Two-sided bounds for eigenvalues of differential operators with applications to Friedrichs, Poincaré, trace and similar constants

Tomáš Vejchodský

Institute of Mathematics
Academy of Sciences
Czech Republic

Ivana Šebestová

Faculty of Mathematics and Physics
Charles University in Prague

Current Problems in Numerical Analysis, Prague, January 25, 2013

Outline

- Abstract theory
  - Hilbert space setting
  - eigenvalue problem
  - abstract complementarity estimate
- Application to Friedrichs' inequality
- Application to Poincaré inequality
- Application to trace inequality

Abstract setting

- V, H Hilbert spaces
- $\gamma : V \to H$ linear, continuous, compact

Eigenproblem: Find $\lambda_i \in \mathbb{R}$, $u_i \in V$, $u_i \neq 0$ such that

$$(u_i, v)_V = \lambda_i (\gamma u_i, \gamma v)_H \quad \forall v \in V$$

Properties:

- $\lambda_i > 0$ and $\gamma u_i \neq 0$
- $(\gamma u_i, \gamma u_j)_H = \delta_{ij} \quad \forall i, j = 1, 2, \dots$
- $\{\lambda_i : \lambda_i \leq M\}$ is finite for all $M > 0$
- $\lambda_1 = \inf_{v \in V, v \neq 0} \|v\|_V^2 / \|\gamma v\|_H^2$ is the smallest eigenvalue

Theorem (abstract inequality):
There exists $C_\gamma > 0$ such that $\|\gamma v\|_H \leq C_\gamma \|v\|_V \quad \forall v \in V$.
Moreover, $C_\gamma = \lambda_1^{-1/2}$ is optimal.

Example: Friedrichs' inequality

Setting

- $V = H_0^1(\Omega)$, $\quad (u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$
- $H = L^2(\Omega)$, $\quad (u, v)_H = \int_\Omega uv \, \mathrm{d}x$
- $\gamma : H^1(\Omega) \to L^2(\Omega)$ is identical (compact by Rellich theorem)

Abstract inequality

$\Rightarrow$ Exists $C_{\mathrm{F}} > 0$: $\quad \|\nabla v\|_{L^2(\Omega)} \leq C_{\mathrm{F}} \|v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$

$\Rightarrow$ Moreover, $C_{\mathrm{F}} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue of

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Upper bound on λ_1 (Galerkin method)

$V^h \subset V$

Discrete eigenproblem:
Find $\lambda_i^h \in \mathbb{R}$, $u_i^h \in V^h$, $u_i^h \neq 0$ such that

$$(u_i^h, v^h)_V = \lambda_i^h (\gamma u_i^h, \gamma v^h)_H \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$
Proof:

$$\lambda_1 = \inf_{0 \neq v \in V} \frac{\|v\|_V^2}{\|v\|_H^2} \leq \inf_{0 \neq v^h \in V^h} \frac{\|v\|_V^2}{\|v\|_H^2} = \lambda_1^h$$

$\square$

Lower bound on λ_1 (a priori-a posteriori inequalities)

Lemma (Parseval's identity): $\|\gamma u_*\|_H^2 = \sum_{i=1}^{\infty} |(\gamma u_*, \gamma u_i)_H|^2 \quad \forall u_* \in V$

Theorem (Kuttler, Sigillito, 1978):

- $u_* \in V$, $\lambda_* \in \mathbb{R}$ arbitrary
- $w \in V: \quad (w,v)_V = (u_*,v)_V - \lambda_*(\gamma u_*, \gamma v)_H \quad \forall v \in V$

Then

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|\gamma w\|_H}{\|\gamma u_*\|_H}$$

Proof:

$$\begin{aligned}
\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right|^2 \|\gamma u_*\|_H^2 &\leq \sum_{i=1}^{\infty} \left| \frac{\lambda_i - \lambda_*}{\lambda_i} (\gamma u_*, \gamma u_i)_H \right|^2 \\
&= \sum_{i=1}^{\infty} \left| \frac{(u_i, u_*)_V}{\lambda_i} - \frac{(u_* - w, u_i)_V}{\lambda_i} \right|^2 \\
&= \sum_{i=1}^{\infty} \left| \frac{(w, u_i)_V}{\lambda_i} \right|^2 = \sum_{i=1}^{\infty} |(\gamma w, \gamma u_i)_H|^2 = \|\gamma w\|_H^2
\end{aligned}$$

Abstract complementarity

Theorem: If

- $u_* \in V$, $\lambda_* \in \mathbb{R}$ arbitrary
- $w \in V: \quad (w, v)_V = (u_*, v)_V - \lambda_*(\gamma u_*, \gamma v)_H \quad \forall v \in V$
- $\left|\frac{\lambda_1 - \lambda_*}{\lambda_1}\right| \leq \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \quad \forall i = 1, 2, \dots$.
- $\|w\|_V \leq A + C_\gamma B, \quad B < \lambda_* \|\gamma u_*\|_H$

then

$$X_2^2 \leq \lambda_1,$$

$$X_2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4(\lambda_* - \beta)}\right), \quad \alpha = \frac{A}{\|\gamma u_*\|_H}, \quad \beta = \frac{B}{\|\gamma u_*\|_H}.$$

Proof:

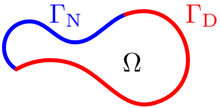
$$\lambda_* C_\gamma^2 - 1 = \frac{\lambda_* - \lambda_1}{\lambda_1} \leq \min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \leq \frac{\|\gamma w\|_H}{\|\gamma u_*\|_H} \leq C_\gamma \frac{\|w\|_V}{\|\gamma u_*\|_H} \leq C_\gamma \alpha + C_\gamma^2 \beta$$

$$\Leftrightarrow \quad 0 \leq C_\gamma^2(\beta - \lambda_*) + C_\gamma \alpha + 1 \quad \Rightarrow \quad C_\gamma \leq 1/X_2 \qquad \square$$

Application to Friedrichs' inequality

Notation and assumptions

- $\mathcal{A} \in [L^\infty(\Omega)]^{d\times d}$ symmetric, $\quad c \in L^\infty(\Omega), \quad \alpha \in L^\infty(\Gamma_\mathrm{N})$
- $\boldsymbol{\xi}^T \mathcal{A}(x)\boldsymbol{\xi} \geq C|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,$ a.e. $x \in \Omega,$
- $c \geq 0, \quad \alpha \geq 0$
- $H^1_{\Gamma_\mathrm{D}}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_\mathrm{D}\}$
- $a(u,v) = \int_\Omega (\nabla u)^T \mathcal{A} \nabla v \,\mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} \alpha uv \,\mathrm{d}s.$
- $\|v\|_a^2 = a(v,v)$
- $a(\cdot,\cdot)$ scalar product in $H^1_{\Gamma_\mathrm{D}}(\Omega)$



Friedrichs' inequality – application of abstract results

Setting

- $V = H^1_{\Gamma_\mathrm{D}}(\Omega), \quad (u,v)_V = a(u,v)$
- $H = L^2(\Omega), \quad (u,v)_H = (u,v)$
- $\gamma : H^1_{\Gamma_\mathrm{D}}(\Omega) \to L^2(\Omega)$ identity mapping

Conclusions

- $\exists C_\mathrm{F} > 0 : \quad \|v\|_{L^2(\Omega)} \le C_\mathrm{F} \|v\|_a \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$
- $C_\mathrm{F} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue:

$$\lambda_i \in \mathbb{R},\ 0 \neq u_i \in H^1_{\Gamma_\mathrm{D}}(\Omega) : \quad a(u_i, v) = \lambda_i (u_i, v) \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$$

Friedrichs' inequality – complementarity

- $\mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{q} \in [L^2(\Omega)]^d : \mathrm{div}\, \mathbf{q} \in L^2(\Omega)\}$
- $\|\mathbf{q}\|_{\mathcal{A}}^2 = (\mathcal{A}\mathbf{q}, \mathbf{q})$ a norm in $[L^2(\Omega)]^d$

Theorem: If

- $\lambda_* \in \mathbb{R}, \quad u_* \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$
- $w \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad a(w, v) = a(u_*, v) - \lambda_*(u_*, v) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$

Then

$$\|w\|_a \leq \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + C_{\mathrm{F}} \|\lambda_* u_* - c u_* + \mathrm{div}\, \mathbf{q}\|_{L^2(\Omega)} \quad \forall \mathbf{q} \in W,$$

where $W = \{\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{q} \cdot \mathbf{n} = -\alpha u_* \text{ on } \Gamma_{\mathrm{N}}\}$

Proof: $\mathbf{q} \in W$, $v = w$

$$\begin{aligned}
\|w\|_a^2 &= (\mathcal{A}\nabla u_*, \nabla w) + (c u_*, w) + (\alpha u_*, w)_{\Gamma_{\mathrm{N}}} - \lambda_*(u_*, w) \\
&\qquad - (\mathbf{q}, \nabla w) - (\mathrm{div}\, \mathbf{q}, w) + (\mathbf{q} \cdot \mathbf{n}, w)_{\Gamma_{\mathrm{N}}} \\
&= \left(\mathcal{A}(\nabla u_* - \mathcal{A}^{-1}\mathbf{q}), \nabla w\right) - (\lambda_* u_* - c u_* + \mathrm{div}\, \mathbf{q}, w) \\
&\leq \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} \underbrace{\|\nabla w\|_{\mathcal{A}}}_{\|w\|_a} + C_{\mathrm{F}} \|\lambda_* u_* - c u_* + \mathrm{div}\, \mathbf{q}\|_{L^2(\Omega)} \|w\|_a.
\end{aligned}$$

Choice of $\mathbf{q} \in W$

- $\overline{\mathbf{q}} \in W$ arbitrary
- $W = \overline{\mathbf{q}} + W_0, \quad W_0 = \{\mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega) : \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{N}}\}$
- $W_0^h \subset W_0$ Raviart-Thomas finite element space
- $\|w\|_a \leq A + C_{\mathrm{F}} B \leq (1+\varrho^{-1})A^2 + (1+\varrho)C_{\mathrm{F}}^2 B^2 \quad \forall \varrho > 0$
- Minimize

$$(1+\varrho^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}^2 + (1+\varrho)(\lambda_1^h)^{-1}\|\lambda_1^h u_1^h - c u_1^h + \operatorname{div}\mathbf{q}\|_{L^2(\Omega)}^2$$

over $\overline{\mathbf{q}} + W_0^h$ with a fixed $\varrho > 0$.

- Equivalent to

$$\mathbf{q}_0^h \in W_0^h : \quad \mathcal{B}(\mathbf{q}_0^h, \mathbf{w}_0^h) = \mathcal{F}(\mathbf{w}_0^h) - \mathcal{B}(\overline{\mathbf{q}}, \mathbf{w}_0^h) \quad \forall \mathbf{w}_0^h \in W_0^h$$

where

$$\mathcal{B}(\mathbf{q}, \mathbf{w}) = (\operatorname{div}\mathbf{q}, \operatorname{div}\mathbf{w}) + \frac{\lambda_1^h}{\varrho}(\mathcal{A}^{-1}\mathbf{q}, \mathbf{w}),$$

$$\mathcal{F}(\mathbf{w}) = \frac{\lambda_1^h}{\varrho}(\nabla u_1^h, \mathbf{w}) - (\lambda_1^h u_1^h - c u_1^h, \operatorname{div}\mathbf{w})$$

Friedrichs' inequality – numerical example

- $\Omega = (-1,1)^2$
- $\Gamma_\mathrm{N} = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = 1 \text{ and } -1 < x_2 < 1\}$
- $\Gamma_\mathrm{D} = \partial\Omega \setminus \overline{\Gamma}_\mathrm{N}$
- $c = 0, \quad \alpha = 0$
- $\mathcal{A}(x_1,x_2) = \begin{cases} 1 & \text{for } x_1 x_2 \le 0 \\ \mathcal{A}^* & \text{for } x_1 x_2 > 0 \end{cases}$
- $u_1^h \in V^h = \{v^h \in H^1_{\Gamma_\mathrm{D}}(\Omega) : v^h|_K \in P^1(K),\ \forall K \in \mathcal{T}_h\}$
- $\mathbf{q}^h \in W_0^h = \{\mathbf{w}_h \in W_0 : \mathbf{w}_h \in [P^2(K)]^2,\ \forall K \in \mathcal{T}_h\}$
- $\eta_K^2 = (1+\varrho^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}^h\|^2_{\mathcal{A},K} + (1+\varrho)(\lambda_1^h)^{-1}\|\lambda_1^h u_1^h - c u_1^h + \operatorname{div}\mathbf{q}^h\|^2_{L^2(K)}$
- Adaptive algorithm driven by η_K
- Stop if $E_\mathrm{REL} = \dfrac{C_\mathrm{F}^\mathrm{up} - C_\mathrm{F}^\mathrm{low}}{C_\mathrm{F}^\mathrm{avg}} \le E_\mathrm{TOL}$

Friedrichs' inequality – numerical results

$\mathcal{A}^*$	C_F^{low}	C_F^{up}	E_{REL}	N_{DOF}
0.001	9.0086	9.0939	0.94 %	4 832
0.01	2.8697	2.8971	0.95 %	5 003
0.1	1.0035	1.0124	0.88 %	7 866
1	0.5693	0.5743	0.86 %	4 802
10	0.3173	0.3201	0.88 %	7 866
100	0.2870	0.2897	0.95 %	5 003
1000	0.2849	0.2876	0.94 %	4 832

Note: $C_{\text{F}} = 4/(\pi\sqrt{5}) \approx 0.5694$ for $\mathcal{A}^* = 1$.

Application to Poincaré inequality

Setting

- $c = 0, \quad \alpha = 0, \quad \Gamma_\mathrm{D} = \emptyset$
- $V = \overline{H}^1(\Omega) = \{v \in H^1(\Omega) : \int_\Omega v \,\mathrm{d}x = 0\}$
- $(u,v)_V = a(u,v) = (\mathcal{A}\nabla u, \nabla v)$
- $H = L^2(\Omega), \quad (u,v)_H = (u,v)$
- $\gamma : H^1_{\Gamma_\mathrm{D}}(\Omega) \to L^2(\Omega)$ identity mapping

Conclusions

- $\exists C_\mathrm{P} > 0 : \quad \|v\|_{L^2(\Omega)} \leq C_\mathrm{P} \|v\|_a \quad \forall v \in \overline{H}^1(\Omega)$
- $C_\mathrm{P} = \lambda_2^{-1/2}$, where λ_2 is the smallest positive eigenvalue:

$$\lambda_i \in \mathbb{R},\ 0 \neq u_i \in H^1(\Omega) : \quad a(u_i, v) = \lambda_i (u_i, v) \quad \forall v \in H^1(\Omega)$$

Complementarity

$$\|w\|_a \leq \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_\mathcal{A} + C_\mathrm{P} \|\lambda_* u_* + \operatorname{div} \mathbf{q}\|_{L^2(\Omega)} \quad \forall \mathbf{q} \in W_0,$$

Poincaré inequality – numerical results

$\mathcal{A}^*$	$C_{\mathrm{P}}^{\mathrm{low}}$	$C_{\mathrm{P}}^{\mathrm{up}}$	E_{REL}	N_{DOF}
0.001	14.2390	14.3690	0.91 %	3 400
0.01	4.5199	4.5623	0.93 %	3 510
0.1	1.4849	1.4989	0.94 %	4 382
1	0.6365	0.6424	0.92 %	3 009
10	0.4696	0.4740	0.94 %	4 382
100	0.4520	0.4562	0.93 %	3 510
1000	0.4503	0.4544	0.91 %	3 400

Note: $C_{\mathrm{P}} = 2/\pi \approx 0.6366$ for $\mathcal{A}^* = 1$

Application to trace inequality

Setting

- $\text{meas}_{d-1} \Gamma_\mathrm{N} > 0$
- $V = H^1_{\Gamma_\mathrm{D}}(\Omega), \quad (u,v)_V = a(u,v)$
- $H = L^2(\Gamma_\mathrm{N}), \quad (u,v)_H = (u,v)_{\Gamma_\mathrm{N}}$
- $\gamma : H^1_{\Gamma_\mathrm{D}}(\Omega) \to L^2(\Gamma_\mathrm{N})$ trace operator, compact [Biegert, 2009]

Conclusions

- $\exists C_\mathrm{T} > 0 : \quad \|v\|_{L^2(\Gamma_\mathrm{N})} \le C_\mathrm{T} \|v\|_a \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$
- $C_\mathrm{T} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue:

$$\lambda_i \in \mathbb{R},\ 0 \ne u_i \in H^1_{\Gamma_\mathrm{D}}(\Omega) : \quad a(u_i, v) = \lambda_i (u_i, v)_{\Gamma_\mathrm{N}} \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$$

Complementarity

$$\|w\|_a \le \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_\mathcal{A} + C_\mathrm{F} \|c u_* - \operatorname{div} \mathbf{q}\|_{L^2(\Omega)} + C_\mathrm{T} \|\alpha u_* - \lambda_* u_* + \mathbf{q}\cdot\mathbf{n}\|_{L^2(\Gamma_\mathrm{N})}$$

$$\forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Trace inequality – numerical results

$\mathcal{A}^*$	C_T^{low}	C_T^{up}	E_{REL}	N_{DOF}
0.001	17.8110	17.9760	0.92 %	5 523
0.01	5.6490	5.7047	0.98 %	5 418
0.1	1.8433	1.8593	0.86 %	7 775
1	0.7963	0.8033	0.88 %	5 499
10	0.5829	0.5880	0.86 %	7 775
100	0.5649	0.5705	0.98 %	5 421
1000	0.5632	0.5685	0.92 %	5 523

Note: $C_T = \sqrt{2/(\pi \coth \pi)} \approx 0.7964$ for $\mathcal{A}^* = 1$

Conclusions

- General method for two-sided bounds of principal eigenvalues
- Straightforward applications
- Guaranteed bounds if
  - no round-off errors
  - all integrals evaluated exactly
  - domain Ω represented exactly
  - Galerkin method requires exact solution of matrix eigenproblem, but complementarity does not.
- Crucial assumption: $\left|\dfrac{\lambda_1 - \lambda_*}{\lambda_1}\right| \leq \left|\dfrac{\lambda_i - \lambda_*}{\lambda_i}\right| \quad \forall i = 1, 2, \dots .$
- Generalizations to nonlinear and nonsymmetric problems?

Thank you for your attention

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