y|^{-2} \cdot |g(y)|$ is integrable on $\mathbb{R}^3$. We may thus define $Q(g)(x) := \int_{\mathbb{R}^3} E(x-y) \cdot g(y)\, dy$*

for $x \in \mathbb{R}^3$.

Then $\|Q(g)\|_{(1/p-1/3)^{-1}} \le C(p) \cdot \|g\|_p$. *If* $g \in C_0^\infty(\mathbb{R}^3)^3$, *we have* $Q(g) \in C^\infty(\mathbb{R}^3)$ *and* $\partial^\alpha Q(g) = Q(\partial^\alpha g)$ *for* $\alpha \in \mathbb{N}_0^3$.

Moreover $Q(g) \in W^{1,1}_{loc}(\mathbb{R}^3)$, *and if in addition* $g \in L^s(\mathbb{R}^3)^3$ *for some* $s \in (1,\infty)$, *the inequality* $\|\nabla Q(g)\|_s \le C(s) \cdot \|g\|_s$ *holds.*

Proof: The first part of Theorem 2.7, up to and including the estimate of $Q(g)$ in the norm of $L^{(1/p-1/3)^{-1}}(\mathbb{R}^3)$, follows from the Hardy-Littlewood-Sobolev inequaltiy ([57, p. 119, Theorem 1]). The claims pertaining to the case $g \in C_0^\infty(\mathbb{R}^3)^3$ hold by Theorem 2.5. The last part of Theorem 2.7 follows from the same reference by a density argument. □

Next we turn to the fundamental solution Γ of the time-dependent variant of (1.8).

Lemma 2.7 *The functions* $\mathfrak{K}$, Ψ *and* Γ_{jk} *belong to* $C^\infty\big(\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)\big)$ *for* $1 \le j,k \le 3$. *Moreover*

$$\Gamma(y,z,t) = \big(\mathfrak{K}(y,z,t) \cdot \delta_{rs} + \partial y_r \partial y_s \Psi(y,z,t)\big)_{1 \le r,s \le 3} \cdot e^{-t \cdot \Omega} \quad \textit{for } y,z \in \mathbb{R}^3,\ t > 0. \quad (2.3)$$

Proof: Obviously $\mathfrak{K}$ is a C^∞-function on $\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)$. In view of [37, (3.11)], the same is true of Ψ. Equation (2.3) holds according to [37, (3.9)]. The relation $\Gamma \in C^\infty\big(\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)\big)^{3\times 3}$ now follows from (2.3). Note that in [6], we proved this relation in a different way; see [6, Corollary 3.1]. □

Lemma 2.8 $\sum_{k=1}^3 \partial z_k \Gamma_{jk}(y,z,t) = 0$ *for* $y,z \in \mathbb{R}^3$, $t \in (0,\infty)$, $j \in \{1, 2, 3\}$.

Proof: See [37, Theorem 1.3] or verify by a direct computation. □

Lemma 2.9 $|\partial_x^\alpha K(x,t)| \le C \cdot (|x|^2 + t)^{-3/2-|\alpha|/2}$ *for* $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$.

Proof: The reader may either refer to [56], or prove the lemma directly by distinguishing the cases $|x|^2 \le t$ and $|x|^2 > t$. □

Lemma 2.10 ([6, Lemma 3.2]) *For* $y,z \in \mathbb{R}^3$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 1$, *we have* $|\partial_y^\alpha \Gamma(y,z,t)| + |\partial_z^\alpha \Gamma(y,z,t)| \le C \cdot (|y - \tau \cdot t \cdot e_1 - e^{-t \cdot \Omega} \cdot z|^2 + t)^{-3/2-|\alpha|/2}$.

Corollary 2.2 *For* $y,z \in \mathbb{R}^3$ *with* $y \ne z$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *we have* $\int_0^\infty \big(|\partial_y^\alpha \mathfrak{K}(y,z,t)| + |\partial_z^\alpha \mathfrak{K}(y,z,t)|\big)\, dt < \infty$.

Proof: Lemma 2.9 and Theorem 2.4. □

Corollary 2.3 $\int_0^\infty \big(|\partial_y^\alpha \Gamma(y,z,t)| + |\partial_z^\alpha \Gamma(y,z,t)|\big)\, dt < \infty$ *for* $y,z \in \mathbb{R}^3$ *with* $y \ne z$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$. *This means in particular that the function* $\mathfrak{Z}$ *is well defined.*

Proof: Lemma 2.10 and Theorem 2.4. □

Corollary 2.4 *If* $y,z \in \mathbb{R}^3$ *with* $y \ne z$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, $j,k \in \{1, 2, 3\}$, *we have* $\int_0^\infty \big(|\partial_y^\alpha \partial y_j \partial y_k \Psi(y,z,t)| + |\partial_z^\alpha \partial y_j \partial y_k \Psi(y,z,t)|\big)\, dt < \infty$.

Proof: Equation (2.3), Corollary 2.2 and 2.3. □

Lemma 2.11 *$\mathfrak{Z} \in C^1\big((\mathbb{R}^3 \times \mathbb{R}^3) \setminus diag\,(\mathbb{R}^3 \times \mathbb{R}^3) \big)^{3\times 3}$. If $y, z \in \mathbb{R}^3$ with $y \neq z$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, then*

$$\partial_y^\alpha \mathfrak{Z}(y,z) = \int_0^\infty \partial_y^\alpha \Gamma(y,z,t)\,dt, \quad \partial_z^\alpha \mathfrak{Z}(y,z) = \int_0^\infty \partial_z^\alpha \Gamma(y,z,t)\,dt, \tag{2.4}$$

$$|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)| \leq C \cdot \int_0^\infty \big(|y - \tau \cdot t \cdot e_1 - e^{-t \cdot \Omega} \cdot z|^2 + t \big)^{-3/2 - |\alpha|/2}\,dt. \tag{2.5}$$

Proof: See [7, Lemma 2.15] for the first statement of Lemma 2.11 and for (2.4). Inequality (2.5) follows from (2.4) and Lemma 2.10. □

Corollary 2.5 *Let $R \in (0,\infty)$. Then $|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)| \leq C(\tau, \varrho, R) \cdot |y - z|^{-1-|\alpha|/2}$ for $y, z \in B_R$ with $x \neq y$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Proof: (2.5) and Theorem 2.4. □

Corollary 2.6 *Let R, $\delta \in (0,\infty)$. Then*

$|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)| \leq C(\tau, R, \delta) \cdot \big(|z| \cdot \nu(z) \big)^{-1-|\alpha|/2}$ for $z \in B_{(1+\delta)\cdot R}^c$, $y \in \overline{B_R}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. If instead $y \in B_{(1+\delta)\cdot R}^c$, $z \in \overline{B_R}$, and α is given in the same way, then

$|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)| \leq C(\tau, R, \delta) \cdot \big(|y| \cdot \nu(y) \big)^{-1-|\alpha|/2}$.

Proof: (2.5) and Theorem 2.3. □

Lemma 2.12 *Let $S \in (0,\infty)$, $g \in L^1(\partial B_S)$, $y \in (\partial B_S)^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then $\int_{\partial B_S} \big(|\partial_z^\alpha \mathfrak{Z}(y,z)| + |E(y-z)| \big) \cdot |g(z)|\,do_z < \infty$.*

Proof: First statement of Lemma 2.11. □

Next we introduce a volume potential with $\mathfrak{Z}$ as kernel. Due to its role in the representation formula in Theorem 4.1 below, this potential is the key mathematical object of this study.

Lemma 2.13 ([7, Lemma 3.1]) *Let $A \subset \mathbb{R}^3$ be measurable, $p \in (1,2)$, $f \in L^p(A)^3$. Let $\widetilde{f}$ denote the zero extension of f to $\mathbb{R}^3$. Then $\int_{\mathbb{R}^3} (|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)|) \cdot |\widetilde{f}(z)|\,dz < \infty$ for a. e. $y \in \mathbb{R}^3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Define $\mathfrak{R}(f)(y) := \int_{\mathbb{R}^3} \mathfrak{Z}(y,z) \cdot \widetilde{f}(z)\,dz$ for $y \in \mathbb{R}^3$. Then $\mathfrak{R}(f) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$ and $\partial_y^\alpha \mathfrak{R}(f)(y) = \int_{\mathbb{R}^3} \partial_y^\alpha \mathfrak{Z}(y,z) \cdot \widetilde{f}(z)\,dz$ for y, α as above.

Let us list some properties of $\mathfrak{R}(f)$.

Lemma 2.14 *Let $p \in (1,2)$, $R \in (0,\infty)$, $f \in L^p(\mathbb{R}^3)^3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Then $\|\partial^\alpha \mathfrak{R}(f)|B_R\|_p \leq C(\tau, \varrho, R, p) \cdot \|f\|_p$.*

Proof: [7, (3.6)] and Lemma 2.13. □

Corollary 2.7 *Let $p \in (1,2)$, $f \in L^p(\mathbb{R}^3)^3$. Take sequences (ϕ_n), (φ_n) as in Theorem 2.6. Then there is a subsequence (g_n) of $(\phi_n + \nabla\varphi_n)$ such that $\partial^\alpha \mathfrak{R}(g_n)(x) \to \partial^\alpha \mathfrak{R}(f)(x)$ $(n \to \infty)$ for a. e. $x \in \mathbb{R}^3$ and for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$.*

Proof: Theorem 2.6 and Lemma 2.14 imply that $\|\big(\partial^\alpha \mathfrak{R}(f) - \partial^\alpha \mathfrak{R}(\phi_n + \nabla\varphi_n) \big) \,|\, B_R\|_p \to 0$ when n tends to ∞, for any $R > 0$ and for α as above. □

Lemma 2.15 *Let $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, $\phi \in C_0^\infty(\mathbb{R}^3)^3$ with $div\,\phi = 0$, $\varphi \in C_0^\infty(\mathbb{R}^3)$, $y \in \mathbb{R}^3$. Then $\int_0^\infty \int_{\mathbb{R}^3} |\partial_y^\alpha \mathfrak{K}(y,z,t) \cdot (e^{-t\cdot\Omega} \cdot \phi)(z)|\,dz\,dt < \infty$ and*

$$\partial^\alpha \mathfrak{R}(\phi + \nabla\varphi)(y) = \int_0^\infty \int_{\mathbb{R}^3} \partial_y^\alpha \mathfrak{K}(y,z,t) \cdot (e^{-t\cdot\Omega} \cdot \phi)(z)\,dz\,dt.$$

Proof: There is $R > 0$ with $supp(\phi + \nabla\varphi) \subset B_R$. Therefore, by (2.5) and Theorem 2.4, we see that $|\partial_y^\alpha \Gamma(y,z,t)| \cdot |(\phi + \nabla\varphi)(z)|$ as a function of $z \in \mathbb{R}^3$, $t \in (0,\infty)$ is integrable. It follows by Lemma 2.13 and 2.11 that

$$\partial^\alpha \mathfrak{R}(\phi + \nabla\varphi)(y) = \int_0^\infty \int_{\mathbb{R}^3} \partial_y^\alpha \Gamma(y,z,t) \cdot (\phi + \nabla\varphi)(z)\,dz\,dt. \tag{2.6}$$

In view of the first claim in Lemma 2.7 and by Lemma 2.8, we obtain by an integration by parts that $\int_{\mathbb{R}^3} \partial_y^\alpha \Gamma(y,z,t) \cdot \nabla\varphi(z)\,dz = 0$ for $t \in (0,\infty)$. Moreover, we use the splitting of $\partial_y^\alpha \Gamma(y,z,t)$ in (2.3), thus arriving at the equation $\partial^\alpha \mathfrak{R}(\phi + \nabla\varphi)(y) = \mathfrak{A}_1 + \mathfrak{A}_2$, with $\mathfrak{A}_m := \int_0^\infty \int_{\mathbb{R}^3} H_m(y,z,t) \cdot (e^{-t\cdot\Omega} \cdot \phi)(z)\,dz\,dt$ for $m \in \{1, 2\}$, where $H_1(y,z,t) := \mathfrak{K}(y,z,t)$ and $H_2(y,z,t) := \big(\partial_y^\alpha \partial y_j \partial y_k \Psi(y,z,t)\big)_{1 \le j,k \le 3}$. Once more using the first claim in Lemma 2.7 in order to perform an integration by parts, taking into account the equation $\nabla_y \Psi(y,z,t) = -\nabla_z \Psi(y,z,t) \cdot e^{t\cdot\Omega}$, and recalling the assumption $\operatorname{div}\phi = 0$, we obtain $\mathfrak{A}_2 = 0$. Therefore Lemma 2.15 follows from (2.6). □

Lemma 2.16 *Let $a \in (1/2,\infty)$, $b \in [0,3/2]$, $R,S \in (0,\infty)$ with $R < S$, $y \in B_S^c$ and $l \in \{1, 2, 3\}$. Then*

$$\int_{B_R^c} (|\partial y_l \mathfrak{Z}(y,z)| + |\partial z_l \mathfrak{Z}(y,z)|) \cdot |z|^{-a} \cdot \nu(z)^{-b}\,dz \le C(\tau,\varrho,R,S,a,b) \cdot \Big(|y|^{-a} \cdot \nu(y)^{-b} + \int_{\mathbb{R}^3} \big[\,(1+|y-z|) \cdot \nu(y-z)\,\big]^{-3/2} \cdot (1+|z|)^{-a} \cdot \nu(z)^{-b}\,dz\Big).$$

Proof: Replicate the proof of [7, (3.13)], with S_1, A, B replaced by R, a, b, respectively, and with $\alpha = e_l$. □

Lemma 2.17 *Let $A \subset \mathbb{R}^3$ be measurable, with $\Omega \cdot A = A$. Let $g : A \mapsto \mathbb{R}$ be a measurable function, and let $y \in \mathbb{R}^3$, $l \in \{1, 2, 3\}$. Then*

$$\int_A (|\partial y_l \mathfrak{Z}(y,z)| + |\partial z_l \mathfrak{Z}(y,z)|) \cdot |g(z)|\,dz \le C(\tau,\varrho) \cdot \int_0^\zeta \int_A |g(e^{t\cdot\Omega} \cdot z)| \cdot W(y-z,t)\,dz\,dt$$

where $W(x,t)$, for $x \in \mathbb{R}^3$, $t \in (0,\zeta)$ is defined by

$$W(x,t) := (|x - \tau \cdot t \cdot e_1|^2 + t)^{-2} + \int_\zeta^\infty (|x - \tau \cdot s \cdot e_1|^2 + s)^{-2}\,ds,$$

and where the quantity ζ was introduced at the beginning of this section.

Proof: Let the left-hand side of the estimate in Lemma 2.17 be denoted by $\mathfrak{V}$. By (2.5),

$$\mathfrak{V} \le C(\tau,\varrho) \cdot \int_0^\infty \int_A (|y - \tau \cdot t \cdot e_1 - e^{-t\cdot\Omega} \cdot z|^2 + t)^{-2} \cdot |g(z)|\,dz\,dt.$$

On performing a change of variables, and taking account of the fact that the components of the matrix $e^{t\cdot\Omega}$ are continuous functions of t with period ζ (Lemma 2.3), and in particular are bounded, we may conclude that $\mathfrak{V} \le C(\tau,\varrho)\cdot(\mathfrak{V}_1+\mathfrak{V}_2)$, with

$$\mathfrak{V}_1 := \int_0^\zeta \int_A \sum_{k=1}^{\infty} (|y-x-\tau\cdot(k\cdot\zeta+t)\cdot e_1|^2 + k\cdot\zeta+t)^{-2}\cdot|g(e^{t\cdot\Omega}\cdot x)|\,dx\,dt, \tag{2.7}$$

and with $\mathfrak{V}_2$ defined in the same way as $\mathfrak{V}_1$, except that the sum with respect to k is replaced by the term $(|y-x-\tau\cdot t\cdot e_1|^2+t)^{-2}$. In other words, $\mathfrak{V}_2$ is related to the index $k=0$ in the sum appearing in (2.7). By Lemma 2.1 with $s=t, r=0$, we obtain

$$\mathfrak{V}_1 \le C(\tau,\varrho)\cdot\int_A G(x)\cdot\sum_{k=1}^{\infty}(|y-x-\tau\cdot k\cdot\zeta\cdot e_1|^2+k\cdot\zeta)^{-2}\,dx,$$

where we used the abbreviation $G(x) := \int_0^\zeta |g(e^{t\cdot\Omega}\cdot x)|\,dt$, for $x\in A$. Applying the identity $a=\zeta^{-1}\cdot\int_0^\zeta a\,ds$, valid for any $a\in\mathbb{R}$, and exploiting Lemma 2.1 again, this time with $s=0$, we arrive at the inequality

$$\mathfrak{V}_1 \le C(\tau,\varrho)\cdot\int_A G(x)\cdot\sum_{k=1}^{\infty}\int_0^\zeta(|y-x-\tau\cdot(k\cdot\zeta+r)\cdot e_1|^2+k\cdot\zeta+r)^{-2}\,dr\,dx.$$

But the sum in the preceding estimate equals $\int_\zeta^\infty(|y-x-\tau\cdot r\cdot e_1|^2+r)^{-2}\,dr$. Lemma 2.17 follows with the inequality $\mathfrak{V}\le C(\tau,\varrho)\cdot(\mathfrak{V}_1+\mathfrak{V}_2)$ derived above. □

In the rest of this section, we show that the potential $\mathfrak{R}(f)$ (Lemma 2.13) is the velocity part, and $Q(f)$ (Theorem 2.7) the pressure part, of a solution to (1.8) in the whole space $\mathbb{R}^3$, under the assumption that f is smooth. We start with a representation formula for the "velocity part" of C^∞-solutions in the whole space $\mathbb{R}^3$ to the system adjoint to the momentum equation of the linear system (1.8), with vanishing "pressure". Such a "velocity" need not be solenoidal.

Theorem 2.8 *Let $\widetilde{\Gamma}$ be defined in the same way as Γ, but with τ and ϱ replaced by $-\tau$ and $-\varrho$, respectively. Then $\int_0^\infty|\widetilde{\Gamma}(y,z,t)|\,dt \le C(\tau,\varrho,R)\cdot|y-z|^{-1}$ for $y,z\in B_R,\ y\ne z,\ R\in(0,\infty)$. Thus, for $\varphi\in C_0^\infty(\mathbb{R}^3)^3$, we may define $\widetilde{\mathfrak{R}}(\varphi)$ in the same manner as $\mathfrak{R}(\varphi)$ (Lemma 2.11), but with Γ replaced by $\widetilde{\Gamma}$.*

For $\varphi\in C_0^\infty(\mathbb{R}^3)^3$, the equation $\varphi=\widetilde{\mathfrak{R}}\big(\widetilde{L}(\varphi)\big)-\nabla\mathfrak{N}(div\,\varphi)$ holds, with $\mathfrak{N}(div\,\varphi)$ introduced in Theorem 2.5.

Proof: We refer to [6, Theorem 3.1] with $U=\tau\cdot e_1$ and with $-\omega$ instead of ω, and to [6, Theorem 4.3], with $\mathfrak{D}=\emptyset,\ \pi=0,\ u=\varphi,\ U=\tau\cdot e_1$, and again with $-\omega$ in the place of ω. Note that the term $\int_{\mathbb{R}^3}(4\cdot\pi\cdot|y-z|^3)^{-1}\cdot(y-z)_j\cdot\operatorname{div}\varphi(z)\,dz$ arising due to that latter reference equals $-\partial_j\mathfrak{N}(\operatorname{div}\varphi)(x)$, for $y\in\mathbb{R}^3,\ 1\le j\le 3$ (Theorem 2.5). □

Lemma 2.18 *Let $g\in C_0^\infty(\mathbb{R}^3)^3$. Then $\mathfrak{R}(g)\in C^\infty(\mathbb{R}^3)^3$.*

Proof: Obviously, for $t\in(0,\infty)$, the function $z\mapsto g(e^{t\cdot\Omega}\cdot z)$ is C^∞ on $\mathbb{R}^3$. Moreover, we see by Lemma 2.3 that for $\alpha\in\mathbb{N}_0^3$, there is $c_\alpha>0$ with $|\partial_z^\alpha g(e^{t\cdot\Omega}\cdot z)|\le c_\alpha$ for $z\in\mathbb{R}^3,\ t>0$.

Fix $R > 0$ with $supp(g) \subset B_R$, and let $S > 0$. By Lemma 2.3, we have $|e^{t \cdot \Omega} \cdot (y - z)| \geq R$ for $y \in B_S,\ z \in B^c_{R+S},\ t > 0$, so that $|\partial^\alpha_y g\big(e^{t \cdot \Omega} \cdot (y - z)\big)| \leq C(\alpha, c_\alpha) \cdot \chi_{B_{R+S}}(z)$ for y, z, t as before and for $\alpha \in \mathbb{N}^3_0$.

On the other hand, $|\Lambda(z, t)|$ is bounded by $C \cdot (|z|^2 + t)^{-3/2}$ for $z \in \mathbb{R}^3,\ t > 0$ ([6, Lemma 3.2]), so with Lemma 2.6, $|\Lambda(z - \tau \cdot t \cdot e_1, t)| \leq C(\tau, R, S) \cdot (|z|^2 + t)^{-3/2}$ for $z \in B_{R+S},\ t > 0$. Therefore the term $|\Lambda(z - \tau \cdot t \cdot e_1, t)| \cdot \chi_{B_{R+S}}(z)$ is integrable with respect to $(z, t) \in \mathbb{R}^3 \times (0, \infty)$.

Now it follows by Lebesgue's and Fubini's theorem that $\mathfrak{R}(g)|B_S \in C^\infty(B_S)^3$. This proves the lemma. □

Corollary 2.8 *Let $g \in C^\infty_0(\mathbb{R}^3)^3$. Then $L\big(\mathfrak{R}(g)\big) + \nabla Q(g) = g$, with $Q(g)$ defined in Theorem 2.7.*

Proof: Let $\varphi \in C^\infty_0(\mathbb{R}^3)^3$. Then

$$\int_{\mathbb{R}^3} \varphi \cdot \big[\, L\big(\mathfrak{R}(g)\big) + \nabla Q(g) \,\big] \, dz = \int_{\mathbb{R}^3} \big(\widetilde{L}(\varphi) \cdot \mathfrak{R}(g) - \operatorname{div} \varphi \cdot Q(g) \big) \, dz. \tag{2.8}$$

But $\Gamma(x, y, t) = e^{-t \cdot \Omega} \cdot \Lambda(y + \tau \cdot t \cdot e_1 - e^{t \cdot \Omega} \cdot x,\ t)$ for $x, y \in \mathbb{R}^3,\ t \in (0, \infty)$, and $\big(e^{-t \cdot \Omega}\big)^T = e^{t \cdot \Omega},\ \Lambda^T = \Lambda$. Thus, noting that $\widetilde{L}(\varphi) \in C^\infty_0(\mathbb{R}^3)^3$, we have $\int_{\mathbb{R}^3} \widetilde{L}(\varphi) \cdot \mathfrak{R}(g)\ dz = \int_{\mathbb{R}^3} \widetilde{\mathfrak{R}}\big(\widetilde{L}(\varphi)\big) \cdot g\ dz$. Moreover we have $\int_{\mathbb{R}^3} \operatorname{div} \varphi \cdot Q(g)\ dz = \int_{\mathbb{R}^3} \nabla \mathfrak{N}(\operatorname{div} \varphi) \cdot g\ dz$ by Theorem 2.5. Thus the left-hand side of (2.8) equals $\int_{\mathbb{R}^3} \big[\, \widetilde{\mathfrak{R}}\big(\widetilde{L}(\varphi)\big) - \nabla \mathfrak{N}(\operatorname{div} \varphi) \,\big] \cdot g\ dx$. Now Theorem 2.8 implies that the left-hand side of (2.8) coincides with $\int_{\mathbb{R}^3} \varphi \cdot g\ dx$. This proves the corollary. □

3. L^p-estimates of the volume potential $\mathfrak{R}(f)$.

The ensuing theorem, which constitutes the main difficulty of our argument, will be proved by reduction to a multiplier theorem by Lizorkin [52].

Theorem 3.1 *Let $p \in (1, 2),\ q \in (1, 4),\ r \in (1, \infty)$. Then there are constants $C_1(p) = C_1(p, \varrho, \tau),\ C_2(q) = C_2(q, \varrho, \tau),\ C_3(r) = C_3(r, \varrho, \tau) > 0$ such that*

$$\|\mathfrak{R}(\Phi + \nabla \varphi)\|_{(1/p - 1/2)^{-1}} \leq C_1(p) \cdot \|\Phi + \nabla \varphi\|_p,$$
$$\|\nabla \mathfrak{R}(\Phi + \nabla \varphi)\|_{(1/q - 1/4)^{-1}} \leq C_2(q) \cdot \|\Phi + \nabla \varphi\|_q,$$
$$\|\partial_1 \mathfrak{R}(\Phi + \nabla \varphi)\|_r + \|\nabla^2 \mathfrak{R}(\Phi + \nabla \varphi)\|_r \leq C_3(r) \cdot \|\Phi + \nabla \varphi\|_r,$$

for $\Phi \in C^\infty_0(\mathbb{R}^3)^3$ with $div\, \Phi = 0,\ \varphi \in C^\infty_0(\mathbb{R}^3)$.

Proof: We will establish the second inequality. The other ones may be shown by a similar reasoning. There is a minor additional difficulty related to the estimate of the second derivatives of $\mathfrak{R}(\Phi + \nabla \varphi)$. We will give an indication in this respect at the end of this proof.

For a rapidly decreasing function $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$, we define its Fourier transform $\widehat{\gamma}$ by $\widehat{\gamma}(\xi) := (2 \cdot \pi)^{-3/2} \cdot \int_{\mathbb{R}^3} e^{-i \cdot \xi \cdot x} \cdot \gamma(x)\ dx\ (\xi \in \mathbb{R}^3)$, and consequently its inverse Fourier transform $\check{\gamma}$ by $\check{\gamma}(\xi) := (2 \cdot \pi)^{-3/2} \cdot \int_{\mathbb{R}^3} e^{i \cdot \xi \cdot x} \cdot \gamma(x)\, dx$ $(\xi \in \mathbb{R}^3)$. It is well known that $\big[K(\,\cdot\,,t) \big]^{\wedge}(\xi) = (2 \cdot \pi)^{-3/2} \cdot e^{-|\xi|^2 \cdot t}$ for $t > 0,\ \xi \in \mathbb{R}^3$.

Let $l \in \{1, 2, 3\}$, $\Phi \in C_0^\infty(\mathbb{R}^3)^3$ with $\operatorname{div} \Phi = 0$, $\varphi \in C_0^\infty(\mathbb{R}^3)$, $y \in \mathbb{R}^3$. Setting $h(x,t) := e^{-t \cdot \Omega} \cdot \Phi(e^{t \cdot \Omega} \cdot x)$ for $x \in \mathbb{R}^3$, $t \in (0, \infty)$, we get with Lemma 2.15, 2.3 and a change of variables that

$$\partial_l \mathfrak{R}(\Phi + \nabla\varphi)(y) = \int_0^\infty \int_{\mathbb{R}^3} \partial y_l K(y - \tau \cdot t \cdot e_1 - x,\, t) \cdot h(x,t)\, dx\, dt.$$

By Plancherel's theorem and the above formula for $\big[K(\,\cdot\,,t) \big]^{\wedge}$, we may conclude that

$$\partial_l \mathfrak{R}(\Phi + \nabla\varphi)(y) = \int_0^\infty \int_{\mathbb{R}^3} e^{-i \cdot \xi \cdot y} \cdot \mathfrak{S}(\xi,t) \cdot \big[h(\,\cdot\,,t) \big]^{\wedge}(\xi)\, d\xi\, dt,$$

where $\mathfrak{S}(\xi,t) := \mathfrak{S}_l(\xi,t) := -(2 \cdot \pi)^{-3/2} \cdot i \cdot \xi_l \cdot e^{t \cdot (i \cdot \tau \cdot \xi_1 - |\xi|^2)}$ for $\xi \in \mathbb{R}^3$, $t > 0$. By Lemma 2.3, we have

$$\big[h(\,\cdot\,,t) \big]^{\wedge}(\xi) = \big[h(\,\cdot\,,t+\zeta) \big]^{\wedge}(\xi) \quad \text{for } \xi \in \mathbb{R}^3,\ t > 0 \tag{3.1}$$

so we arrive at the following equation:

$$\partial_l \mathfrak{R}(\Phi + \nabla\varphi)(y) = \sum_{k=0}^\infty \int_0^\zeta \int_{\mathbb{R}^3} e^{-i \cdot \xi \cdot y} \cdot \mathfrak{S}(\xi, t + k \cdot \zeta) \cdot \big[h(\,\cdot\,,t) \big]^{\wedge}(\xi)\, d\xi\, dt. \tag{3.2}$$

Observe that $|\mathfrak{S}(\xi,t)| \le |\xi| \cdot e^{-|\xi|^2 \cdot t}$ and $|\big[h(\,\cdot\,,t) \big]^{\wedge}(\xi)| \le |\widehat{\Phi}(e^{t \cdot \Omega} \cdot \xi)|$ $(\xi \in \mathbb{R}^3,\ t > 0)$, where the last inequality follows with Lemma 2.3. Therefore

$$\begin{aligned} &\sum_{k=0}^\infty \int_0^\zeta \int_{\mathbb{R}^3} \big| e^{-i \cdot \xi \cdot y} \cdot \mathfrak{S}(\xi, t + k \cdot \zeta) \cdot \big[h(\,\cdot\,,t) \big]^{\wedge}(\xi) \big|\, d\xi\, dt \\ &\le C \cdot \int_0^\infty \int_{\mathbb{R}^3} |\eta| \cdot e^{-|\eta|^2 \cdot t} \cdot |\widehat{\Phi}(\eta)|\, d\eta\, dt \le C \cdot \int_{\mathbb{R}^3} |\eta|^{-1} \cdot |\widehat{\Phi}(\eta)|\, d\eta, \end{aligned} \tag{3.3}$$

where we used Lemma 2.3 again. But since $\Phi \in C_0^\infty(\mathbb{R}^3)^3$, the function $\widehat{\Phi}$ is rapidly decreasing, so $\int_{\mathbb{R}^3} |\eta|^{-1} \cdot |\widehat{\Phi}(\eta)|\, d\eta < \infty$. Hence on the right-hand side of (3.2), we may reorder integration and summation, thus arriving at the equation

$$\partial_l \mathfrak{R}(\Phi + \nabla\varphi)(y) = \int_0^\zeta \int_{\mathbb{R}^3} e^{-i \cdot \xi \cdot y} \cdot \sum_{k=0}^\infty \mathfrak{S}(\xi, t + k \cdot \zeta) \cdot \big[h(\,\cdot\,,t) \big]^{\wedge}(\xi)\, d\xi\, dt. \tag{3.4}$$

The idea now is to apply the multiplier theorem [52, Theorem 8] to the inner integral on the right-hand side of (3.4). To this end, we have to derive a suitable estimate of $\sum_{k=0}^\infty \mathfrak{S}(\xi, t + k \cdot \zeta)$. So let $t \in (0, \zeta]$, and define

$$\begin{aligned} h_k^{(\kappa)}(\xi) &:= i \cdot \xi_l \cdot (-\tau \cdot i + 2 \cdot \xi_1)^{\kappa_1} \cdot (2 \cdot \xi_2)^{\kappa_2} \cdot (2 \cdot \xi_3)^{\kappa_3} \cdot (-t - k \cdot \zeta)^{|\kappa|}, \\ \widetilde{h}_k^{(\kappa)}(\xi) &:= \delta_{1,\kappa_l} \cdot i \cdot (-\tau \cdot i + 2 \cdot \xi_1)^{\kappa_1 - \delta_{1l}} \cdot (2 \cdot \xi_2)^{\kappa_2 - \delta_{2l}} \cdot (2 \cdot \xi_3)^{\kappa_3 - \delta_{3l}} \cdot (-t - k \cdot \zeta)^{|\kappa| - 1} \end{aligned}$$

for $\xi \in \mathbb{R}^3,\ k \in \mathbb{N}_0$ and $\kappa \in \{0,1\}^3$. Note that for such κ, we have $\kappa_j = 0$ or $\kappa_j = 1$ for $j \in \{1,\, 2,\, 3\}$, as well as $|\kappa| \le 3$. Since

$$|\big(\, h_k^{(\kappa)}(\xi) + \widetilde{h}_k^{(\kappa)}(\xi)\,\big) \cdot e^{(t+k\cdot\zeta)\cdot(\tau\cdot i\cdot\xi_1 - |\xi|^2)}| \le C(\tau,\varrho)\cdot R^4 \cdot (1+k)^3 \cdot e^{-k\cdot\zeta\cdot\delta^2}$$

for $\kappa \in \{0,1\}^3,\ k \in \mathbb{N}_0,\ \delta \in (0,1),\ R \in (1,\infty),\ \xi \in B_R \backslash B_\delta$, and because $\sum_{k=0}^{\infty}(1+k)^3 \cdot e^{-k\cdot\zeta\cdot\delta^2} < \infty$ for $\delta > 0$, we may conclude that the derivative $\partial_\xi^\kappa\big(\sum_{k=0}^{\infty} \mathfrak{S}(\xi, t+k\cdot\zeta)\big)$ exists, is continuous, and

$$\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty} \mathfrak{S}(\xi, t+k\cdot\zeta)\Big) = -(2\cdot\pi)^{-3/2}\cdot\sum_{k=0}^{\infty}\big(\, h_k^{(\kappa)}(\xi) + \widetilde{h}_k^{(\kappa)}(\xi)\,\big)\cdot e^{(t+k\cdot\zeta)\cdot(\tau\cdot i\cdot\xi_1 - |\xi|^2)} \tag{3.5}$$

for $\kappa \in \{0,1\}^3$ and $\xi \in \mathbb{R}^3\backslash\{0\}$. Moreover, we may choose $\epsilon_0 = \epsilon_0(\tau,\varrho) \in (0,1]$ such that for $\xi \in B_{\epsilon_0}$

$$\begin{aligned} &|\sin(\tau\cdot\zeta\cdot\xi_1)| \ge \tau\cdot\zeta\cdot|\xi_1|/2, \quad 1 - e^{-\zeta\cdot|\xi|^2} \ge \zeta\cdot|\xi|^2/2, \quad e^{-\zeta\cdot|\xi|^2} \ge 1/2, \\ &1 - \cos(\tau\cdot\zeta\cdot\xi_1) \ge \tau^2\cdot\zeta^2\cdot\xi_1^2/4. \end{aligned} \tag{3.6}$$

Let $\kappa \in \{0,1\}^3$ and $\xi \in \mathbb{R}^3\backslash\{0\}$. In view of estimating the right-hand side of (3.5), we distinguish the cases $|\xi| \le \epsilon_0$ and $|\xi| > \epsilon_0$. First suppose that $|\xi| \le \epsilon_0$. Then, observing that $|-\tau\cdot i + 2\cdot\xi_1| \le \tau + \epsilon_0$ and $|e^{t\cdot(\tau\cdot i\cdot\xi_1 - |\xi|^2)} \le 1$, we get from (3.5)

$$\begin{aligned} \Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty} \mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big| \le C(\tau,\varrho)\cdot\Big(&|\xi|^{1+\kappa_2+\kappa_3}\cdot\Big|\sum_{k=0}^{\infty}(-t-k\cdot\zeta)^{|\kappa|}\cdot e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)\cdot k}\Big| \\ &+\delta_{1,\kappa_l}\cdot|\xi|^{\kappa_2+\kappa_3-\delta_{2l}-\delta_{3l}}\cdot\Big|\sum_{k=0}^{\infty}(-t-k\cdot\zeta)^{|\kappa|-1}\cdot e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)\cdot k}\Big|\Big). \end{aligned} \tag{3.7}$$

Next we note that for $k \in \mathbb{N}_0$, we have

$$(-t-k\cdot\zeta)^{|\kappa|} = \sum_{m=0}^{|\kappa|} a_{m,|\kappa|}\cdot\prod_{j=0}^{m-1}(k-j), \tag{3.8}$$

with coefficients $a_{0,|\kappa|}$, ..., $a_{|\kappa|,|\kappa|}$ that are polynomials in t and ζ and do not depend on k, and which may thus be estimated by some constant $C(\tau,\varrho) > 0$. In fact, if $|\kappa| = 0$, we take $a_{0,|\kappa|} = 1$. In the case $|\kappa| = 1$, we choose $a_{0,|\kappa|} = -t$ and $a_{1,|\kappa|} = -\zeta$, and if $|\kappa| = 2$, we may set $a_{0,|\kappa|} = t^2,\ a_{1,|\kappa|} = 2\cdot t\cdot\zeta + \zeta^2,\ a_{2,|\kappa|} = \zeta^2$. We leave it to the reader to determine $a_{0,|\kappa|}$, ..., $a_{3,|\kappa|}$ if $|\kappa| = 3$. In the case $\kappa_l = 1$, an equation analogous to (3.8) holds for $(-t-k\cdot\zeta)^{|\kappa|-1}$. Thus we get from (3.7) that

$$\begin{aligned} \Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty} \mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big| \le C(\tau,\varrho)\cdot\Big(&|\xi|^{1+\kappa_2+\kappa_3}\cdot\sum_{m=0}^{|\kappa|}\Big|\sum_{k=0}^{\infty}\Big(\prod_{j=0}^{m-1}(k-j)\Big)\cdot e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)\cdot k}\Big| \\ &+\delta_{1,\kappa_l}\cdot|\xi|^{\kappa_2+\kappa_3-\delta_{2l}-\delta_{3l}}\cdot\sum_{m=0}^{|\kappa|-1}\Big|\sum_{k=0}^{\infty}\Big(\prod_{j=0}^{m-1}(k-j)\Big)\cdot e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)\cdot k}\Big|\Big). \end{aligned}$$

Note that for $k, m \in \mathbb{N}_0$ with $k < m$, the product $\prod_{j=0}^{m-1}(k-j)$ vanishes, so we may start summation with respect to k only with $k = m$ instead of $k = 0$. On applying the formula $\sum_{k=m}^{\infty}\Big(\prod_{j=0}^{m-1}(k-j)\Big) \cdot a^{k-m} = m! \cdot (1-a)^{-m-1}$, which is valid for $a \in \mathbb{C}$ with $|a| < 1$ and for $m \in \mathbb{N}_0$, we may thus deduce from the preceding inequality that

$$\Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty}\mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big| \le C(\tau,\varrho)\cdot\Big(|\xi|^{1+\kappa_2+\kappa_3}\cdot\sum_{m=0}^{|\kappa|}\Big|m!\cdot(1-e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)})^{-m-1}\Big| \tag{3.9}$$
$$+\delta_{1,\kappa_l}\cdot|\xi|^{\kappa_2+\kappa_3-\delta_{2l}-\delta_{3l}}\cdot\sum_{m=0}^{|\kappa|-1}\Big|m!\cdot(1-e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)})^{-m-1}\Big|\Big).$$

On the other hand, recalling the choice of ϵ_0 in (3.6), and observing that $(a^2+b^2)^{1/2} \ge (a+b)\cdot 2^{-1/2}$ for $a, b \in [0,\infty)$, we find

$$|1-e^{\zeta\cdot(\tau\cdot i\cdot\xi_1-|\xi|^2)}| \ge \big[\,|\sin(\zeta\cdot\tau\cdot\xi_1)|\cdot e^{-\zeta\cdot|\xi|^2} + \big(1-\cos(\zeta\cdot\tau\cdot\xi_1)\cdot e^{-\zeta\cdot|\xi|^2}\big)\big]\cdot 2^{-1/2}$$
$$\ge C(\tau,\varrho)\cdot(|\xi_1|+|\xi|^2+\xi_1^2) \ge C(\tau,\varrho)\cdot(|\xi_1|+|\xi|^2).$$

Moreover, because of the assumption $|\xi| \le \epsilon_0$, we find for $m \in \{0, ..., |\kappa|-1\}$ that the inequality $(|\xi_1|+|\xi|^2)^{-m-1} \le C(\tau,\varrho)\cdot(|\xi_1|+|\xi|^2)^{-a}$ holds with $a = |\kappa|+1$ and $a = |\kappa|$. Thus, from (3.9)

$$\Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty}\mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big| \le C(\tau,\varrho)\cdot\big[\,(|\xi_1|+|\xi|^2)^{-|\kappa|-1/2+(\kappa_2+\kappa_3)/2}$$
$$+\delta_{1,\kappa_l}\cdot(|\xi_1|+|\xi|^2)^{-|\kappa|+(\kappa_2+\kappa_3)/2-(\delta_{2l}+\delta_{3l})/2}\,\big] \le C(\tau,\varrho)\cdot(|\xi_1|+|\xi|^2)^{-|\kappa|-1/2+(\kappa_2+\kappa_3)/2},$$

where in the case $l = 1$, $\kappa_l = 1$, the last inequality follows with the assumption $|\xi| \le \epsilon_0$. Therefore

$$|\xi_1|^{\kappa_1+1/4}\cdot\Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty}\mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big| \le C(\tau,\varrho)\cdot(|\xi_1|+|\xi|^2)^{-|\kappa|-1/4+\kappa_1+(\kappa_2+\kappa_3)/2}$$
$$\le C(\tau,\varrho)\cdot|\xi|^{-2\cdot|\kappa|-1/2+2\cdot\kappa_1+\kappa_2+\kappa_3} = C(\tau,\varrho)\cdot|\xi|^{-\kappa_2-\kappa_3-1/2}.$$

Hence

$$\prod_{j=1}^{3}|\xi_j|^{\kappa_j+1/4}\cdot\Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty}\mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big| \le C(\tau,\varrho). \tag{3.10}$$

Now suppose that $|\xi| > \epsilon_0$. Then we have $|-\tau\cdot i+2\cdot\xi_1| \le C(\tau,\varrho)\cdot|\xi|$, so we may deduce from (3.5) that

$$\Big|\partial_\xi^\kappa\Big(\sum_{k=0}^{\infty}\mathfrak{S}(\xi, t+k\cdot\zeta)\Big)\Big|$$
$$\le C(\tau,\varrho)\cdot\sum_{k=0}^{\infty}\Big(|\xi|^{1+|\kappa|}\cdot(t+k\cdot\zeta)^{|\kappa|}+\delta_{1,\kappa_l}\cdot|\xi|^{|\kappa|-1}\cdot(t+k\cdot\zeta)^{|\kappa|-1}\Big)\cdot e^{-(t+k\cdot\zeta)\cdot|\xi|^2}.$$

Thus, when we multiply the left-hand side of the preceding inequality by $\prod_{j=1}^{3} |\xi_j|^{\kappa_j+1/4}$, we get an upper bound of the form of a series with respect to $k \in \mathbb{N}_0$ of the products

$$\big(|\xi|^2 \cdot (t + k \cdot \zeta) \big)^{|\kappa|-1/8} \cdot \big(|\xi|^2 \cdot (t + k \cdot \zeta) + \delta_{1,\kappa_l} \big) \cdot (t + k \cdot \zeta)^{-7/8}.$$

But for $m \in \{0, 1\}$, the term $\big((t + k \cdot \zeta) \cdot |\xi|^2 \big)^{|\kappa|-1/8+m} \cdot e^{-(t+k\cdot\zeta)\cdot|\xi|^2/2}$ is bounded by a constant only depending on τ and ϱ. Thus we obtain

$$\prod_{j=1}^{3} |\xi_j|^{\kappa_j+1/4} \cdot \Big| \partial_\xi^\kappa \Big(\sum_{k=0}^{\infty} \mathfrak{S}(\xi, t + k \cdot \zeta) \Big) \Big| \le C(\tau, \varrho) \cdot \sum_{k=0}^{\infty} (t + k \cdot \zeta)^{-7/8} \cdot e^{-(t+k\cdot\zeta)\cdot|\xi|^2/2} \tag{3.11}$$

$$\le C(\tau, \varrho) \cdot t^{-7/8} \cdot \sum_{k=0}^{\infty} e^{-k\cdot\zeta\cdot\epsilon_0^2/2} \le C(\tau, \varrho) \cdot t^{-7/8},$$

where in the second inequality, we again used the assumption $|\xi| \ge \epsilon_0$. In view of (3.10), we may thus conclude that the preceding estimate holds for any $\xi \in \mathbb{R}^3 \backslash \{0\}$. Recall that t was chosen arbitrarily in $(0, \zeta]$. Now, for such t, define $\mathfrak{H}(t) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by

$$\mathfrak{H}(t)(y) := \int_{\mathbb{R}^3} e^{-i\cdot\xi\cdot y} \cdot \sum_{k=0}^{\infty} \mathfrak{S}(\xi, t + k \cdot \zeta) \cdot \big[h(\,\cdot\,, t) \big]^\wedge(\xi)\, d\xi$$

for $y \in \mathbb{R}^3$. Then inequality (3.11) and [52, Theorem 8] imply that $\|\mathfrak{H}(t)\|_{(1/q-1/4)^{-1}} \le C(\tau, \varrho) \cdot t^{-7/8} \cdot \|h(\,\cdot\,, t)\|_q$ for $t \in (0, \zeta]$. Therefore with (3.4) and Minkowski's inequality

$$\|\partial_l \mathfrak{R}(\Phi + \nabla\varphi)\|_{(1/q-1/4)^{-1}} \le C(\tau, \varrho) \cdot \int_0^\zeta t^{-7/8} \cdot \|h(\,\cdot\,, t)\|_q\, dt \tag{3.12}$$

for t as before. But $\|h(\,\cdot\,, t)\|_q = \|\Phi\|_q$ by Lemma 2.3 and a change of variables. Since in addition $\|\Phi\|_q \le C(q) \cdot \|\Phi + \nabla\varphi\|_q$ by [24, Theorem III.1.2], the second inequality in Theorem 3.1 follows from (3.12).

Concerning an estimate of the second derivatives of $\mathfrak{R}(\Phi + \nabla\varphi)$, there is an additional problem which consists in establishing an analogue to (3.2) for these derivatives although no analogue of the formula at the end of Lemma 2.13 is available for them. In this respect, we indicate that

$$|\partial y_m(e^{i\cdot\xi\cdot y}) \cdot \mathfrak{S}(\xi, t + k \cdot \zeta) \cdot \big[h(\,\cdot\,, t) \big]^\wedge(\xi)| \le C \cdot |\xi|^2 \cdot e^{-|\xi|^2\cdot(t+k\cdot\zeta)} \cdot |\widehat{\Phi}(e^{t\cdot\Omega} \cdot \xi)|$$

for $y, \xi \in \mathbb{R}^3$, $t \in (0, \zeta]$, $1 \le l, m \le 3$, where the function $\mathfrak{S} = \mathfrak{S}_l$ is defined as above. But the term $\sum_{k=0}^{\infty} \int_0^\zeta \int_{\mathbb{R}^3} |\xi|^2 \cdot e^{-|\xi|^2\cdot(t+k\cdot\zeta)} \cdot |\widehat{\Phi}(e^{t\cdot\Omega} \cdot \xi)|\, d\xi\, dt$ is finite, as follows by the same reasoning as used in (3.3). In view of (3.2), this means that $\partial_m \partial_l \mathfrak{R}(\Phi + \nabla\varphi)(y)$ equals the right-hand side of (3.2), but with the term $e^{i\cdot\xi\cdot y}$ replaced by $\partial y_m(e^{i\cdot\xi\cdot y})$. Now we may estimate $\|\partial_m \partial_l \mathfrak{R}(\Phi + \nabla\varphi)\|_r$ by the same techniques as used above in order to find an upper bound for $\|\partial_l \mathfrak{R}(\Phi + \nabla\varphi)\|_{(1/q-1/4)^{-1}}$. □

Corollary 3.1 *Let $p \in (1, 2)$ and $f \in L^p(\mathbb{R}^3)^3$. Then $\mathfrak{R}(f) \in W^{2,p}_{loc}(\mathbb{R}^3)^3$ and the inequality $\|\mathfrak{R}(f)\|_{(1/p-1/2)^{-1}} \le C_1(p) \cdot \|f\|_p$ holds.*

If $q \in (1, 4)$ and $f \in L^p(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$, we have $\|\nabla\mathfrak{R}(f)\|_{(1/q-1/4)^{-1}} \le C_2(q) \cdot \|f\|_q$.

If $r \in (1, \infty)$ and $f \in L^p(\mathbb{R}^3)^3 \cap L^r(\mathbb{R}^3)^3$, we further have $\|\partial_1\mathfrak{R}(f)\|_r + \|\nabla^2\mathfrak{R}(f)\|_r \le C_3(r) \cdot \|f\|_r$.

Proof: By Theorem 2.6, we may choose sequences (Φ_n), (φ_n) with $\Phi_n \in C_0^\infty(\mathbb{R}^3)^3$, $\varphi_n \in C_0^\infty(\mathbb{R}^3)$, $\operatorname{div}\Phi_n = 0$ for $n \in \mathbb{N}$, and $\|f - \Phi_n - \nabla\varphi_n\|_p \to 0$ $(n \to \infty)$. Therefore, by Corollary 2.7, there is a subsequence (g_n) of $(\Phi_n + \nabla\varphi_n)$ such that $\mathfrak{R}(g_n)(x) \to \mathfrak{R}(f)(x)$ $(n \to \infty)$ for a. e. $x \in \mathbb{R}^3$. At this point we may conclude from Theorem 3.1 that $\mathfrak{R}(f) \in W^{2,p}_{loc}(\mathbb{R}^3)^3$ and $\|\partial^\alpha\mathfrak{R}(f) - \partial^\alpha\mathfrak{R}(\Phi_n + \nabla\varphi_n)\|_s \to 0$ $(n \to \infty)$, $\|\partial^\alpha\mathfrak{R}(f)\|_s \le c(\alpha, p) \cdot \|f\|_p$, for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$, where $s = (1/p - 1/2)^{-1}$, $c(\alpha, p) = C_1(p)$ if $\alpha = 0$, $s = (1/p - 1/4)^{-1}$, $c(\alpha, p) = C_2(p)$ if $|\alpha| = 1$, and $s = p$, $c(\alpha, p) = C_3(p)$ in the case $|\alpha| = 2$.

Now take $q \in (1, 4)$, and suppose in addition that $f \in L^q(\mathbb{R}^3)^3$. Again referring to Theorem 2.6, we choose sequences (Φ_n), (φ_n), $(\widetilde{\varphi}_n)$, with (Φ_n), (φ_n) having properties as above, and such that $\widetilde{\varphi}_n \in C_0^\infty(\mathbb{R}^3)$ for $n \in \mathbb{N}$ and $\|f - \Phi_n - \nabla\widetilde{\varphi}_n\|_q \to 0$ $(n \to \infty)$. Let $l \in \{1, 2, 3\}$. As shown above, $\|\partial_l\mathfrak{R}(f) - \partial_l\mathfrak{R}(\Phi_n + \nabla\varphi_n)\|_{(1/p-1/4)^{-1}} \to 0$ $(n \to \infty)$. In addition we deduce from Theorem 3.1 there is a function $G \in L^{(1/q-1/4)^{-1}}(\mathbb{R}^3)^3$ such that $\|G - \partial_l\mathfrak{R}(\Phi_n + \nabla\widetilde{\varphi}_n)\|_{(1/q-1/4)^{-1}} \to 0$ $(n \to \infty)$. In particular, there exists a strictly increasing function $\sigma : \mathbb{N} \mapsto \mathbb{N}$ such that $\partial_l\mathfrak{R}(\Phi_{\sigma(n)} + \nabla\varphi_{\sigma(n)})(x) \to \partial_l\mathfrak{R}(f)(x)$ and $\partial_l\mathfrak{R}(\Phi_{\sigma(n)} + \nabla\widetilde{\varphi}_{\sigma(n)})(x) \to G(x)$ $(n \to \infty)$ for a. e. $x \in \mathbb{R}^3$. But we know from Lemma 2.15 that $\mathfrak{R}(\Phi_n + \nabla\varphi_n) = \mathfrak{R}(\Phi_n + \nabla\widetilde{\varphi}_n)$ for $n \in \mathbb{N}$. Thus we obtain $\partial_l\mathfrak{R}(f) = G$, and we may now conclude with Theorem 3.1 and the relation $\|f - \Phi_n - \nabla\widetilde{\varphi}_n\|_q \to 0$ that $\|\partial_l\mathfrak{R}(f)\|_{(1/q-1/4)^{-1}} \le C_2(q) \cdot \|f\|_q$. The last statement of the corollary may be shown in an analogous way. □

The preceding results immediately imply Theorem 1.2. In fact, we may give the following

Proof of Theorem 1.2: Combine Corollary 3.1, Theorem 2.7 and Corollary 2.8. □

4. L^p-estimates of the velocity and its gradient.

We begin by recalling that the quantities τ and ϱ were fixed at the beginning of Section 2.

Take a set $\mathfrak{D}$, parameters γ, S_1, p_0, A, B and functions f, u and π as in Theorem 1.1. The set $\mathfrak{D}$ and the preceding parameters and functions (in particular u and π) are to be fixed for the rest of this article. Since $\overline{\mathfrak{D}} \subset B_{S_1}$, we may choose $S_0 \in (0, S_1)$ with $\overline{\mathfrak{D}} \subset B_{S_0}$. In the following, we will use the letter $\mathfrak{C}$ for constants that may depend on τ, ϱ, γ, S_0, S_1, p_0, A, B, $\|f|B_{S_1}\|_1$, u and π. If such a constant additionally depends on γ_1, ..., $\gamma_n \in (0, \infty)$ for some $n \in \mathbb{N}$, we will denote it by $\mathfrak{C}(\gamma_1, ..., \gamma_n)$. We recall that by a previous convention (Section 2), we write C for numerical constants, and $C(\gamma_1, ..., \gamma_n)$ for constants only depending on γ_1, ..., γ_n.

The assumptions on f in Theorem 1.1 mean in particular that for any $p > 1$, the function f is L^p-integrable outside B_{S_1}:

Lemma 4.1 $f|B^c_{S_1} \in L^p(S^c_{S_1})^3$ *for* $p \in (1, \infty]$.

Put $B^* := \min\{1, B\}$, $\epsilon := \min\{(A-2)/2,\ (A + B^* - 3)/2\}$. By our choice of A and B in Theorem 1.1, we have $\epsilon > 0$. For $y \in B^c_{S_1}$, we find as in the proof of [4, Lemma 2.12] that

$$|y|^{-A} \cdot \nu(y)^{-B} \le C(A) \cdot |y|^{-2-\epsilon} \cdot \nu(y)^{-(A+B^*-3)/2-1}.$$

Let $p \in (1,\infty)$. Since $A+B^*-3>0$ by our assumptions on A and B, the relation $\big[-(A+B^*-3)/2-1\big]\cdot p<-1$ holds Hence $\int_{B^c_{S_1}}\big(\,|y|^{-A}\cdot\nu(y)^{-B}\,\big)^p\,dy \le C(A,B,S_1,p)$ by Corollary 2.1. Therefore Lemma 4.1 follows with (1.6). □

The preceding lemma means in particular that the potential $\mathfrak{R}(f)$ is well defined and belongs to $W^{1,1}_{loc}(\mathbb{R}^3)^3$ (Lemma 2.13). Since $(u\cdot\nabla)u \in L^{3/2}(\overline{\mathfrak{D}}^c)^3$ by Lemma 2.2, the same is true of $\mathfrak{R}\big((u\cdot\nabla)u\big)$.

The starting point of our estimates is the representation formula from [10] already mentioned in Section 1. This formula refers to the velocity part of Leray solutions to the nonlinear system (1.1) in exterior domains, and may thus be applied to u.

Theorem 4.1 *Put $\widetilde{p} := \min\{3/2,\, p_0\}$, with p_0 from the assumptions on f. Then $u \in W^{2,\widetilde{p}}_{loc}(\overline{\mathfrak{D}}^c)^3$, $\pi \in W^{1,\widetilde{p}}_{loc}(\overline{\mathfrak{D}}^c)$. Define*

$$\begin{aligned}&\mathfrak{B}_j(u,\pi)(y)\\ &:= \int_{\partial B_{S_0}} \sum_{k=1}^{3}\Big[\sum_{l=1}^{3}\Big(\mathfrak{Z}_{jk}(y,z)\cdot\big(-\partial_l u_k(z)+\delta_{kl}\cdot\pi(z)+u_k(z)\cdot(\tau\cdot e_1-\omega\times z)_l\,\big)\\ &\qquad\qquad +\partial z_l\mathfrak{Z}_{jk}(y,z)\cdot u_k(z)\Big)\cdot(z_l/S_0)\;+\;E_j(y-z)\cdot u_k(z)\cdot(z_k/S_0)\Big]\,do_z\end{aligned}$$

for $y \in \overline{B_{S_0}}^{\,c}$, $1\le j\le 3$. Then $u(y) = \mathfrak{R}\big(\,f-\tau\cdot(u\cdot\nabla)u\,|\,B^c_{S_0}\big)(y)+\mathfrak{B}(u,\pi)(y)$, for y as before.

Proof: The first statement of Theorem 4.1 follows from the interior regularity theory for the Stokes system. For more details, we refer to the proof of [7, Theorem 5.5]. The assumptions on f in that reference hold according to Lemma 4.1. ¿From the first claim in Theorem 4.1, we may conclude that $u|\partial B_{S_0} \in W^{2-1/\widetilde{p},\widetilde{p}}(\partial B_{S_0})^3$. Due to this observation and Lemma 2.12, the function $\mathfrak{B}(u,\pi)$ is well defined. The main claim of Theorem 4.1, that is, the representation formula in the last sentence of this theorem, may now be deduced from [10, Theorem 5] with $\mathfrak{D}$ replaced by B_{S_0}, and from Lemma 4.1. □

As a consequence, we have

Theorem 4.2 *Let $S \in (S_1,\infty)$. Then*

$$|\partial^\alpha u(x)-\partial^\alpha\mathfrak{R}\big(\,\tau\cdot(u\cdot\nabla)u\,|\,B^c_{S_0}\big)(x)| \le C(S_0,S_1,S,A,B)\cdot\mathfrak{M}\cdot\big(\,|x|\cdot\nu(x)\,\big)^{-1-|\alpha|/2} \quad (4.1)$$

for $x\in B^c_S$, $\alpha\in\mathbb{N}^3_0$ with $|\alpha|\le 1$, where we used the abbreviation

$$\mathfrak{M} := \gamma+\|f|B_{S_1}\|_1+\|u|\partial B_{S_0}\|_1+\|\nabla u|\partial B_{S_0}\|_1+\|\pi|\partial B_{S_0}\|_1.$$

In particular,

$$u-\mathfrak{R}\big(\,\tau\cdot(u\cdot\nabla)u\,|\,B^c_{S_0}\big)\,|\,B^c_S \in L^q(B^c_S)^3 \quad \textit{for } q\in(2,\infty), \quad (4.2)$$

$$\nabla u-\nabla\mathfrak{R}\big(\,\tau\cdot(u\cdot\nabla)u\,|\,B^c_{S_0}\big)\,|\,B^c_S \in L^q(B^c_S)^9 \quad \textit{for } q\in(4/3,\,\infty). \quad (4.3)$$

Proof: Consider the function $\mathfrak{B}(u,\,\pi)$ from Theorem 4.1. By [9, Corollary 1] with $\mathfrak{D}$ replaced by B_{S_0}, we see that $|\partial^\alpha\mathfrak{B}(u,\pi)(x)|$ bounded by

$$C(S_0,S_1,S)\cdot(\|u|\partial B_{S_0}\|_1+\|\nabla u|\partial B_{S_0}\|_1+\|\pi|\partial B_{S_0}\|_1)\cdot\big(\,|x|\cdot\nu(x)\,\big)^{-1-|\alpha|/2}$$

for x, α as in (4.1). On the other hand, according to [7, Theorem 3.3], we have

$$|\partial^\alpha \mathfrak{R}(f|B^c_{S_0})(x)| \le C(S_1, S, A, B) \cdot (\gamma + \|f|B_{S_1}\|_1) \cdot \big(\, |x| \cdot \nu(x)\,\big)^{-1-|\alpha|/2},$$

again for x, α as in (4.1). Now (4.1) follows from Theorem 4.1. The remaining statements of Theorem 4.2 follow from (4.1) and Corollary 2.1. □

Lemma 4.2 $\nabla u|B^c_S \in L^{12/5}(B^c_S)^9$ *for* $S \in (S_1, \infty)$.

Proof: Since $(u \cdot \nabla)u \in L^{3/2}(\overline{\mathfrak{D}}^c)^3$ (Lemma 2.2), Corollary 3.1 with $q = 3/2$ yields $\nabla \mathfrak{R}\big(\,(u\cdot\nabla)u \,|\, B^c_{S_0}\,\big) \in L^{12/5}(\mathbb{R}^3)^3$. The lemma follows from this observation and (4.3). □

Lemma 4.2 and Theorem 2.1 imply

Corollary 4.1 $u|B^c_S \in L^{12}(B^c_S)^3$ *for* $S \in (S_1, \infty)$.

Theorem 4.3 $u \in L^4(\overline{\mathfrak{D}}^c)^3$.

Proof: Due to the assumption $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$, we may choose $r \in (S_1, \infty)$ such that

$$\|\nabla u|B^c_r\|_2 \le \min\{\, \big(\,2\cdot\tau\cdot C_0(4,2)\cdot C_1(4/3)\,\big)^{-1},\ \big(\,2\cdot\tau\cdot C_0(6,2)\cdot C_1(3/2)\,\big)^{-1}\,\}, \quad (4.4)$$

with $C_0(4,2)$, $C_0(6,2)$ from Lemma 2.2, and $C_1(4/3)$, $C_1(3/2)$ from Theorem 3.1. For $\phi \in L^4(B^c_r)^3$, we define $\mathfrak{A}(\phi)(x) := u(x)$ for $x \in B_r\backslash B_{S_0}$, $\mathfrak{A}(\phi)(x) := \phi(x)$ for $x \in B^c_r$. Since $u \in L^6(\overline{\mathfrak{D}}^c)^3$, we have $\mathfrak{A}(\phi) \in L^4(B^c_{S_0})^3$. Therefore Lemma 2.2 with $p = 4$, $q = 2$ yields $\big(\mathfrak{A}(\phi)\cdot\nabla\big)u \in L^{4/3}(B^c_{S_0})^3$ if $\phi \in L^4(B^c_r)^3$, so by Corollary 3.1 with $p = 4/3$ and (4.2), we may define a mapping $\mathfrak{T} : L^4(B^c_r)^3 \mapsto L^4(B^c_r)^3$ by setting

$$\mathfrak{T}(\phi) := \Big(u - \tau\cdot\mathfrak{R}\big(\,(u\cdot\nabla)u \,|\, B^c_{S_0}\,\big) + \tau\cdot\mathfrak{R}\big(\,\big[\,\mathfrak{A}(\phi)\cdot\nabla\,\big]u\,\big)\Big) \,|\, B^c_r \quad \text{for } \phi \in L^4(B^c_r)^3.$$

By the same references, we find for w_1, $w_2 \in L^4(B^c_r)^3$ that

$$\begin{aligned}
&\|\mathfrak{T}(w_1) - \mathfrak{T}(w_2)\|_4 \le \tau\cdot\|\mathfrak{R}\big(\,\big[\,(w_1-w_2)\cdot\nabla\,\big]u \,|\, B^c_r\,\big)\|_4 \\
&= \tau\cdot C_1(4/3)\cdot\|\big[\,(w_1-w_2)\cdot\nabla\,\big]u \,|\, B^c_r\|_{4/3} \\
&\le \tau\cdot C_1(4/3)\cdot C_0(4,2)\cdot\|w_1-w_2\|_4\cdot\|\nabla u|B^c_r\|_2,
\end{aligned}$$

so that by (4.4) $\|\mathfrak{T}(w_1) - \mathfrak{T}(w_2)\|_4 \le \|w_1 - w_2\|_4/2$. Thus there is $v \in L^4(B^c_r)^3$ with $\mathfrak{T}(v) = v$, that is,

$$v = \Big(u - \tau\cdot\mathfrak{R}\big(\,(u\cdot\nabla)u \,|\, B^c_{S_0}\,\big) + \tau\cdot\mathfrak{R}\big(\,\big[\,\mathfrak{A}(v)\cdot\nabla\,\big]u\,\big)\Big) \,|\, B^c_r. \quad (4.5)$$

Lemma 2.2 with $p = 4$, $q = 12/5$ and $p = 6$, $q = 2$, respectively, yields

$$\|\big(\,\mathfrak{A}(v)\cdot\nabla\,\big)u\|_{3/2} \le C_0(4,\, 12/5)\cdot\|v\|_4\cdot\|\nabla u|B^c_r\|_{12/5} + C_0(6,2)\cdot\|u\|_6\cdot\|\nabla u\|_2,$$

so $\big(\,\mathfrak{A}(v)\cdot\nabla\,\big)u \in L^{3/2}(B^c_{S_0})^3$ in view of Lemma 4.2. It follows with Corollary 3.1 with $p = 3/2$ that $\mathfrak{R}\big(\,\big[\,\mathfrak{A}(v)\cdot\nabla\,\big]u\,\big) \in L^6(\mathbb{R}^3)^3$. We may conclude by (4.2) and (4.5) that $v \in L^6(B^c_r)^3$, so $\mathfrak{A}(v) \in L^6(B^c_{S_0})^3$. Now we return to Lemma 2.2 and to Corollary 3.1, applying the former one with $p = 6,\ q = 2$, and the latter one with $p = 3/2$, to deduce from (4.5) and (4.4) that

$$\|v - u|B_r^c\|_6 \le \tau \cdot \|\mathfrak{R}\big(\,[\,(v-u)\cdot\nabla\,]u\,|\,B_r^c\,\big)\|_6$$
$$\le \tau \cdot C_1(3/2)\cdot C_0(6,2)\cdot \|v-u|B_r^c\|_6 \cdot \|\nabla u|B_r^c\|_2 \le \|v-u|B_r^c\|_6/2.$$

This means that $v - u|B_r^c = 0$, so $u|B_r^c = v \in L^4(B_r^c)^3$. Since $u \in L^6(\overline{\mathfrak{D}}^c)^3$, we thus have proved that $u \in L^4(\overline{\mathfrak{D}}^c)^3$. □

Theorem 4.4 *$u \in L^3(\overline{\mathfrak{D}}^c)^3$ and $\nabla u \in L^{12/7}(\overline{\mathfrak{D}}^c)^9$.*

Proof: Since $u \in L^6(\overline{\mathfrak{D}}^c)^3$ by assumption and $u \in L^4(\overline{\mathfrak{D}}^c)^3$ by Theorem 4.3, we may choose $r \in (S_1, \infty)$ with

$$\|u|B_r^c\|_6 + \|u|B_r^c\|_4 \le \min\big\{\,(a_j\cdot\tau\cdot A_j\cdot B_j)^{-1}\ :\ j \in \{1, ..., 4\}\,\big\}, \tag{4.6}$$

with $A_1 := C_1(6/5),\ A_2 := C_2(6/5),\ B_1 := B_2 := C_0(4, 12/7),\ a_1 := a_2 := 4,\ A_3 := C_1(3/2),\ B_3 := C_0(6,2),\ a_3 := 1,\ A_4 := C_2(4/3),\ B_4 := C_0(4,2),\ a_4 := 2$. (The preceding constants were introduced in Lemma 2.2 and Theorem 3.1, respectively.) Put

$$\mathfrak{W} := \{\phi \in L^3(\overline{B_r}^{\,c})^3 \cap W^{1,1}_{loc}(\overline{B_r}^{\,c})^3\ :\ \nabla\phi \in L^{12/7}(\overline{B_r}^{\,c})^9\},\ \|\phi\|_{\mathfrak{W}} := \|\phi\|_3 + \|\nabla\phi\|_{12/7}$$

for $\phi \in \mathfrak{W}$. The mapping $\|\ \ \|_{\mathfrak{W}}$ is a norm on $\mathfrak{W}$, and $\mathfrak{W}$ equipped with this norm is a Banach space. For $\phi \in \mathfrak{W}$, define $\mathfrak{A}(\phi)(x) := \big(\,\partial_k u_l(x)\,\big)_{1\le k,l\le 3}$ for $x \in B_r\backslash B_{S_0}$, $\mathfrak{A}(\phi)(x) := \big(\,\partial_k \phi_l(x)\,\big)_{1\le k,l\le 3}$ for $x \in B_r^c$. Since $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ and by the definition of $\mathfrak{W}$, we have $\mathfrak{A}(\phi) \in L^{12/7}(B_{S_0}^c)^9$ $(\phi \in \mathfrak{W})$. Therefore by Lemma 2.2 with $p = 4,\ q = 12/7$, and by Theorem 4.3, we get $u^T \cdot \mathfrak{A}(\phi) \in L^{6/5}(B_{S_0}^c)^3$ for $\phi \in \mathfrak{W}$. Thus Corollary 3.1 with $p = q = 6/5$ implies $\mathfrak{R}\big(\,u^T\cdot\mathfrak{A}(\phi)\,\big) \in L^3(\mathbb{R}^3)^3$ and $\nabla\mathfrak{R}\big(\,u^T\cdot\mathfrak{A}(\phi)\,\big) \in L^{12/7}(\mathbb{R}^3)^9$. In view of (4.2) and (4.3), we may thus define $\mathfrak{T} : \mathfrak{W} \mapsto \mathfrak{W}$ by setting

$$\mathfrak{T}(\phi) := \big[\,u - \tau\cdot\mathfrak{R}\big(\,(u\cdot\nabla)u\,|\,B_{S_0}^c\,\big) + \tau\cdot\mathfrak{R}\big(\,u^T\cdot\mathfrak{A}(\phi)\,\big)\,\big]\,|\,B_r^c \quad (\phi \in \mathfrak{W}).$$

For $w_1,\ w_2 \in \mathfrak{W}$, we obtain with Lemma 2.2 $(p = 4,\ q = 12/7)$, Corollary 3.1 $(p = q = 6/5)$ and (4.6) that

$$\|\mathfrak{T}(w_1) - \mathfrak{T}(w_2)\|_3 \le \tau\cdot\|\mathfrak{R}\big(\,(u|B_r^c\cdot\nabla)(w_1-w_2)\,\big)\|_3$$
$$\le \tau\cdot C_1(6/5)\cdot C_0(4,12/7)\cdot\|u|B_r^c\|_4\cdot\|\nabla(w_1-w_2)\|_{12/7} \le \|w_1-w_2\|_{\mathfrak{W}}/4,$$
$$\|\nabla\big(\,\mathfrak{T}(w_1)-\mathfrak{T}(w_2)\,\big)\|_{12/7} \le \tau\cdot\|\nabla\mathfrak{R}\big(\,(u|B_r^c\cdot\nabla)(w_1-w_2)\,\big)\|_{12/7}$$
$$\le \tau\cdot C_2(6/5)\cdot C_0(4,12/7)\cdot\|u|B_r^c\|_4\cdot\|\nabla(w_1-w_2)\|_{12/7} \le \|w_1-w_2\|_{\mathfrak{W}}/4.$$

Therefore $\|\mathfrak{T}(w_1)-\mathfrak{T}(w_2)\|_{\mathfrak{W}} \le \|w_1-w_2\|_{\mathfrak{W}}/2$ for $w_1,\ w_2 \in \mathfrak{W}$, so there is $v \in \mathfrak{W}$ with $v = \mathfrak{T}(v)$, that is,

$$v = \big[\,u - \tau\cdot\mathfrak{R}\big(\,(u\cdot\nabla)u\,|\,B_{S_0}^c\,\big) + \tau\cdot\mathfrak{R}\big(\,u^T\cdot\mathfrak{A}(v)\,\big)\,\big]\,|\,B_r^c. \tag{4.7}$$

As mentioned above, we have $\mathfrak{A}(\phi) \in L^{12/7}(B_{S_0}^c)^3$ for $\phi \in \mathfrak{W}$. Therefore Lemma 2.2 with $p = 6,\ q = 12/7$ implies $u^T\cdot\mathfrak{A}(v) \in L^{4/3}(B_{S_0}^c)^3$, so by Corollary 3.1 with $q = 4/3$, the relation $\nabla\mathfrak{R}\big(u^T\cdot\mathfrak{A}(v)\big) \in L^2(\mathbb{R}^3)^9$ holds. It follows with (4.3) and (4.7) that $\nabla v \in L^2(B_r^c)^9$. But then we may conclude that $\mathfrak{A}(v) \in L^2(B_{S_0}^c)^9$, so Lemma 2.2 with $p = 6,\ q = 2$ yields $u^T \cdot \mathfrak{A}(v) \in L^{3/2}(B_{S_0}^c)^3$. Hence $\mathfrak{R}\big(u^T \cdot \mathfrak{A}(v)\big) \in L^6(\mathbb{R}^3)^3$ due to Corollary 3.1 with $p = 3/2$. Referring to (4.2) and (4.7), we may conclude that $v \in L^6(B_r^c)^3$. We may now apply (4.7), and then Corollary 3.1 with $p = 3/2,\ q = 4/3$, Lemma 2.2 with $p = 6,\ q = 2$ and $p = 4,\ q = 2$, respectively, as well as (4.6), to obtain

$$\begin{aligned}
&\|v - u|B_r^c\|_6 \le \tau \cdot \|\mathfrak{R}\big((u\cdot\nabla)(v - u|B_r^c)\big)\|_6 \\
&\le \tau \cdot C_1(3/2) \cdot C_0(6,2) \cdot \|u|B_r^c\|_6 \cdot \|\nabla(v - u|B_r^c)\|_2 \le \|\nabla(v - u|B_r^c)\|_2,
\end{aligned}$$

$$\begin{aligned}
&\|\nabla(v - u|B_r^c)\|_2 \le \tau \cdot \|\nabla\mathfrak{R}\big((u\cdot\nabla)(v - u|B_r^c)\big)\|_2 \\
&\le \tau \cdot C_2(4/3) \cdot C_0(4,2) \cdot \|u|B_r^c\|_4 \cdot \|\nabla(v - u|B_r^c)\|_2 \le \|\nabla(v - u|B_r^c)\|_2/2.
\end{aligned}$$

The second estimate implies $\|\nabla(v - u|B_r^c)\|_2 = 0$, so the first yields $\|v - u|B_r^c\|_6 = 0$. Therefore $u|B_r^c = v \in L^3(B_r^c)^3$, and $\nabla(u|B_r^c) = \nabla v \in L^{12/7}(B_r^c)^9$. Since $u \in L^6(\overline{\mathfrak{D}}^c)^3$ and $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$, Theorem 4.4 follows. □

Corollary 4.2 *$u \in L^p(\overline{\mathfrak{D}}^c)^3$ for $p \in [12/5, 6]$, in particular for $p = 8/3$ and $p = 5/2$.*

Proof: Theorem 4.4 and Lemma 2.2 with $p = 3,\ q = 12/7$ imply $(u\cdot\nabla)u \in L^{12/11}(\overline{\mathfrak{D}}^c)^3$, so Corollary 3.1 with $p = 12/11$ yields $\mathfrak{R}\big((u\cdot\nabla)u\big) \in L^{12/5}(\mathbb{R}^3)^3$. Now the corollary follows from (4.2). □

Lemma 4.3 *Let $S \in (S_1, \infty)$. Then $\nabla u|B_S^c \in L^4(B_S^c)^9$.*

Proof: Put $S_2 := S_1 + (S - S_1)/3,\ S_3 := S_1 + 2\cdot(S - S_1)/3$. Let $l \in \{1,\ 2,\ 3\}$. By Lemma 2.13,

$$\partial_l\mathfrak{R}\big((u\cdot\nabla)u|B_{S_0}^c\big) = \mathfrak{A} + \partial_l\mathfrak{R}\big((u\cdot\nabla)u|B_{S_2}^c\big), \tag{4.8}$$

with $\mathfrak{A}(y) := \int_{B_{S_2}\backslash B_{S_0}} \partial y_l\mathfrak{Z}(y,z)\cdot\big((u\cdot\nabla)u\big)(z)\, dz$ for $y \in \mathbb{R}^3$. For $y \in B_{S_3}^c$, we get by Corollary 2.6 with R, δ replaced by S_2, $S_3/S_2 - 1$, respectively, that

$$|\mathfrak{A}(y)| \le C(\tau, S_1, S)\cdot\|(u\cdot\nabla)u\,|\,B_{S_2}\backslash B_{S_0}\|_1\cdot\big(|y|\cdot\nu(y)\big)^{-3/2}.$$

But $\|(u\cdot\nabla)u\,|\,B_{S_2}\backslash B_{S_0}\|_1 \le C(S_1,S)\cdot\|(u\cdot\nabla)u\|_{3/2} \le C(S_1,S)\cdot\|u\|_6\cdot\|\nabla u\|_2 \le \mathfrak{C}(S)$, so $|\mathfrak{A}(y)| \le \mathfrak{C}(S)\cdot\big(|y|\cdot\nu(y)\big)^{-3/2}$ for $y \in B_{S_3}^c$. Corollary 2.1 now implies $\mathfrak{A}|B_{S_3}^c \in L^3(B_{S_3}^c)^3$. Since $\nabla u|B_{S_2}^c \in L^{12/5}(B_{S_2}^c)^9$ (Lemma 4.2) and $u \in L^6(\overline{\mathfrak{D}}^c)^3$, we may apply Lemma 2.2 with $p = 6,\ q = 12/5$ and Corollary 3.1 with $q = 12/7$ and $g = (u\cdot\nabla)u\,|\,B_{S_2}^c$, to obtain $\partial_l\mathfrak{R}\big((u\cdot\nabla)u\,|\,B_{S_2}^c\big) \in L^3(\mathbb{R}^3)^3$. On recalling (4.3) and (4.8), we thus see that $\partial_l u|B_{S_3}^c \in L^3(B_{S_3}^c)^3$. Again take $l \in \{1,\ 2,\ 3\}$. Then

$$\partial_l\mathfrak{R}\big((u\cdot\nabla)u|B_{S_0}^c\big) = \widetilde{\mathfrak{A}} + \partial_l\mathfrak{R}\big((u\cdot\nabla)u|B_{S_3}^c\big), \tag{4.9}$$

with $\widetilde{\mathfrak{A}}(y)$ defined in the same way as $\mathfrak{A}(y)$, except that the domain of integration $B_{S_2}\backslash B_{S_0}$ is replaced by $B_{S_3}\backslash B_{S_0}$. Applying Corollary 2.6 again, but this time with R, δ replaced by S_3, $S/S_3 - 1$, respectively, we get that $|\widetilde{\mathfrak{A}}(y)| \le \mathfrak{C}(S)\cdot\big(|y|\cdot\nu(y)\big)^{-3/2}$ for $y \in B_S^c$, in particular $\widetilde{\mathfrak{A}}|B_S^c \in L^4(B_S^c)^3$ by Corollary 2.1. Moreover, referring to Lemma 2.2 with

$p = 6,\ q = 3$ and to Corollary 3.1 with $q = 2$ and g replaced by $(u\cdot\nabla)u\,|\,B^c_{S_3}$, we may conclude that $\partial_l\mathfrak{R}\big(\,(u\cdot\nabla)u\,|\,B^c_{S_3}\,\big) \in L^4(\mathbb{R}^3)^3$. Lemma 4.3 now follows with (4.3) and (4.9). $\square$

5. Pointwise estimates of the velocity.

In a first step, we transform $\mathfrak{R}_j\big(\,(u\cdot\nabla)u\,|\,B^c_{S_0}\,\big)$ by an integration by parts:

Lemma 5.1 *Let $j \in \{1, 2, 3\},\ y \in \overline{B_{S_0}}^{\,c}$. Then $\mathfrak{R}_j\big(\,(u\cdot\nabla)u\,|\,B^c_{S_0}\,\big)(y) = -\mathfrak{F}_j(y) - \widetilde{\mathfrak{B}}_j(y)$, with*

$$\mathfrak{F}_j(y) := \sum_{k,l=1}^{3} \int_{B^c_{S_0}} \partial z_l \mathfrak{Z}_{jk}(y,z)\cdot(u_l\cdot u_k)(z)\,dz,$$

$$\widetilde{\mathfrak{B}}_j(y) := \sum_{k,l=1}^{3} \int_{\partial B_{S_0}} \mathfrak{Z}_{jk}(y,z)\cdot(u_l\cdot u_k)(z)\cdot z_l/S_0\,do_z.$$

Proof: Since $(u\cdot\nabla)u$ and $u_k\cdot u$ belong to $L^{3/2}(\overline{\mathfrak{D}}^c)^3$ (Lemma 2.2 and Theorem 4.4), Lemma 2.13 yields that $\int_{B_{S_0}} |\partial^\alpha_z \mathfrak{Z}_{jk}(y,z)|\cdot|g(z)|\,dz < \infty$ for $\alpha = 0,\ g = u_l\cdot\partial_l u_k$, and for $\alpha = e_l,\ g = u_l\cdot u_k$, with $1 \le k, l \le 3$.

Let $\psi \in C^\infty_0(\mathbb{R}^3)$ with $\psi|B_1 = 0,\ \psi|B^c_2 = 1,\ 0 \le \psi \le 1$. For $\epsilon > 0$, put $\psi_\epsilon(z) := \psi(\epsilon^{-1}\cdot z)$ for $z \in \mathbb{R}^3$. We may conclude with the first sentence of this proof and with Lebesgue's theorem that

$$\mathfrak{R}\big(\,(u\cdot\nabla)u\,|\,B^c_{S_0}\,\big)(y) = \lim_{R\to\infty}\lim_{\epsilon\downarrow 0}\int_{B_R\backslash B_{S_0}} \psi_\epsilon(y-z)\cdot\mathfrak{Z}(y,z)\cdot(u(z)\cdot\nabla)u(z)\,dz. \tag{5.1}$$

Put $\delta := \min\{(|y|-S_0)/2,\ S_0/2\}$, and note that for $R \in (S_0+|y|, \infty),\ \epsilon \in (0,\delta)$, we have $\overline{B_{S_0}} \subset B_R$ and $B_{2\cdot\epsilon}(y) \subset B_R\backslash B_{S_0}$. Moreover, $\psi_\epsilon(y-\ \cdot\ )\cdot\mathfrak{Z}_{jk}(y-\ \cdot\ ) \in C^1(\mathbb{R}^3)$ for $1 \le j,k \le 3,\ \epsilon > 0$ by Lemma 2.11. Thus we may deduce from (5.1) by an integration by parts that

$$\mathfrak{R}_j\big(\,(u\cdot\nabla)u\,|\,B^c_{S_0}\,\big)(y) = \lim_{R\to\infty,\ R>|y|+S_0}\ \lim_{\epsilon\downarrow 0,\ \epsilon<\delta}\big(-\widetilde{\mathfrak{B}}_j(y) + \mathfrak{F}_{1,\epsilon} + \mathfrak{F}_{2,R} - \mathfrak{F}_{3,R,\epsilon}\big), \tag{5.2}$$

where

$$\mathfrak{F}_{1,\epsilon} := \sum_{k,l=1}^{3} \int_{B_{2\cdot\epsilon}(y)\backslash B_\epsilon(y)} (\partial_l\psi_\epsilon)(y-z)\cdot\mathfrak{Z}_{jk}(y,z)\cdot(u_l\cdot u_k)(z)\,dz,$$

$$\mathfrak{F}_{2,R} := \sum_{k,l=1}^{3} \int_{\partial B_R} \mathfrak{Z}_{jk}(y,z)\cdot(u_l\cdot u_k)(z)\cdot z_l/R\,do_z,$$

and where $\mathfrak{F}_{3,R,\epsilon}$ is defined as the term $\mathfrak{F}_j(y)$ in the lemma, but with the additional factor $\psi_\epsilon(y-z)$ under the integral, and with the domain of integration $B^c_{S_0}$ replaced by $B_R\backslash B_{S_0}$. For $\epsilon \in (0,\delta),\ z \in B_{2\cdot\epsilon}(y)\backslash B_\epsilon(y)$, we have $\epsilon \le |y-z|$ and $y, z \in B_R$ (see above), so Corollary 2.5 yields $|\mathfrak{Z}(y,z)| \le C(\tau,\varrho,R)\cdot|y-z|^{-1} \le C(\tau,\varrho,R)\cdot\epsilon^{-1}$. Since by Hölder's inequality $\int_{B_{2\cdot\epsilon}(y)\backslash B_\epsilon(y)} |u(z)|^2\,dz \le C\cdot\epsilon^2\cdot\|u|B_{2\cdot\epsilon}(y)\|_6^2$ for $\epsilon\in(0,\delta)$, and because $|\nabla\psi(x)| \le C\cdot\epsilon^{-1}$ for $x\in\mathbb{R}^3,\ \epsilon>0$, we may conclude that $|\mathfrak{F}_{1,\epsilon}| \le C(\tau,\varrho,R)\cdot\|u|B_{2\cdot\epsilon}(y)\|_6^2$, so $\mathfrak{F}_{1,\epsilon}\to 0$ for $\epsilon\downarrow 0$. Moreover, Lebesgue's theorem and the first sentence of this proof yield $\mathfrak{F}_{3,R,\epsilon}\to\mathfrak{F}_{3,R}$ $(\epsilon\downarrow 0)$, with $\mathfrak{F}_{3,R}$ defined in the same way as $\mathfrak{F}_j(y)$, but with the domain of integration $B_{S_0}^c$ replaced by $B_R\backslash B_{S_0}$ $(R\in(|y|+S_0,\infty)$. Therefore, from (5.2),

$$\mathfrak{R}_j\big((u\cdot\nabla)u\,|\,B_{S_0}^c\big)(y) = \lim_{R\to\infty,\,R>|y|+S_0}\big(-\widetilde{\mathfrak{B}}_j(y)+\mathfrak{F}_{2,R}-\mathfrak{F}_{3,R}\big). \tag{5.3}$$

Next we observe that $\int_{B_R^c}|u(z)|^6\,dz = \int_R^\infty\int_{\partial B_r}|u(z)|^6\,do_z\,dr$ for $R\in(S_0,\infty)$. Therefore the assumption that there is $R_0\in(S_0,\infty)$ with $\int_{\partial B_r}|u(z)|^6\,do_z \ge 1/r$ for any $r\in[R_0,\infty)$ immediately leads to a contradiction to the relation $u\in L^6(\overline{\mathfrak{D}}^c)^c$. Thus there is a sequence (R_n) in $[2\cdot(S_0+|y|),\infty)$ with $R_n\to\infty$ and $\int_{\partial B_{R_n}}|u(z)|^6\,do_z \le 1/R_n$ for $n\in\mathbb{N}$. On the other hand, by Corollary 2.6 with $\delta=1,\ R=S_0+|y|$, we have

$$|\mathfrak{Z}(y,z)| \le C(\tau,S_0+|y|)\cdot\big(R_n\cdot\nu(z)\big)^{-1} \quad \text{for } z\in B_{R_n},\ n\in\mathbb{N}.$$

As a consequence $|\mathfrak{F}_{2,R_n}| \le C(\tau,S_0+|y|)\cdot R_n^{-1}\cdot\int_{\partial B_{R_n}}\nu(z)^{-1}\cdot|u(z)|^2\,do_z$ for $n\in\mathbb{N}$. Therefore by Hölder's inequality and Corollary 2.1 with $a=0,\ b=3/2$, we get $|\mathfrak{F}_{2,R_n}| \le C(\tau,S_0+|y|)\cdot R_n^{-1/3}\cdot\|u|\partial B_{R_n}\|_6^2$, so $\mathfrak{F}_{2,R_n}\to 0$ $(n\to\infty)$ by the choice of the sequence (R_n). Finally, Lebesgue's theorem and the first sentence of this proof imply $\mathfrak{F}_{3,R_n}\to\mathfrak{F}_j(y)$ $(n\to\infty)$. Lemma 5.1 thus follows from (5.3). □

Corollary 5.1 *Let $S\in(S_1,\infty)$. Then*

$$|u(x)-\tau\cdot\mathfrak{F}(x)| \le C(S_0,S_1,S,A,B)\cdot(\mathfrak{M}+\|u|\partial B_{S_0}\|_2^2)\cdot\big(|x|\cdot\nu(x)\big)^{-1} \quad \textit{for } x\in B_S^c,$$

with $\mathfrak{M}$ from Theorem 4.2.

Proof: Combine (4.1) with $\alpha=0$, Lemma 5.1 and Corollary 2.6 with $R=S_0,\ \delta=S/S_0-1$. □

Theorem 5.1 *Put $\varphi(S) := \sup\{|u(y)|\ :\ y\in B_S^c\}$ for $S\in(S_1,\infty)$.*

Then $\varphi(S)<\infty$ for $S\in(S_1,\infty)$, $\varphi(S)\to 0$ $(S\to\infty)$ and $\varphi(S)\le\mathfrak{C}\cdot\big(S^{-1}+\varphi(S/2)^{7/6}\big)$ for $S\in(2\cdot S_1,\infty)$.

Proof: We note that $e^{t\cdot\Omega}\cdot B_R^c = B_R^c$ for $t\in\mathbb{R},\ R>0$ (Lemma 2.3).

Let $S\in(S_1,\infty)$, and put $S_2 := (S+S_1)/2,\ S_3 := \min\{S_1/2,\ (S-S_1)/2\}$. Let $y\in B_S^c$. Then Lemma 2.17 yields $|\mathfrak{F}(y)| \le C(\tau,\varrho)\cdot\sum_{m=1}^3\mathfrak{K}_m(y)$, with $\mathfrak{F}(y)$ defined in Lemma 5.1, and $\mathfrak{K}_m(y) := \int_0^\zeta\int_{A_m}W(y-z,t)\cdot|u(e^{t\cdot\Omega}\cdot z)|^2\,dz$ for $m\in\{1,2,3\}$, with $A_1 := B_{S_2}\backslash B_{S_0},\ A_2 := B_{S_2}^c\cap B_{S_1}(y),\ A_3 := B_{S_2}^c\backslash B_{S_1}(y)$. The function W was introduced in Lemma 2.17.

For $z\in B_{S_2}$, we have $|y-z|\ge S-S_2 = (S-S_1)/2 \ge S_3$. Thus we may apply Lemma 2.6 with $M=S_3$ as well as Theorem 2.3 with $z=0,\ a=2,\ R=S_3/2,\ \delta=1$, and with y replaced by $y-z$, to obtain for $z\in B_{S_2},\ t\in(0,\zeta)$ that

$$\begin{aligned} W(y-z,t) &\le C(\tau,S_3)\cdot\Big[\big(|y-z|\cdot\nu(y-z)+t\big)^{-2}+\big(|y-z|\cdot\nu(y-z)\big)^{-3/2}\Big] \\ &\le C(\tau,S_3)\cdot\big(|y-z|\cdot\nu(y-z)\big)^{-3/2}. \end{aligned}$$

Therefore, according to Hölder's inequality, the term $\mathfrak{K}_1(y)$ is bounded by

$$C(\tau, S_3) \cdot \int_0^\zeta \Big(\int_{B_{S_2} \backslash B_{S_0}} |u(e^{t \cdot \Omega} \cdot z)|^{8/3} \, dz \Big)^{3/4} \cdot \Big(\int_{B_{S_2}} \big(|y-z| \cdot \nu(y-z) \big)^{-6} \, dz \Big)^{1/4} \, dt.$$

But the first of the preceding integrals with respect to z admits the upper bound $\|u\|_{8/3}^2$, as follows by a change of variables and Lemma 2.3. Since $|y-z| \geq (S-S_1)/2$ for $z \in B_{S_2}$, as observed above, so that $B_{S_2} \subset B_{(S-S_1)/2}(y)^c$, we may use Corollary 2.1 with $a = b = 6$, obtaining that the second integral with respect to z is bounded by $C \cdot (S-S_1)^{-4}$. Recalling Corollary 4.2, we thus get

$$\mathfrak{K}_1(y) \leq C(\tau, S_3) \cdot \|u\|_{8/3}^2 \cdot (S-S_1)^{-1} \leq \mathfrak{C}(S_3) \cdot (S-S_1)^{-1}. \tag{5.4}$$

For $z \in B_{S_1}(y)$ with $z \neq y, \ t \in (0, \zeta)$, Lemma 2.6 with $M = S_1$ yields

$$\begin{aligned} W(y-z,t) &\leq C(\tau, S_1) \cdot \Big((|y-z|^2 + t)^{-2} + \int_0^\infty (|y-z|^2 + s)^{-2} \, ds \Big) \\ &\leq C(\tau, S_1) \cdot \big((|y-z| + t^{1/2})^{-4} + |y-z|^{-2} \big). \end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned} &\mathfrak{K}_2(y) \leq C(\tau, S_1) \\ &\quad \cdot \int_0^\zeta \Big(\int_{B_{S_1}(y)} \big((|y-z| + t^{1/2})^{-4} + |y-z|^{-2} \big)^{6/5} \, dz \Big)^{5/6} \cdot \Big(\int_{B_{S_2}^c} |u(e^{t \cdot \Omega} \cdot z)|^{12} \, dz \Big)^{1/6} \, dt. \end{aligned}$$

The preceding integral over $B_{S_2}^c$ equals $\|u|B_{S_2}^c\|_{12}^2$. Concerning the integral over $B_{S_1}(y)$, it is bounded by $C(S_1) \cdot (t^{-9/10} + 1)$. Thus, on integrating with respect to t, we get $\mathfrak{K}_2(y) \leq C(\tau, S_1) \cdot \|u|B_{S_2}\|_{12}^2$.

Lemma 2.6 with $M = S_1$ and Theorem 2.3 with $z = 0, \ R = S_1/2, \ \delta = 1, \ a = 2$ and y replaced by $y - z$ imply the following estimate, for $t \in (0, \zeta), \ z \in B_{S_1}(y)^c$:

$$\begin{aligned} W(y-z,t) &\leq C(\tau, S_1) \cdot \big[\big(|y-z| \cdot \nu(y-z) + t \big)^{-2} + \big(|y-z| \cdot \nu(y-z) \big)^{-3/2} \big] \\ &\leq C(\tau, S_1) \cdot \big(|y-z| \cdot \nu(y-z) \big)^{-3/2}. \end{aligned}$$

We may conclude with Hölder's inequality that

$$\begin{aligned} &\mathfrak{K}_3(y) \leq C(\tau, S_1) \\ &\quad \cdot \int_0^\zeta \Big(\int_{B_{S_1}(y)^c} \big(|y-z| \cdot \nu(y-z) \big)^{-9/4} \, dz \Big)^{2/3} \cdot \Big(\int_{B_{S_2}^c} |u(e^{t \cdot \Omega} \cdot z)|^6 \, dz \Big)^{1/3} \, dt. \end{aligned}$$

Corollary 2.1 with $a = b = 9/4$ yields that the preceding integral over $B_{S_1}(y)^c$ is bounded by a constant $C(S_1)$. Moreover, the integral over $B_{S_2}^c$ equals $\|u|B_{S_2}^c\|_6^6$. Thus we have found that $\mathfrak{K}_3(y) \leq C(\tau, S_1) \cdot \|u|B_{S_2}^c\|_6^2$.

Combining the preceding estimates of $\mathfrak{K}_1(y)$, $\mathfrak{K}_2(y)$ and $\mathfrak{K}_3(y)$, we arrive at the inequality

$$|\mathfrak{F}(y)| \leq \mathfrak{C}(S_3) \cdot \big((S-S_1)^{-1} + \|u|B_{S_2}^c\|_{12}^2 + \|u|B_{S_2}^c\|_6^2 \big), \tag{5.5}$$

where y was arbitrary from $B_S(y)^c$, and S arbitrary from (S_1, ∞). Abbreviating $A(S) := \sup\{|\mathfrak{F}(y)| \,:\, y \in B_S^c\}$ for $S \in (S_1, \infty)$, we may conclude from (5.5) and Corollary 4.1 that $A(S) < \infty$ for $S \in (S_1, \infty)$. We further observe that the parameter S_2 in (5.5) satisfies the relation $S_2 > S/2$, and the quantity S_3 coincides with $S_1/2$ if $S \in (2 \cdot S_1, \infty)$. Thus we may deduce from (5.5) that

$$|A(S)| \le \mathfrak{C} \cdot \big(S^{-1} + \|u|B_{S/2}^c\|_{12}^2 + \|u|B_{S/2}^c\|_6^2 \big) \quad \text{for } S \in (2 \cdot S_1, \infty). \tag{5.6}$$

Again referring to Corollary 4.1, we see that (5.6) implies $A(S) \to 0$ for $S \to \infty$. On the other hand, put $B(S) := \sup\{|u(y) - \tau \cdot \mathfrak{F}(y)| \,:\, y \in B_S^c\}$ for $S \in (S_1, \infty)$. Then Corollary 5.1 implies that $B(S) < \infty$ for $S \in (S_1, \infty)$. In addition, the same reference yields $|u(y) - \tau \cdot \mathfrak{F}(y)| \le \mathfrak{C} \cdot |y|^{-1}$ for $y \in B_{2 \cdot S_1}^c$. Hence $B(S) \to 0$ for $S \to \infty$. Thus we have shown that $\varphi(S) < \infty$ for $S \in (S_1, \infty)$, and $\varphi(S) \to 0$ for $S \to \infty$.

Now let $S \in (2 \cdot S_1, \infty)$ and $y \in B_S^c$. Lemma 2.17 and Lemma 2.3 yield

$$|\mathfrak{F}(y)| \le C(\tau, \varrho) \cdot \big[\mathfrak{K}_1(y) + \varphi(S_2)^{7/6} \cdot (\widetilde{\mathfrak{K}}_2(y) + \widetilde{\mathfrak{K}}_3(y)) \big],$$

where $\widetilde{\mathfrak{K}}_2(y)$, $\widetilde{\mathfrak{K}}_3(y)$ are defined as $\mathfrak{K}_2(y)$, $\mathfrak{K}_3(y)$, respectively, but with the term $|u(e^{\tau \cdot \Omega} \cdot z)|^2$ replaced by $|u(e^{\tau \cdot \Omega} \cdot z)|^{5/6}$. The quantity S_2 is defined as at the beginning of this proof. Since $S \ge 2 \cdot S_1$, inequality (5.4) yields $\mathfrak{K}_1(y) \le \mathfrak{C} \cdot S^{-1}$. The terms $\widetilde{\mathfrak{K}}_2(y)$ and $\widetilde{\mathfrak{K}}_3(y)$ may be estimated in the same way as $\mathfrak{K}_2(y)$ and $\mathfrak{K}_3(y)$, respectively. We obtain

$$\widetilde{\mathfrak{K}}_2(y) + \widetilde{\mathfrak{K}}_3(y) \le C(\tau, S_1) \cdot (\|u|B_{S_2}^c\|_5^{5/6} + \|u|B_{S_2}^c\|_{5/2}^{5/6}).$$

Observing that $S_2 \ge S/2$, hence $\varphi(S_2) \le \varphi(S/2)$, and recalling Corollary 4.2, we thus get

$$|\mathfrak{F}(y)| \le \mathfrak{C} \cdot \big(S^{-1} + \varphi(S/2)^{7/6} \cdot (\|u\|_5^{5/6} + \|u\|_{5/2}^{5/6}) \big) \le \mathfrak{C} \cdot \big(S^{-1} + \varphi(S/2)^{7/6} \big).$$

On the other hand, Corollary 5.1 with S replaced by $2 \cdot S_1$ yields $|u(y) - \tau \cdot \mathfrak{F}(y)| \le \mathfrak{C} \cdot |y|^{-1}$. Altogether we have $|u(y)| \le \mathfrak{C} \cdot \big(S^{-1} + \varphi(S/2)^{7/6} \big)$. Therefore $\varphi(S) \le \mathfrak{C} \cdot \big(S^{-1} + \varphi(S/2)^{7/6} \big)$. □

Theorem 5.2 *Let $S \in (S_1, \infty)$. Then $|u(x)| \le \mathfrak{C}(S) \cdot |x|^{-1}$ for $x \in B_S^c$.*

Proof: The theorem follows from Theorem 5.1 by an argument due to Babenko [2]. For details, we refer to the proof of [3, Theorem 4.1]. □

Now we are in a position to establish the looked-for decay estimate of u:

Theorem 5.3 *Let $S \in (S_1, \infty)$. Then $|u(y)| \le \mathfrak{C}(S) \cdot \big(|y| \cdot \nu(y) \big)^{-1}$ for $y \in B_S^c$.*

Proof: By Theorem 5.1, we have $|u(y)| \le \mathfrak{C}(S)$ for $y \in B_S^c$, so

$$|u(y)| \le \mathfrak{C}(S) \cdot \big(|y| \cdot \nu(y) \big)^{-1} \quad \text{for } y \in B_{2 \cdot S} \backslash B_S. \tag{5.7}$$

Let $y \in B_{2 \cdot S}^c$. Then by Corollary 5.1,

$$|u(y)| \le \mathfrak{C}(S) \cdot \big(|y| \cdot \nu(y) \big)^{-1} + \mathfrak{A}_1(y) + \mathfrak{A}_2(y), \tag{5.8}$$

with $\mathfrak{A}_1(y) := \sum_{j,k,l=1}^{3} \int_{B_S \backslash B_{S_0}} |\partial z_l \mathfrak{Z}_{jk}(y,z) \cdot u_l(z) \cdot u_k(z)|\, dz$, and with $\mathfrak{A}_2(y)$ defined in the same way as $\mathfrak{A}_1(y)$, but with the domain of integration $B_S \backslash B_{S_0}$ replaced by B_S^c. Corollary 2.6 with $R = S,\ \delta = 1,\ \alpha = e_l$ for $1 \le l \le 3$ yields

$$\mathfrak{A}_1(y) \le C(\tau, S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-1} \cdot \|u|B_S \backslash B_{S_0}\|_2^2 \le \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-1}. \tag{5.9}$$

Note that $\|u|B_S \backslash B_{S_0}\|_2 \le C(S) \cdot \|u\|_6$. Moreover, by Theorem 5.2,

$$\mathfrak{A}_2(y) \le \mathfrak{C}(S) \cdot \sum_{j,k,l=1}^{3} \int_{B_S^c} |\partial z_l \mathfrak{Z}_{jk}(y,z)| \cdot |z|^{-2}\, dz. \tag{5.10}$$

Hence with Lemma 2.16 and Theorem 2.2 with $a = 2,\ b = 0$,

$$\mathfrak{A}_2(y) \le \mathfrak{C}(S) \cdot \big(\, |y|^{-2} + |y|^{-1} \cdot \nu(y)^{-1/2}\,\big) \le \mathfrak{C}(S) \cdot |y|^{-1} \cdot \nu(y)^{-1/2},$$

where the last inequality holds by Lemma 2.5. This estimate, (5.8) and (5.9) yield $|u(y)| \le \mathfrak{C}(S) \cdot |y|^{-1} \cdot \nu(y)^{-1/2}$ for $y \in B_{2\cdot S}^c$. Due to (5.7), the preceding inequality even holds for $y \in B_S^c$. With this result at hand, we return to the term $\mathfrak{A}_2(y)$, which may now be estimated as in (5.10), but with an additional factor $\nu(z)^{-1}$ under the integral. Observing that $\nu(z)^{-1} \le \nu(z)^{-1/2}$ ($z \in \mathbb{R}^3$), we again apply Lemma 2.16, and then Theorem 2.2, this time with $a = 2,\ b = 1/2$, to obtain $\mathfrak{A}_2(y) \le \mathfrak{C}(S) \cdot \big(\, |y|^{-2} \cdot \nu(y)^{-1/2} + |y|^{-5/4} \cdot \nu(y)^{-3/4}\,\big)$ for $y \in B_{2\cdot S}^c$, and thus $\mathfrak{A}_2(y) \le \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-1}$ due to Lemma 2.5. Recalling (5.8) and (5.9), we see that $|u(y)| \le \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-1}$ for $y \in B_{2\cdot S}^c$. Theorem 5.3 now follows with (5.7). □

6. Pointwise estimates of the gradient of the velocity.

Let us first show that ∇u is bounded outside any ball B_S with $S > S_1$.

Theorem 6.1 *Let $S \in (S_1, \infty)$. Then $|\nabla u(y)| \le \mathfrak{C}(S)$ for $y \in B_S^c$.*

Proof: Put $S_2 := (S + S_1)/2$. Let $l \in \{1, 2, 3\},\ y \in B_S^c$. By (4.1) and Lemma 2.13,

$$|\partial_l u(y)| \le \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-3/2} + \mathfrak{A}_1(y) + \mathfrak{A}_2(y), \tag{6.1}$$

with $\mathfrak{A}_1(y) := \int_{B_{S_2} \backslash B_{S_0}} |\partial y_l \mathfrak{Z}(y,z) \cdot (u(z) \cdot \nabla) u(z)|\, dz$, and with $\mathfrak{A}_2(y)$ defined in the same way as $\mathfrak{A}_1(y)$, but with the domain of integration $B_{S_2} \backslash B_{S_0}$ replaced by $B_{S_2}^c$. Corollary 2.6 with $R = S_2,\ \delta = S/S_2 - 1,\ \alpha = e_l$ yields

$$\begin{aligned} \mathfrak{A}_1(y) &\le C(\tau, S_1, S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-3/2} \cdot \|(u \cdot \nabla) u \,|\, B_{S_2} \backslash B_{S_0}\|_1 \qquad (6.2) \\ &\le C(\tau, S_0, S_1, S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-3/2} \cdot \|u\|_6 \cdot \|\nabla u\|_2 \le \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y)\,\big)^{-3/2}. \end{aligned}$$

Moreover, we deduce from Lemma 2.3 and 2.17 with $A = B_{S_2}^c,\ g(z) := \tau \cdot |(u(z) \cdot \nabla) u(z)|$ for $z \in B_{S_2}^c$ that

$$\mathfrak{A}_2(y) \le C(\tau, \varrho) \cdot \big(\, \mathfrak{K}_1(y) + \mathfrak{K}_2(y)\,\big), \tag{6.3}$$

where $\mathfrak{K}_1(y) := \int_0^\zeta \int_{B^c_{S_2} \cap B_{S_1}(y)} W(y-z,t) \cdot g(e^{t \cdot \Omega} \cdot z)\; dz\; dt$, and where $\mathfrak{K}_2(y)$ is defined in the same way as $\mathfrak{K}_1(y)$, but with the domain of integration $B^c_{S_2} \cap B_{S_1}(y)$ replaced by $B^c_{S_2} \backslash B_{S_1}(y)$. The function W was introduced in Lemma 2.17. For $z \in B_{S_1}(y),\ z \neq y,\ t \in (0,\zeta)$, we deduce from Lemma 2.6 with $M = S_1$ that

$$W(y-z,t) \le C(S_1) \cdot \big((|y-z|^2 + t)^{-2} + |y-z|^{-2} \big).$$

Therefore by Hölder's inequality

$$\mathfrak{K}_1(y) \le C(\tau, S_1) \tag{6.4}$$
$$\cdot \int_0^\zeta \Big(\int_{B^c_{S_2}} g(e^{t \cdot \Omega} \cdot z)^4\, dz \Big)^{1/4} \cdot \Big(\int_{B_{S_1}(y)} \big((|y-z|^2 + t)^{-2} + |y-z|^{-2} \big)^{4/3} dz \Big)^{3/4} dt.$$

By a change of variables and Lemma 2.3, we see that the preceding integral over $B^c_{S_2}$ is bounded by $\|(u \cdot \nabla)u \,|\, B^c_{S_2}\|_4$. But $|u(z)| \le \mathfrak{C}(S)$ for $z \in B^c_{S_2}$ (Theorem 5.2), so with Lemma 4.3 $\|(u \cdot \nabla)u \,|\, B^c_{S_2}\|_4 \le \mathfrak{C}(S) \cdot \|\nabla u | B^c_{S_2}\|_4 \le \mathfrak{C}(S)$. We further observe that

$$\int_{B_{S_1}(y)} (|y-z| + t^{1/2})^{-16/3}\, dz \le C(S_1) \cdot t^{-7/6} \quad \text{and} \quad \int_{B_{S_1}(y)} |y-z|^{-8/3}\, dz \le C(S_1).$$

Thus we may deduce from (6.4) that $\mathfrak{K}_1(y) \le \mathfrak{C}(S) \cdot \int_0^\zeta (t^{-7/8} + 1)\, dt \le \mathfrak{C}(S)$. Turning to $\mathfrak{K}_2(y)$, we remark that for $z \in B_{S_1}(y)^c$, we have

$$\int_\zeta^\infty (|y-z-\tau \cdot t \cdot e_1|^2 + t)^{-2}\, dt \le C(\tau, S_1) \cdot \big(|y-z| \cdot \nu(y-z) \big)^{-3/2},$$

as follows from Theorem 2.3 with $R = S_1/2,\ \delta = 1,\ z = 0,\ a = 2$ and with $y - z$ in the place of y. Therefore by Lemma 2.6 with $M = S_1$, we see that for $z \in B_{S_1}(y)^c$, the term $W(y-z,t)$ is bounded by $C(\tau, S_1) \cdot \sum_{\sigma \in \{3/2,\, 2\}} \big(|y-z| \cdot \nu(y-z) \big)^{-\sigma}$, and hence by $C(\tau, S_1) \cdot \big(|y-z| \cdot \nu(y-z) \big)^{-3/2}$. It follows with Hölder's inequality

$$\mathfrak{K}_2(y) \le C(\tau, S_1)$$
$$\cdot \int_0^\zeta \Big(\int_{B^c_{S_2}} g(e^{t \cdot \Omega} \cdot z)^3\, dz \Big)^{1/3} \cdot \Big(\int_{B_{S_1}(y)^c} \big(|y-z| \cdot \nu(y-z) \big)^{-9/4} dz \Big)^{2/3} dt.$$

By the same reasoning as used above for $\mathfrak{K}_1(y)$, we see that the preceding integral over $B^c_{S_2}$ is bounded by $\mathfrak{C}(S) \cdot \|\nabla u | B^c_{S_2}\|_3$, and hence by $\mathfrak{C}(S) \cdot (\|\nabla u | B^c_{S_2}\|_2 + \|\nabla u | B^c_{S_2}\|_4) \le \mathfrak{C}(S)$. Concerning the integral over $B_{S_1}(y)^c$, it is bounded by a constant $C(S_1)$, according to Corollary 2.1. Therefore $\mathfrak{K}_2(y) \le \mathfrak{C}(S)$. Theorem 6.1 follows from (6.1) – (6.3) and the preceding estimate of $\mathfrak{K}_1(y)$ and $\mathfrak{K}_2(y)$. □

Now we may derive the looked-for estimate of the gradient of u. Together with Theorem 5.3, it implies Theorem 1.1.

Theorem 6.2 *Let $S \in (S_1, \infty)$. Then $|\nabla u(y)| \le \mathfrak{C}(S) \cdot \big(|y| \cdot \nu(y) \big)^{-3/2}$ for $y \in B^c_S$.*

Proof: Recalling that $|\nabla u(y)| \leq \mathfrak{C}(S)$ for $y \in B_S^c$ by the preceding theorem, we see that

$$|\nabla u(y)| \leq \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} \quad \text{for } \; y \in B_{2 \cdot S} \backslash B_S. \tag{6.5}$$

Inequality (4.1) and Lemma 2.13 yield

$$|\nabla u(y)| \leq \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} + \mathfrak{A}_1(y) + \mathfrak{A}_2(y) \quad \text{for } \; y \in B_S^c, \tag{6.6}$$

with $\mathfrak{A}_1(y) := \sum_{l=1}^{3} \int_{B_S \backslash B_{S_0}} |\partial y_l \mathfrak{Z}(y,z) \cdot (u(z) \cdot \nabla) u(z)| \; dz$, and with $\mathfrak{A}_2(y)$ defined in the same way as $\mathfrak{A}_1(y)$, but with the domain of integration $B_S \backslash B_{S_0}$ replaced by B_S^c. By Corollary 2.6 with $R = S,\ \delta = 1,\ \alpha = e_l$ for $1 \leq l \leq 3$, we find

$$\begin{aligned} &\mathfrak{A}_1(y) \leq C(\tau, S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} \cdot \|(u \cdot \nabla) u \,|\, B_S \backslash B_{S_0}\|_1 \\ &\leq C(\tau, S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} \cdot \|u\|_6 \cdot \|\nabla u\|_2 \leq \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} \quad \text{for } \; y \in B_{2 \cdot S}^c, \end{aligned} \tag{6.7}$$

where we used that $\|u|B_S \backslash B_{S_0}\|_2 \leq C(S) \cdot \|u\|_6 \cdot \|\nabla u\|_2$. Let $y \in B_{2 \cdot S}^c$. By Theorem 5.3 and 6.1, we get $\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \sum_{l=1}^{3} \int_{B_S^c} |\partial y_l \mathfrak{Z}(y,z)| \cdot \big(\, |z| \cdot \nu(z) \,\big)^{-1} dz$. It follows by Lemma 2.16 and Theorem 2.2 with $a = b = 1$ that

$$\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \big[\, \big(\, |y| \cdot \nu(y) \,\big)^{-1} + |y|^{-1/2} \cdot \nu(y)^{-1} \,\big] \leq \mathfrak{C}(S) \cdot |y|^{-1/2} \cdot \nu(y)^{-1}.$$

This inequality and (6.5) – (6.7) imply that

$$|\nabla u(y)| \leq \mathfrak{C}(S) \cdot |y|^{-1/2} \cdot \nu(y)^{-1} \leq \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-1/2} \quad \text{for } \; y \in B_S^c.$$

On using this inequality and again Theorem 5.3, we find

$$\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \sum_{l=1}^{3} \int_{B_S^c} |\partial y_l \mathfrak{Z}(y,z)| \cdot \big(\, |z| \cdot \nu(z) \,\big)^{-3/2} dz \quad \text{for } \; y \in B_{2 \cdot S}^c.$$

Hence by Lemma 2.16 and Theorem 2.2 with $a = b = 3/2$,

$$\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \big[\, \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} + |y|^{-1} \cdot \nu(y)^{-3/2} \,\big] \leq \mathfrak{C}(S) \cdot |y|^{-1} \cdot \nu(y)^{-3/2}$$

$(y \in B_{2 \cdot S}^c)$. Again referring to (6.5) – (6.7), we conclude $|\nabla u(y)| \leq \mathfrak{C}(S) \cdot |y|^{-1} \cdot \nu(y)^{-3/2}$ for $y \in B_S^c$. Returning to $\mathfrak{A}_2(y)$ a third time, we deduce from the preceding estimate and Theorem 5.3 that $\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \sum_{l=1}^{3} \int_{B_S^c} |\partial y_l \mathfrak{Z}(y,z)| \cdot |z|^{-2} \cdot \nu(z)^{-3/4} \, dz \quad (y \in B_{2 \cdot S}^c)$. Hence with Lemma 2.16 and Theorem 2.2 with $a = 2,\ b = 3/4$,

$$\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \big(\, |y|^{-2} \cdot \nu(y)^{-3/4} + |y|^{-11/8} \cdot \nu(y)^{-7/8} \,\big) \leq \mathfrak{C}(S) \cdot |y|^{-11/8} \cdot \nu(y)^{-7/8}$$

$(y \in B_{2 \cdot S}^c)$, where we used Lemma 2.5 in the last inequality. Now (6.5) – (6.7) yield

$$|\nabla u(y)| \leq \mathfrak{C}(S) \cdot |y|^{-11/8} \cdot \nu(y)^{-7/8} \leq \mathfrak{C}(S) \cdot |y|^{-11/8} \cdot \nu(y)^{-1/2} \quad \text{for } \; y \in B_S^c.$$

Finally we use the same sequence of references again, in particular Theorem 2.2 – this time with $a = 19/8$ and $b = 3/2$ –, obtaining

$$\mathfrak{A}_2(y) \leq \mathfrak{C}(S) \cdot \big[\, |y|^{-19/8} \cdot \nu(y)^{-3/2} + \big(\, |y| \cdot \nu(y) \,\big)^{-3/2} \,\big] \leq \mathfrak{C}(S) \cdot \big(\, |y| \cdot \nu(y) \,\big)^{-3/2}$$

$(y \in B_{2 \cdot S}^c)$. Theorem 6.2 follows with (6.5) – (6.7). $\square$

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