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1 Introduction

We consider the system of equations

$$\left.\begin{array}{rcl} -\mu\Delta u(z) - (U + \omega \times z) \cdot \nabla u(z) + \omega \times u(z) + u \cdot \nabla u(z) + \nabla \pi(z) & = & f(z), \\ \operatorname{div} u(z) & = & 0, \end{array}\right\} \tag{1.1}$$

with $z \in \mathbb{R}^3 \setminus \overline{\mathfrak{D}}$. This system describes the stationary flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating at a constant angular velocity. We refer to [19] for more details on the physical background of (1.1). Here we only indicate that $\mathfrak{D} \subset \mathbb{R}^3$ is an open bounded set describing the rigid body, the vector $U \in \mathbb{R}^3 \backslash \{0\}$ represents the constant translational velocity of this body, the vector $\omega \in \mathbb{R}^3 \backslash \{0\}$ stands for its constant angular velocity, and μ denotes the constant kinematic viscosity of the fluid. The given function $f : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ describes a body force, and the unknowns $u : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ and $\pi : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}$ correspond respectively to the velocity and pressure field of the fluid. We assume that $U \cdot \omega \neq 0$. Then, according to [21], without loss of generality we may replace (1.1) by the normalized system

$$L(u) + \tau(u \cdot \nabla)u + \nabla\pi = f, \;\; \operatorname{div} u = 0 \quad \text{in } \; \mathbb{R}^3 \backslash \overline{\mathfrak{D}}, \tag{1.2}$$

where the differential operator L is defined by

$$\begin{array}{l} L(u)(z) := -\Delta u(z) + \tau\,\partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z) \\ \text{for } \; u \in W^{2,1}_{loc}(U)^3, \; z \in U, \; U \subset \mathbb{R}^3 \text{ open,} \end{array}$$

with $\tau \in (0, \infty)$ (Reynolds number) and $\omega = \varrho(1, 0, 0)$ for some $\varrho \in \mathbb{R}$ (Taylor number). The linearized version of (1.1) we are interested in is the following:

$$L(u) + \nabla\pi = f, \;\; \operatorname{div} u = 0 \quad \text{in } \; \mathbb{R}^3 \backslash \overline{\mathfrak{D}}, \tag{1.3}$$

It is well known ([20]) that for data of arbitrary size, both problem (1.2) and (1.3) admit a "Leray solution" characterized by the relations

$$u \in L^6(\mathbb{R}^3 \setminus \overline{\mathfrak{D}})^3, \nabla u \in L^2(\mathbb{R}^3 \setminus \overline{\mathfrak{D}})^9, \pi \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\mathfrak{D}}). \tag{1.4}$$

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Galdi, Kyed [21] showed that if the right-hand side f in the nonlinear problem (1.2) has bounded support, then the velocity part u of such a solution (u, π) decays as follows:

$$|\partial^\alpha u(x)| = O\big[\,\big(\,|x| \cdot s_\tau(x)\,\big)^{-1-|\alpha|/2}\,\big] \quad (|x| \to \infty), \tag{1.5}$$

where $\alpha \in \mathbb{N}_0^3$ with$|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 1$ (decay of u and ∇u). The term $s_\tau(x)$ in (1.5) is defined by

$$s_\tau(x) := 1 + \tau(|x| - x_1) \quad (x \in \mathbb{R}^3).$$

Its presence in (1.5) may be considered as a mathematical manifestation of the wake extending downstream behind the rigid body.

In [1] – [4], we considered a different kind of solution, which we called "weak solution", and which satisfies the relations

$$u_{|B_S^c} \in L^6(B_S^c)^3,\ \nabla u_{|B_S^c} \in L^2(B_S^c)^9,\ \pi|_{B_S^c} \in L^2(B_S^c) \tag{1.6}$$

for some $S > 0$ with $\overline{\mathfrak{D}} \subset B_S$, as well as some additional regularity assumptions, which require in particular that $\pi \in W^{1,p}_{\text{loc}}(\mathbb{R}^3 \backslash \overline{\mathfrak{D}}^c)$ and $\pi_{|B_T \backslash \overline{\mathfrak{D}}} \in L^p(B_T \backslash \overline{\mathfrak{D}})$ for some $p \in (1, \infty)$ and for any $T \in (0, \infty)$ with $\overline{\mathfrak{D}} \subset B_T$. Although these additional regularity assumptions and (1.6) do not imply (1.4) if $p < 6/5$, it is clear that weak solutions should be considered as less general than Leray ones, in particular in view of the requirement on π in (1.4) and (1.6), respectively.

The main results in [1] – [4] concern representation formulas, decay behaviour and asymptotic expansions for weak solutions of the linear problem (1.3), and a representation formula for weak solutions of the nonlinear system (1.2). In particular, the relation in (1.5) is established in the linear case for a right-hand side f which decays sufficiently fast, but need not have bounded support. In the work at hand, we revisit the theory in [1] – [4], showing that it remains valid when the condition $\pi|_{B_S^c} \in L^2(B_S^c)$ in (1.6) is dropped. Again due to our additional regularity assumptions, this does not mean that the theory in [1] – [4] is generalized to the Leray case. But this improved theory implies by some additional arguments that the relation in (1.5) extends to Leray solutions of the linear problem (1.3), even if the right-hand side f in (1.3) does not have bounded support, but decays sufficiently fast (Corollary 3.16).

It will not be necessary to rework the theory in [1] – [4] from beginning to end. Instead, we will only show that the representation formula established in [2, Theorem 4.6] for the velocity part of a solution to (1.3) remains valid without the assumption imposed in (1.6) on the pressure π. Since we used this assumption only in the proof of that formula, we may thus drop it everywhere in [1] – [4].

We remark that our linear theory extends to pairs of functions (u, π) that do not necessarily solve (1.3) because the divergence of u need not vanish. In this respect, however, there is a feature of the theory in [1] – [4] we cannot reproduce here. In fact, we will always assume $\operatorname{div} u$ to have bounded support, whereas in [2] – [4], we obtained some results under the assumption that $\operatorname{div} u$ decays sufficiently fast, but need not have bounded support (see [2, Theorem 5.3], [4, Theorem 6] for example).

The argument of the work at hand is based on a uniqueness result proved by Galdi, Kyed [21, Lemma 4.1] for the linear problem (1.3) in the whole space $\mathbb{R}^3$, and on an existence result for the same problem in $\mathbb{R}^3$ established by Farwig [7] and also, more recently, by ourselves in [5]. These references enter into the proof of Theorem 2.1 below, which provides a regularity result for the pressure part of solutions to the linear problem (1.3).

Readers interested in further results on flows around rotating bodies are referred to [5], [7] – [17], and [22] – [44]. We mention in particular that in [5], we prove (1.5) also for the nonlinear case, presenting a second access to that result besides the one previously provided by Galdi and Kyed [21].

2 Notation, definitions and auxiliary results

The open bounded set $\mathfrak{D} \subset \mathbb{R}^3$ introduced in Section 1 will be kept fixed throughout. We assume its boundary $\partial\mathfrak{D}$ to be of class C^2, and we denote its outward unit normal by $n^{(\mathfrak{D})}$. The numbers τ and ϱ and the vector ω also introduced in Section 1 will be kept fixed, too. Define the matrix $\Omega \in \mathbb{R}^{3\times 3}$ by

$$\Omega := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

so that $\omega \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$.

We recall that the function s_τ was defined in Section 1, as was the notation $|\alpha|$ for the length of a multi-index $\alpha \in \mathbb{N}_0^3$. If $A \subset \mathbb{R}^3$, we write A^c for the complement $\mathbb{R}^3 \backslash A$ of A. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. Put $e_1 := (1, 0, 0)$. Let $x \times y$ denote the usual vector product of $x, y \in \mathbb{R}^3$. For $T \in (0, \infty)$, set $\mathfrak{D}_T := B_T \backslash \overline{\mathfrak{D}}$ ("truncated exterior domain"). By the symbol $\mathfrak{C}$, we denote constants only depending on $\mathfrak{D}$, τ or ω. We write $\mathfrak{C}(\gamma_1, ..., \gamma_n)$ for constants that additionally depend on parameters γ_1, ..., $\gamma_n \in \mathbb{R}$, for some $n \in \mathbb{N}$. Put

$$\begin{aligned} X_q = &\{(v, p) | v \in L^{\frac{2q}{2-q}}(\mathbb{R}^3)^3,\ \nabla' v \in L^{\frac{4q}{4-q}}(\mathbb{R}^3)^6 \\ &\partial_1 v \in L^q(\mathbb{R}^3)^3, \nabla^2 v \in L^q(\mathbb{R}^3)^{27}, p \in L^{\frac{3q}{3-q}}(\mathbb{R}^3), \nabla p \in L^q(\mathbb{R}^3)^3\}, \end{aligned}$$

where $(\nabla')_{ij} := \partial_i v_j$, $(i = 2, 3, j = 1, 2, 3)$. The key auxiliary result of this article is

Theorem 2.1. *Let $q \in (1, 2)$, $p \in (1, \infty)$, $S > 0$, $f \in L^q(\overline{B_S}^c)^3 + L^{3/2}(\overline{B_S}^c)^3$, $u \in L^6(\overline{B_S}^c)^3 \cap W^{2,p}_{\text{loc}}(\overline{B_S}^c)^3$, $\pi \in W^{1,p}_{\text{loc}}(\overline{B_S}^c)$ with $L(u) + \nabla\pi = f$, $\operatorname{div} u = 0$.*

Then there is $c \in \mathbb{R}$ with $\pi + c_{|B^c_{2S}} \in L^{3q/(3-q)}(B^c_{2S}) + L^3(B^c_{2S})$.

Proof We use the approach from the proof of [21, Theorem 4.4]. By the cut-off procedure from that proof, and since $W^{1,q}_{\text{loc}}(\mathbb{R}^3) \subset L^{3/2}_{\text{loc}}(\mathbb{R}^3)$, we see there are functions $F \in L^q(\mathbb{R}^3)^3 + L^{3/2}(\mathbb{R}^3)^3$, $U \in L^6(\mathbb{R}^3)^3 \cap W^{2,p}_{\text{loc}}(\mathbb{R}^3)^3$, $\Pi \in W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ such that

$$L(U) + \nabla\Pi = F, \quad \operatorname{div} U = 0, \quad U(x) = u(x) + \beta|x|^{-3}x \text{ for } x \in B^c_{2S}, \quad \Pi_{|B^c_{2S}} = \pi_{|B^c_{2S}},$$

with some constant $\beta \in \mathbb{R}$. Note that the argument from that proof strongly simplifies in the present situation because we consider the linear problem (1.3) instead of the nonlinear one (1.2). By the assumptions on F, there is $F^{(1)} \in L^q(\mathbb{R}^3)^3$, $F^{(2)} \in L^{3/2}(\mathbb{R}^3)^3$ such that $F = F^{(1)} + F^{(2)}$. But according to [7] or [5, Theorem 1.2], there are pairs of functions $(U^{(1)}, \Pi^{(1)}) \in X_q$, $(U^{(2)}, \Pi^{(2)}) \in X_{3/2}$ such that

$$L(U^{(\kappa)}) + \nabla \Pi^{(\kappa)} = F^{(\kappa)}, \quad \operatorname{div} U^{(\kappa)} = 0 \quad \text{for } \kappa \in \{1, 2\}.$$

This means in particular that $U^{(1)} \in W^{2,q}_{\text{loc}}(\mathbb{R}^3)^3 \cap L^{2q/(2-q)}(\mathbb{R}^3)^3$, $\Pi^{(1)} \in W^{1,q}_{\text{loc}}(\mathbb{R}^3)$, $U^{(2)} \in L^6(\mathbb{R}^3)^3$ (hence $U - U^{(2)} \in L^6(\mathbb{R}^3)^3$), and $U^{(2)} \in W^{2,3/2}_{\text{loc}}(\mathbb{R}^3)^3$, $\Pi^{(2)} \in W^{1,3/2}_{\text{loc}}(\mathbb{R}^3)$.

We may thus apply [21, Lemma 4.1] with $s = \min\{p, q, 3/2\}$, $q_1 = 2q/(2-q)$, $q_2 = 6$, $f = F^{(1)}$, $(v_1, p_1) = (U^{(1)}, \Pi^{(1)})$, $(v_2, p_2) = (U - U^{(2)}, \Pi - \Pi^{(2)})$. It follows that $U^{(1)} = U - U^{(2)}$, $\Pi^{(1)} = \Pi - \Pi^{(2)} + c$ for some $c \in \mathbb{R}$. Hence $\Pi + c \in L^{3q/(3-q)}(\mathbb{R}^3) + L^3(\mathbb{R}^3)$, so that $\pi + c_{|B^c_{2S}} \in L^{3q/(3-q)}(B^c_{2S}) + L^3(B^c_{2S})$.

We will further use the ensuing estimate, which was proved in [6].

Lemma 2.2. *Let $\beta \in (1, \infty)$. Then $\int_{\partial B_r} s_\tau(x)^{-\beta} \, do_x \le \mathfrak{C}(\beta)\, r$ for $r \in (0, \infty)$.*

3 Linear case

We begin by introducing the fundamental solutions used in what follows. We set

$$\begin{aligned}
&K(x,t) = (4\pi t)^{-3/2} e^{-\frac{|x|^2}{4t}}, \ x \in \mathbb{R}^3, \ t \in (0, \infty),\\
&N_{jk}(x) = x_j x_k |x|^{-2}, \ x \in \mathbb{R}^3 \setminus \{0\},\\
&\Lambda_{jk}(x,t) = K(x,t)\Big(\delta_{jk} - N_{jk}(x) - {}_1F_1\Big(1, 5/2, \frac{|x|^2}{4t}\Big)\left(\delta_{jk}/3 - N_{jk}(x)\right)\Big),\\
&\ x \in \mathbb{R}^3 \setminus \{0\}, \ t \in (0, \infty), \ j, k \in \{1, 2, 3\},\\
&(\Gamma_{jk}(y,z,t))_{1 \le j,k \le 3} := (\Lambda_{rs}(y - \tau\, t\, e_1 - e^{-t\Omega} \cdot z, t))_{1 \le r,s \le 3} \cdot e^{-t\Omega},\\
&y, z \in \mathbb{R}^3, \ t \in (0, \infty) \text{ with } y - \tau\, t\, e_1 - e^{-t\Omega} \cdot z \neq 0,\\
&E_{4j}(x) := (4\pi)^{-1} x_j\, |x|^{-3}, \quad 1 \le j \le 3, \ x \in \mathbb{R}^3 \setminus \{0\}.
\end{aligned}$$

According to [1, Theorem 3.1], we have

Lemma 3.1. *$\int_0^\infty |\Gamma_{jk}(y,z,t)| dt < \infty$ for $y, z \in \mathbb{R}^3$ with $y \neq z$, $1 \le j, k \le 3$.*

Thus we may define

$$\mathfrak{Z}_{jk}(y,z) := \int_0^\infty \Gamma_{jk}(y,z,t) dt$$

for $y, z \in \mathbb{R}^3$ with $y \neq z$, $1 \le j, k \le 3$.

The matrix-valued function $\mathfrak{Z}$ constitutes the velocity part of the fundamental solution introduced by Guenther, Thomann [28] for system (1.3). In the ensuing three theorems, we note those properties of $\mathfrak{Z}$ we will use explicitly. Of course, many more will enter implicitly into our reasoning, via the previous articles we will cite.

Theorem 3.2 ([2, Lemma 2.15])**.** *$\mathfrak{Z}_{jk} \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(x,x) : x \in \mathbb{R}^3\})$ for $1 \le j, k \le 3$.*

Theorem 3.3. *Let $S \in (0, \infty)$. Then $|\mathfrak{Z}(y,z)| \le \mathfrak{C}(S) \left(|z|\, s_\tau(z)\right)^{-1}$ for $y \in \overline{B_S}$, $z \in B^c_{2S}$.*

Proof This theorem is a special case of [2, Theorem 2.19].

Theorem 3.4. *$\sum_{k=1}^{3} \partial z_k \mathfrak{Z}_{jk}(y,z) = 0$ for $1 \le j \le 3,\ y, z \in \mathbb{R}^3$ with $y \neq z$.*

Proof [2, Lemma 2.15], [28, Theorem 1.3].

The next four lemmas serve to introduce the potential functions needed later on.

Lemma 3.5. *Let $p \in (1,\infty),\ q \in (1,2),\ f \in L^p_{loc}(\mathbb{R}^3)^3$ with $f_{|B_S^c} \in L^q(B_S^c)^3$ for some $S \in (0,\infty)$. Then, for $j,k \in \{1, 2, 3\}$, we have*

$$\int_{\mathbb{R}^3} |\mathfrak{Z}_{jk}(y,z)|\ |f_k(z)|\ dy < \infty \quad \textit{for a. e. } y \in \mathbb{R}^3. \tag{3.1}$$

We may thus define $\mathfrak{R}(f) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by

$$\mathfrak{R}_j(f)(y) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z)\ f_k(z)\ dz$$

for $y \in \mathbb{R}^3$ such that (3.1) holds; else we set $\mathfrak{R}_j(f)(y) := 0$ $(1 \le j \le 3)$.

If $p > 3/2$, the relation in (3.1) holds without the restriction "a.e.".

Proof Lemma 3.5 holds by [2, Lemma 3.1, 3.2].

Lemma 3.6. *Let $p \in (1,\infty),\ q \in (1,3),\ g \in L^p_{loc}(\mathbb{R}^3)$ with $g_{|B_S^c} \in L^q(B_S^c)$ for some $S \in (0,\infty)$. Then, for $j \in \{1, 2, 3\}$,*

$$\int_{\mathbb{R}^3} |E_{4j}(y-z)|\ |g(z)|\ dy < \infty \quad \textit{for a. e. } y \in \mathbb{R}^3. \tag{3.2}$$

Thus we may define $\mathfrak{S}(g) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by

$$\mathfrak{S}_j(g)(y) := \int_{\mathbb{R}^3} E_{4j}(y-z)\ g(z)\ dz$$

for $y \in \mathbb{R}^3$ such that (3.2) holds, and $\mathfrak{S}_j(g)(y) := 0$ else $(1 \le j \le 3)$.

If $p > 3$, the relation in (3.2) holds for any $y \in \mathbb{R}^3$ (without the restriction "a. e.").

Proof Lemma 3.6 states some of the assertions of [2, Lemma 3.4].

Lemma 3.7. *Let $R \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_R,\ f \in L^1(\partial\mathfrak{D}_R),\ 1 \le j,\ k \le 3,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Then*

$$\int_{\partial\mathfrak{D}_R} |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z) f(z)| do_z < \infty, \quad \int_{\partial\mathfrak{D}_R} |E_j(y-z) f(z)| do_z < \infty \quad \textit{for } y \in \mathfrak{D}_R.$$

Proof Theorem 3.2, Lebesgue's theorem.

Lemma 3.8. *Let $f \in L^1(\partial\mathfrak{D}),\ 1 \le j,\ k \le 3,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Then*

$$\int_{\partial\mathfrak{D}} |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z) f(z)| do_z < \infty, \quad \int_{\partial\mathfrak{D}} |E_j(y-z) f(z)| do_z < \infty \quad \textit{for } y \in \overline{\mathfrak{D}}^c.$$

Proof Theorem 3.2, Lebesgue's theorem.

In the ensuing lemma, we introduce the function space M_p, which characterizes the regularity of the solutions to (1.3) we consider.

Theorem 3.9 ([2, Theorem 4.4])**.** *Let $p \in (1,\infty)$. Define M_p as the space of all pairs of functions (u,π) such that $u \in W^{2,p}_{\text{loc}}(\overline{\mathfrak{D}}^c)^3$, $\pi \in W^{1,p}_{\text{loc}}(\overline{\mathfrak{D}}^c)$,*

$$\begin{cases} u_{|\mathfrak{D}_T} \in W^{1,p}(\mathfrak{D}_T)^3, \ \pi_{|\mathfrak{D}_T} \in L^p(\mathfrak{D}_T), \\ u_{|\partial\mathfrak{D}} \in W^{2-1/p,p}(\partial\mathfrak{D})^3, \ \operatorname{div} u_{|\mathfrak{D}_T} \in W^{1,p}(\mathfrak{D}_T), \\ L(u) + \nabla\pi_{|\mathfrak{D}_T} \in L^p(\mathfrak{D}_T)^3 \end{cases}$$

for some $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$. Then $u_{|\mathfrak{D}_T} \in W^{2,p}(\mathfrak{D}_T)^3$, $\pi_{|\mathfrak{D}_T} \in W^{1,p}(\mathfrak{D}_T)$ for any $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$.

Next we state a representation formula on $\mathfrak{D}_R$ for solutions of (1.3) belonging to M_p.

Theorem 3.10 ([2, Theorem 4.5])**.** *Let $p \in (1,\infty)$, $(u,\pi) \in M_p$, $j \in \{1,2,3\}$. Put $F := L(u) + \nabla\pi$. Let $R \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_R$, and let $n^{(R)} : \partial B_R \cup \partial\mathfrak{D} \mapsto \mathbb{R}^3$ denote the outward unit normal to $\mathfrak{D}_R$. Then, for a.e. $y \in \mathfrak{D}_R$,*

$$u_j(y) = \mathfrak{R}_j(F_{|\mathfrak{D}_R})(y) + \mathfrak{S}_j(\operatorname{div} u_{|\mathfrak{D}_R})(y) + \int_{\partial\mathfrak{D}_R} U^R_j(u,\pi)(y,z)do_z, \tag{3.3}$$

with

$$\begin{aligned} U^{(R)}_j(u,\pi)(y,z) := \sum_{k=1}^{3} \Bigg[\sum_{l=1}^{3} \Big(& \mathfrak{Z}_{jk}(y,z)(\partial_l u_k(z) - \delta_{kl}\pi(z) \\ & + u_k(z)(-\tau e_1 + \omega \times z)_l) - \partial z_l \mathfrak{Z}_{jk}(y,z)u_k(z)\Big) n^{(R)}_l(z) \\ & - E_{4j}(y-z)u_k(z)n^{(R)}_k(z)\Bigg] \text{ for } y \in \mathfrak{D}_R, z \in \partial\mathfrak{D}_R. \end{aligned}$$

If $p > 3/2$, then (3.3) holds for any $y \in \mathcal{D}_R$ (without the restriction "a.e.").

Now we arrive at the main result of this work – a representation formula on $\overline{\mathfrak{D}}^c$ for solutions of (1.3). Contrary to a corresponding result in [2, Theorem 4.6], this formula is valid without the assumption that the pressure belongs to $L^2(B^c_S)$ for some $S > 0$ with $\overline{\mathfrak{D}} \subset B_S$. Still the formula in question is not proved for Leray solutions because these do not in general belong to M_p for some $p \in (1,\infty)$. But in spite of that, our formula under its present assumptions will yield decay estimates in the Leray case (Corollary 3.16). We further remark that we have to assume the support of the divergence of the velocity to be compact, a condition not present in [2].

Theorem 3.11. *Let $p \in (1,\infty)$, $(u,\pi) \in M_p$. Put $F := L(u) + \nabla\pi$, and suppose there are numbers $q \in (1,2)$, $S \in (0,\infty)$ such that*

$$\begin{aligned} &\overline{\mathfrak{D}} \cup \operatorname{supp}(\operatorname{div} u) \subset B_S, \\ &u_{|B^c_S} \in L^6(B^c_S)^3, \ \nabla u_{|B^c_S} \in L^2(B^c_S)^9, \quad F_{|B^c_S} \in L^q(B^c_S)^3 + L^{3/2}(B^c_S)^3. \end{aligned}$$

Let $j \in \{1,2,3\}$ and put

$$\mathfrak{B}_j(y) := \mathfrak{B}_j(u,\pi)(y) := \int_{\partial\mathfrak{D}} \sum_{k=1}^{3} \Big[\sum_{l=1}^{3} \tag{3.4}$$
$$\Big(\mathfrak{Z}_{jk}(y,z)(-\partial_l u_k(z) + \delta_{kl}\pi(z) + u_k(z)(\tau e_1 - \omega \times z)_l)$$
$$+ \partial z_l \mathfrak{Z}_{jk}(y,z) u_k(z) \Big) n_l^{(\mathfrak{D})}(z) + E_{4j}(y-z) u_k(z) n_k^{(\mathfrak{D})}(z) \Big] do_z$$

for $y \in \overline{\mathfrak{D}}^c$. Then

$$u_j(y) = \mathfrak{R}_j(F)(y) + \mathfrak{S}_j(\operatorname{div} u)(y) + \mathfrak{B}_j(y) \tag{3.5}$$

for a.e. $y \in \overline{\mathfrak{D}}^c$. If $p > 3/2$, (3.5) holds for any $y \in \overline{\mathfrak{D}}^c$, without the restriction "a.e.".

Proof By Theorem 2.1, there is $c \in \mathbb{R}$, $\pi_1 \in L^{3q/(3-q)}(B_{2S}^c)$, $\pi_2 \in L^3(B_{2S}^c)$ such that $\pi_{|B_{2S}^c} = \pi_1 + \pi_2 + c$. ¿From this fact and our assumptions on u it follows that

$$\int_{2S}^{\infty} \int_{\partial B_r} (|u(z)|^6 + |\nabla u(z)|^2 + |\pi_1(z)|^{3q/(3-q)} + |\pi_2(z)|^3) do_z dr < \infty.$$

Thus there is an increasing sequence (R_n) in $(2S, \infty)$ with $R_n \to \infty$ and

$$\int_{\partial B_{R_n}} (|u(z)|^6 + |\nabla u(z)|^2 + |\pi_1(z)|^{3q/(3-q)} + |\pi_2(z)|^3) do_z \le R_n^{-1} \text{ for } n \in \mathbb{N}. \tag{3.6}$$

¿From assumptions on F, there are functions $G^{(1)} \in L^q(B_S^c)^3$, $G^{(2)} \in L^{3/2}(B_S^c)^3$ such that $F|_{B_S^c} = G^{(1)} + G^{(2)}$. Thus by Lemma 3.5

$$\int_{\overline{\mathfrak{D}}^c} \sum_{k=1}^{3} |\mathfrak{Z}_{jk}(y,z)| \Big(\chi_{(0,S)}(|z|)|F_k(z)| + \chi_{(S,\infty)}(|z|)(|G_k^{(1)}(z)| + |G_k^{(2)}(z)|) \Big) dz < \infty \tag{3.7}$$

for a.e. $y \in \overline{\mathfrak{D}}^c$. Moreover, by Lemma 3.6 with $p = q = 2$

$$\int_{\overline{\mathfrak{D}}^c} |E_{4j}(y-z)|\, |\operatorname{div} u(z)| dz < \infty \tag{3.8}$$

for a.e. $y \in \overline{\mathfrak{D}}^c$. Due to these observations and Theorem 3.10, we see there is a subset N of $\overline{\mathfrak{D}}^c$ with measure zero such that the relations in (3.7), (3.8) hold for $y \in \overline{\mathfrak{D}}^c \setminus N$, and such that (3.3) with R replaced by R_n holds for $n \in \mathbb{N}$ and $y \in \mathfrak{D}_{R_n} \setminus N$. In the case $p > 3/2$, Lemma 3.5 yields that (3.7) is valid for any $y \in \overline{\mathfrak{D}}^c$, and Theorem 3.10 states that (3.3) with R replaced by R_n is true for $n \in \mathbb{N}$ and for any $y \in \overline{\mathfrak{D}}^c$. Moreover, since $(u,\pi) \in M_p$ and in view of Theorem 3.9, we obtain $\operatorname{div} u_{|\mathfrak{D}_T} \in W^{1,p}(\mathfrak{D}_T)$ for $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$. But $\operatorname{supp}(\operatorname{div} u) \subset B_S$, so $\operatorname{div} u \in W^{1,p}(\overline{\mathfrak{D}}^c)$. Thus, if $p > 3/2$, a Sobolev inequality implies there is $s \in (3,\infty]$ with $\operatorname{div} u \in L^s(\overline{\mathfrak{D}}^c)$, so Lemma 3.6 yields that the restriction "a. e." may be dropped in (3.8) as well.

Take $y \in \overline{\mathfrak{D}}^c$ in the case $p > 3/2$ and $y \in \overline{\mathfrak{D}}^c \setminus N$ otherwise. Let $n \in \mathbb{N}$ with $R_n > |y|$ (hence $y \in \mathfrak{D}_{R_n}$). Then, by (3.3) with R replaced by R_n and π by $\pi - c$, we get

$$u_j(y) = \mathfrak{R}_j(F_{|\mathfrak{D}_{R_n}})(y) + \mathfrak{S}_j(\operatorname{div} u_{|\mathfrak{D}_{R_n}})(y) + U_{in,j}(y) + \mathfrak{B}_j(y), \tag{3.9}$$

with

$$U_{in,j}(y) := \int_{\partial B_{R_n}} \sum_{k=1}^{3} \Big[\sum_{l=1}^{3} \Big(\mathfrak{Z}_{jk}(y,z)(\partial_l u_k(z) -$$

$$-\delta_{kl}(\pi - c)(z) - \tau\delta_{1l} u_k(z)) - \partial z_l \mathfrak{Z}_{jk}(y,z)(u_k(z)\Big) \frac{z_l}{R_n} - E_{4j}(y-z)u_k(z)\frac{z_k}{R_n}\Big] do_z,$$

where we used the relation $\sum_{l=1}^{3}(\omega \times z)_l z_l = 0$ for $z \in \partial B_{R_n}$. Concerning $\mathfrak{B}_j(y)$, we note that

$$\int_{\partial\mathfrak{D}} \sum_{l=1}^{3} \mathfrak{Z}_{jk}(y,z) n_k^{(\mathfrak{D})}(z) do_z = 0 \quad \text{for } \ y \in \mathfrak{D}_R, \ 1 \le j \le 3;$$

see Theorem 3.2 and 3.4. Thus the definition of $\mathfrak{B}_j(y)$ need not be modified even though we replace π by $\pi - c$.

Let $n \in \mathbb{N}$ with $\frac{R_n}{4} \ge |y|$. Observe that

$$|U_{in,j}(y)| \le \mathfrak{C} \sum_{v=1}^{5} \sum_{k=1}^{3} \mathcal{B}_{v,k}(y),$$

with

$$\begin{aligned}
\mathcal{B}_{1,k}(y) &:= \Big(\int_{\partial B_{R_n}} |\mathfrak{Z}_{jk}(y,z)|^{6/5} do_z \Big)^{5/6} \|u_{|\partial B_{R_n}}\|_6, \\
\mathcal{B}_{2,k}(y) &= \Big(\int_{\partial B_{R_n}} |\mathfrak{Z}_{jk}(y,z)|^{2} do_z \Big)^{1/2} \|\nabla u_{|\partial B_{R_n}}\|_2, \\
\mathcal{B}_{3,k}(y) &= \Big(\int_{\partial B_{R_n}} |\mathfrak{Z}_{jk}(y,z)|^{\frac{3q}{4q-3}} do_z \Big)^{\frac{4q-3}{3q}} \|\pi_{1\ |\partial B_{R_n}}\|_{3q/(3-q)}, \\
\mathcal{B}_{4,k}(y) &= \Big(\int_{\partial B_{R_n}} |\mathfrak{Z}_{jk}(y,z)|^{3/2} do_z \Big)^{2/3} \|\pi_{2\ |\partial B_{R_n}}\|_3, \\
\mathcal{B}_{5,k}(y) &= \sum_{l=1}^{3} \Big(\int_{\partial B_{\mathbb{R}_n}} |\partial z_l \mathfrak{Z}_{jk}(y,z)|^{6/5} do_z \Big)^{5/6} \|u_{|\partial B_{R_n}}\|_6, \\
\mathcal{B}_{6,k}(y) &= \Big(\int_{\partial B_{R_n}} |y-z|^{-12/5} do_z \Big)^{5/6} \|u_{|\partial B_{R_n}}\|_6
\end{aligned}$$

for $k \in \{1,2,3\}$. As in the proof of [2, Theorem 4.6], we get

$$\begin{aligned}
&|\mathcal{B}_{1,k}(k)| \le \mathfrak{C}(|y|)R_n^{-1/3}, \quad |\mathcal{B}_{2,k}(y)| \le \mathfrak{C}(|y|)R_n^{-1}, \quad |\mathcal{B}_{5,k}(y)| \le \mathfrak{C}(|y|)R_n^{-5/6}, \\
&|\mathcal{B}_{6,k}(y)| \le \mathfrak{C}(|y|)R_n^{-1/2}.
\end{aligned}$$

Now applying Theorem 3.3, Lemma 2.2 and (3.6), we obtain

$$\begin{aligned}
\mathcal{B}_{3,k}(y) &\leq \mathfrak{C}(|y|)\bigg(\int_{\partial B_{R_n}} (|z|s_\tau(z))^{-\frac{3q}{4q-3}} do_z \bigg)^{\frac{4q-3}{3q}} \|\pi_1|\partial B_{R_n}\|_{3q/(3-q)} \\
&\leq \mathfrak{C}(|y|)R_n^{-1}\bigg(\int_{\partial B_{R_n}} |s_\tau(z)|^{-\frac{3q}{4q-3}} do_z \bigg)^{\frac{4q-3}{3q}} R_n^{-\frac{3-q}{3q}} \\
&\leq \mathfrak{C}(|y|)R_n^{-1}R_n^{\frac{4q-3}{3q}} R_n^{-\frac{3-q}{3q}} \leq \mathfrak{C}(|y|)R^{\frac{2q-6}{3q}}, \\
\mathcal{B}_{4,k}(y) &\leq \mathfrak{C}(|y|)\bigg(\int_{\partial B_{R_n}} (|z|s_\tau(z))^{-3/2} do_z \bigg)^{2/3} \|\pi_2|\partial B_{R_n}\|_3 \\
&\leq \mathfrak{C}(|y|)R_n^{-1}\bigg(\int_{\partial B_{R_n}} |s_\tau(z)|^{-3/2} do_z \bigg)^{2/3} R_n^{-1/3} \\
&\leq \mathfrak{C}(|y|)R^{-2/3}.
\end{aligned}$$

Thus we conclude that $U_{in,j}(y) \to 0$ for $n \to \infty$. Turning to $\mathfrak{R}_j(F_{|\mathfrak{D}_{R_n}})(y)$ and $\mathfrak{S}_j(\operatorname{div} u_{|\mathfrak{D}_{R_n}})(y)$ and applying (3.7), (3.8) and Lebesgue's theorem, it follows that

$$\mathfrak{R}_j(F_{|\mathfrak{D}_{R_n}})(y) \to \mathfrak{R}_j(F)|(y), \quad \mathfrak{S}_j(\operatorname{div} u_{|\mathfrak{D}_{R_n}})(y) \to \mathfrak{S}_j(\operatorname{div} u)(y)$$

for $n \to \infty$. Theorem 3.11 now follows with (3.9).

Now it is obvious that the theory in [1] – [4] remains valid when we drop the condition requiring that the pressure belongs to $L^2(B_S^c)$ for some $S > 0$ with $\overline{\mathfrak{D}} \subset B_S$. This does not mean that the theory in question now extends to Leray solutions, because in the linear case, we have to keep the assumption $(u, \pi) \in M_p$, and in the nonlinear case, we impose an analogous restriction. The next three theorems and Theorem 4.1 give the details. We begin by considering the decay of the velocity and its gradient in the linear case.

Theorem 3.12. *Let $p \in (1, \infty)$, $(u, \pi) \in M_p$. Put $F := L(u) + \nabla\pi$. Suppose there are numbers $S_1, S, \gamma \in (0, \infty)$, $A \in [2, \infty)$, $B \in \mathbb{R}$ such that $S_1 < S$,*

$$\overline{\mathfrak{D}} \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_1}, \quad u_{|B_S^c} \in L^6(B_S^c)^3, \quad \nabla u_{|B_S^c} \in L^2(B_S^c)^9,$$
$$A + \min\{1, B\} \geq 3, \quad |F(z)| \leq \gamma|z|^{-A}s_\tau(z)^{-B} \text{ for } z \in B_{S_1}^c.$$

Let $i, j \in \{1, 2, 3\}$, $y \in B_S^c$. Then

$$\begin{aligned}
|u_j(y)| \leq &\mathfrak{C}(S_1, S_1, A, B)(\gamma + \|F_{|\mathfrak{D}_{S_1}}\|_1 + \|\operatorname{div} u\|_1 \\
&+ \|u_{|\partial\mathfrak{D}}\|_1 + \|\nabla u_{|\partial\mathfrak{D}}\|_1 + \|\pi_{|\partial\mathfrak{D}}\|_1) \\
&(|y|s_\tau(y))^{-1} l_{A,B}(y)
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
|\partial_i u_j(y)| \leq &\mathfrak{C}(S, S_1, A, B)(\gamma + \|F_{|\mathfrak{D}_{S_1}}\|_1 \\
&+ \|u_{|\partial\mathfrak{D}}\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u_{|\partial\mathfrak{D}}\|_1 + \|\pi_{|\partial\mathfrak{D}}\|_1) \\
&(|y|_{S_\tau}(y))^{-3/2} s_\tau(y)^{\max(0, 7/2-A-B)} l_{A,B}(y),
\end{aligned} \tag{3.11}$$

where $l_{A,B}(y) := 1$ if $A + \min\{1, B\} > 3$, and $l_{A,B}(y) := \max(1, \ln|y|)$ if $A + \min\{1, B\} = 3$.

Proof Theorem 3.12 may be deduced from Theorem 3.11 in exactly same way as [[4], Theorem 6] from [2, Theorem 4.6].

Next we turn to the decay of derivatives of the velocity up to order 2.

Theorem 3.13. *Let $p \in (1, \infty)$, $(u, \pi) \in M_p$. Put $F := L(u) + \nabla\pi$. Suppose there are numbers S_1, $S \in (0, \infty)$, with $S_1 < S$,*

$$\overline{\mathfrak{D}} \cup \operatorname{supp}(F) \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_1}, \;\; u_{|B_S^c} \in L^6(B_S^c)^3, \;\; \nabla u_{|B_S^c} \in L^2(B_S^c)^9.$$

Let $\mathcal{E}_p : W^{2-1/p,p}(\partial\mathfrak{D}) \mapsto W^{2,p}(\mathfrak{D})$ be an extension operator with $\|\mathcal{E}_p(v)\|_{2,p} \le C_p\|v\|_{2-1/2,p}$ for $v \in W^{2-1/p,p}(\partial\mathfrak{D})$, for some $C_p > 0$.

Let $j \in \{1, 2, 3\}$, $y \in B_S^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$. Then

$$\begin{aligned} |\partial^\alpha u_j(y)| \le \mathfrak{C}(S, S_1) \cdot \big(\|F_{|\mathfrak{D}_{S_1}}\|_1 + \|div\, u\|_1 + \|\nabla u_{|\partial\mathfrak{D}}\|_1 + \\ + \|\pi_{|\partial\mathfrak{D}}\|_1 + C_p \cdot \|u_{|\partial\mathfrak{D}}\|_{2-1/p,p} \big) \cdot \big(|y| \cdot s_\tau(y) \big)^{-1-|\alpha|/2}. \end{aligned} \tag{3.12}$$

Proof Theorem 3.13 follows from Theorem 3.11 in exactly the same way as [[3], Theorem 1.2] follows from [2, Theorem 4.6].

Another result from [1] – [4] we extend here concerns an asymptotic profile of the velocity and its gradient in the linear case:

Theorem 3.14. *Let $p \in (1, \infty)$, $(u, \pi) \in M_p$, $S, S_1 \in (0, \infty)$ with $S_1 < S$. Put $F := L(u) + \nabla\pi$. Suppose that*

$$\overline{\mathfrak{D}} \cup \operatorname{supp}(F) \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_1}, \; u|_{B_S^c} \in L^6(B_S^c)^3, \; \nabla u_{|B_S^c} \in L^2(B_S^c)^9.$$

Then there are coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and functions $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in C^1(\overline{B_{S_1}}^c)$ such that for $j \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$,

$$\partial^\alpha u_j(y) = \sum_{k=1}^{3} \beta_k \partial_y^\alpha \mathfrak{Z}_{jk}(y, 0) + \Big(\int_{\partial\mathfrak{D}} u \cdot n^{(\mathfrak{D})} do_z + \int_{\mathfrak{D}_{S_1}} \operatorname{div} u \, dz \Big) \partial^\alpha E_{4j}(y) + \partial^\alpha \mathfrak{F}_j(y)$$

for $y \in \overline{B_{S_1}}^c$, and

$$\begin{aligned} |\partial^\alpha \mathfrak{F}_j(y)| \le & \mathfrak{C}(S, S_1)(\|F\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u_{|\partial\mathfrak{D}}\|_1 \\ & + \|\pi_{|\partial\mathfrak{D}}\|_1 + \|u_{|\partial\mathfrak{D}}\|_1)(|y| s_\tau(y))^{-3/2-|\alpha|/2} \quad \text{for } y \in B_S^c. \end{aligned}$$

Proof Theorem 3.14 may be deduced from Theorem 3.11 in the same way as [[3], Theorem 1.1] from [2, Theorem 4.6].

Remark: An explicit formula for β_i, $\mathfrak{F}_i$, $i = 1, .., 3$ is given in [3], page 473.

We may use Theorem 3.12 and 3.13 in order to derive a decay estimate as in (1.5) for Leray solutions of the linear problem (1.3). This may be done by considering the restriction of such a solution to $B_{S_0}^c$, for some $S_0 > 0$ sufficiently large. The idea is to apply Theorem 3.12 and 3.13 with $\mathfrak{D}$ replaced by B_{S_0}. In this way, the behaviour of the solution in question near the boundary of its domain, and the regularity of that boundary do not matter at all. A crucial technical result in this respect is the ensuing observation on interior regularity of generalized solutions to (1.3).

Theorem 3.15. *Let $U \subset \mathbb{R}^3$ be open and bounded, $p \in (1,\infty)$, $f \in L^p_{\mathrm{loc}}(\overline{U}^c)^3$, $u \in W^{1,2}_{\mathrm{loc}}(\overline{U}^c)^3$, $\pi \in L^p_{\mathrm{loc}}(\overline{U}^c)$,*

$$\int_{\overline{U}^c} (\nabla u \cdot \nabla \varphi + (\tau \partial_1 u - (\omega \times z) \cdot \nabla u + \omega \times u)\varphi - \pi \operatorname{div} \varphi - f\varphi)dx \ = 0 \tag{3.13}$$

for $\varphi \in C_0^\infty(\overline{U}^c)^3$, $\quad \operatorname{div} u = 0$.

Then $u \in W^{2,\min\{2,p\}}_{\mathrm{loc}}(\overline{U}^c)^3$, $\pi \in W^{1,\min\{2,p\}}_{\mathrm{loc}}(\overline{U}^c)$, and $L(u) + \nabla\pi = f$.

Proof This theorem follows from interior regularity of Stokes flows, as stated in [[18], Theorem IV.4.1]. Details of the argument may be found in the proof of [[2], Theorem 5.5].

Now we may establish (1.5) for Leray solutions of (1.3).

Corollary 3.16. *Let $U \subset \mathbb{R}^3$ be open and bounded. Let $p \in (1,\infty)$, $f \in L^p_{\mathrm{loc}}(\overline{U}^c)^3$, $\gamma, S_1 \in (0,\infty)$ with $\overline{U} \subset B_{S_1}$, $A \in [2,\infty)$, $B \in \mathbb{R}$ with $A+\min\{1,B\} > 3$, $|f(z)| \le \gamma \cdot |z|^{-A} s_\tau(z)^{-B}$ for $z \in B^c_{S_1}$.*

Let $u \in W^{1,1}_{\mathrm{loc}}(\overline{U}^c)^3$ with $u \in L^6(\overline{U}^c)^3$ and $\nabla u \in L^2(\overline{U}^c)^9$. Let $\pi \in L^2_{\mathrm{loc}}(\overline{U}^c)$, and suppose that (3.13) holds.

Choose some $S_0 \in (0, S_1)$ with $\overline{U} \subset B_{S_0}$, and let $S \in (S_1,\infty)$. Then, for $z \in B^c_S$, $1 \le i$, $j \le 3$, inequalities (3.10) and (3.11) are valid, but with $\mathfrak{D}$ replaced by B_{S_0}.

Now suppose that $\operatorname{supp}(f) \subset B_{S_1}$, put $s := s(p) := \min\{2,p\}$, and let $\mathcal{E}_s : W^{2-1/s,s}(\partial B_{S_0}) \mapsto W^{2,s}(B_{S_0})$ be a continuous extension operator. Let $C_p > 0$ be a constant with

$$\|\mathcal{E}_s(v)\|_{2,s} \le C_p \cdot \|v\|_{2-1/s,s} \text{ for } v \in W^{2-1/s,s}(\partial B_{S_0}).$$

Then, for $1 \le j \le 3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$, $x \in B^c_S$, inequality (3.12) holds with $\mathfrak{D}$ replaced by B_{S_0}, and with the norm $\| \ \|_{2-1/s,s}$ in the place of $\| \ \|_{2-1/p,p}$.

Proof Theorem 3.15 yields that $u \in W^{2,\min\{2,p\}}_{\mathrm{loc}}(\overline{U}^c)^3$, $\pi \in W^{1,\min\{2,p\}}_{\mathrm{loc}}(\overline{U}^c)$, and $L(u) + \nabla\pi = f$. We may conclude in particular that the pair $(u_{|B^c_{S_0}}, \pi_{|B^c_{S_0}})$ belongs to $M_{\min\{2,p\}}$, with $\overline{B_{S_0}}^c$ in the place of $\overline{\mathfrak{D}}^c$ as the domain of reference in the definition of $M_{\min\{2,p\}}$. Therefore the corollary follows from Theorem 3.12 and 3.13.

4 Nonlinear case

In this section, we show that the representation formula from [2, Theorem 5.5] pertaining to the velocity part of solutions to the nonlinear problem (1.2) remains valid even if the pressure does not belong to $L^2(B^c_S)$ for some $S > 0$ with $\overline{\mathfrak{D}} \subset B_S$.

Theorem 4.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\overline{\mathfrak{D}}^c)^3 \cap L^6(\overline{\mathfrak{D}}^c)^3$ with $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$.*

Let $p \in (1,\infty)$, $q \in (1,2)$, $f : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ a function with $f_{|\mathfrak{D}_T} \in L^p(\mathfrak{D}_T)^3$ for $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$, and $f_{|B^c_S} \in L^q(B^c_S)^3$ for some $S \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_S$.

Further assume that $u_{|\partial\mathfrak{D}} \in W^{2-1/p,p}(\partial\mathfrak{D})^3$ and that $\pi : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}$ is a function with $\pi_{|\mathfrak{D}_T} \in L^p(\mathfrak{D}_T)$ for T as above. Suppose that the pair (u,π) is a generalized solution of

(1.2), that is,

$$\int_{\overline{\mathfrak{D}}^c}\Big((\nabla u\cdot\nabla\varphi)+(\tau(u\cdot\nabla)u+\tau\partial_1 u-(\omega\times z)\cdot\nabla u+\omega\times u)\cdot\varphi+\pi\operatorname{div}\varphi\Big)dz$$
$$=\int_{\overline{\mathfrak{D}}^c} f\cdot\varphi dz,\ \textit{for}\ \varphi\in C_0^\infty(\overline{\mathfrak{D}}^c)^3,$$
$$\operatorname{div} u=0.$$

Then

$$u_j(y)=\mathfrak{R}_j(f-\tau(u\cdot\nabla)u)(y)+\mathfrak{B}_j(u,\pi)(y) \tag{4.1}$$

for $j\in\{1,2,3\}$ *and for a.e.* $y\in\overline{\mathfrak{D}}^c$*, where* $\mathfrak{B}_j(u,\pi)$ *was defined in (3.4).*

Note that in [2, Theorem 5.5], the assumption $u_{|\partial\mathfrak{D}}\in W^{2-1/p,p}(\partial\mathfrak{D})^3$ is missing.

Proof of Theorem 4.1 Since $u\in L^6(\overline{\mathfrak{D}}^c)^3$ and $\nabla u\in L^2(\overline{\mathfrak{D}}^c)^9$, we have $(u\cdot\nabla)u\in L^{3/2}(\overline{\mathfrak{D}}^c)^3$, hence $f-\tau\cdot u\cdot\nabla u_{|\mathfrak{D}_T}\in L^{\min\{p,3/2\}}(\mathfrak{D}_T)^3$ for T as in the theorem. By Theorem 3.15 we thus get $u\in W^{2,\min\{p,3/2\}}_{\text{loc}}(\overline{\mathfrak{D}}^c)^3$, $\pi\in W^{1,\min\{p,3/2\}}_{\text{loc}}(\overline{\mathfrak{D}}^c)$, so we may conclude that $(u,\pi)\in M_{\min\{p,3/2\}}$. Moreover $f-\tau\cdot u\cdot\nabla u_{|B_S^c}\in L^q(B_S^c)^3+L^{3/2}(B_S^c)^3$. Thus (4.1) follows from Theorem 3.11.

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