Computable upper bounds on Friedrichs' and trace constants

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Outline

- Friedrichs', trace, and similar inequalities
- Relationship with the smallest eigenvalue
- Rayleigh–Ritz approximation
- Method of a priori–a posteriori inequalities
- Application for the trace constant
- Examples

Inequalities

Friedrichs' inequality:

$$\|v\|_{0,\Omega} \leq C_{\mathrm{F}} \|\nabla v\|_{0,\Omega} \quad \forall v \in H_0^1(\Omega)$$

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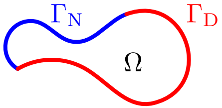

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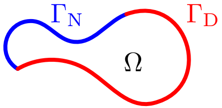

Poincaré inequality: ...

Korn's inequality: ...

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$$u = 0 \quad \text{on } \partial\Omega$$

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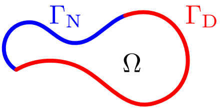

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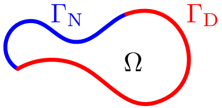

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General form:

$$|\!|\!| v |\!|\!|_c \leq C_{ac} |\!|\!| v |\!|\!|_a \quad \forall v \in V$$

V Hilbert with inner product $a(\cdot,\cdot)$
$|\!|\!| v |\!|\!|_a^2 = a(v,v)$ norm on V
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Relation with the smallest eigenvalue

Space: $V = H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_\mathrm{D}\}$

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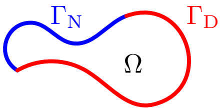

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Theorem:

$$C_{\mathrm{T}}^2 = \frac{1}{\lambda_1}, \quad \lambda_1 = \min_i \lambda_i$$

Steklov eigenproblem:

$$\begin{aligned} -\Delta u_i &= 0 && \text{in } \Omega \\ \partial u_i / \partial n &= \lambda_i u_i && \text{on } \Gamma_{\mathrm{N}} \\ u_i &= 0 && \text{on } \Gamma_{\mathrm{D}} \end{aligned}$$

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Proof:

Weak formulation: $u_i \in V : \quad (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_{\mathrm{N}}} \quad \forall v \in V$

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|^2_{0,\Omega}}{\|v\|^2_{0,\Gamma_{\mathrm{N}}}} \quad \Leftrightarrow \quad \frac{1}{\lambda_1} = \sup_{v \in V} \frac{\|v\|^2_{0,\Gamma_{\mathrm{N}}}}{\|\nabla v\|^2_{0,\Omega}} \qquad \square$$

Rayleigh–Ritz approximation of λ_1

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Rayleigh–Ritz method: $V^h \subset V$, $\dim V^h < \infty$

$$u_i^h \in V^h : \quad (\nabla u_i^h, \nabla v^h)_\Omega = \lambda_i^h \langle u_i^h, v^h \rangle_{\Gamma_\mathrm{N}} \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$

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Theorem: $\lambda_1 \leq \lambda_1^h$

Proof:

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_{0,\Omega}^2}{\|v\|_{0,\Gamma_N}^2} \leq \inf_{v^h \in V^h} \frac{\|\nabla v\|_{0,\Omega}^2}{\|v\|_{0,\Gamma_N}^2} = \lambda_1^h$$

□

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□

Corollary: $C_\mathrm{T}^h \leq C_\mathrm{T}$

Method of *a priori-a posteriori inequalities*

Theorem (Kuttler and Sigillito, 1978):

- H a separable Hilbert space
- $A : H \mapsto H$ symmetric with pure point spectrum
- $Au_i = \lambda_i u_i, \quad \{u_i\}$ ON basis of H
- $\lambda_* \in \mathbb{R}$, $u_* \in H$ arbitrary, $\|u_*\|_H = 1$
- $w \in H$ such that $Aw = Au_* - \lambda_* u_*$

Then

$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \le \|w\|_H$$

Proof

- $Au_i = \lambda_i u_i, \qquad \lambda_* \in \mathbb{R},\ u_* \in H$ arbitrary, $\|u_*\|_H = 1$
- $w \in H: \quad Aw = Au_* - \lambda_* u_*$
- $1 = \|u_*\|_H^2 = \sum_i |\langle u_*, u_i\rangle_H|^2$

$$
\begin{aligned}
\min_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right|^2 \|u_*\|_H^2 &\le \sum_i \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\langle u_*, u_i\rangle_H\right|^2 \\
&= \sum_i \frac{1}{\lambda_i^2} \left|\langle u_*, \lambda_i u_i\rangle_H - \langle \lambda_* u_*, u_i\rangle_H\right|^2 \\
&= \sum_i \frac{1}{\lambda_i^2} \left|\langle u_*, Au_i\rangle_H - \langle Au_*, u_i\rangle_H + \langle Aw, u_i\rangle_H\right|^2 \\
&= \sum_i \frac{1}{\lambda_i^2} \left|\langle Aw, u_i\rangle_H\right|^2 = \sum_i \frac{1}{\lambda_i^2} \left|\langle w, Au_i\rangle_H\right|^2 \\
&= \sum_i \frac{1}{\lambda_i^2} \left|\langle w, \lambda_i u_i\rangle_H\right|^2 = \sum_i \left|\langle w, u_i\rangle_H\right|^2 = \|w\|_H^2
\end{aligned}
$$

Application to C_T

Steklov eigenproblem: $u_i \in V: \quad (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v \rangle_{\Gamma_\mathrm{N}} \quad \forall v \in V$

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Residual: $w \in V: \quad (\nabla w, \nabla v)_\Omega = (\nabla u_1^h, \nabla v)_\Omega - \lambda_1^h \langle u_1^h, v \rangle_{\Gamma_\mathrm{N}} \quad \forall v \in V$

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Theorem $\Rightarrow$ $\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_\mathrm{N}}$

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Lemma: $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\|\nabla w\|_{0,\Omega} \leq \|\nabla u_1^h - \mathbf{q}\|_{0,\Omega} + C_\mathrm{F} \|\operatorname{div} \mathbf{q}\|_{0,\Omega} + C_\mathrm{T} \|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_\mathrm{N}}$$

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Proof:

$$\begin{aligned} \|\nabla w\|^2 &= (\nabla u_1^h, \nabla w) - \langle \lambda_1^h u_1^h, w \rangle - (\mathbf{q}, \nabla w) - (\operatorname{div} \mathbf{q}, w) + \langle \mathbf{q} \cdot \mathbf{n}, w \rangle \\ &= (\nabla u_1^h - \mathbf{q}, \nabla w) - (\operatorname{div} \mathbf{q}, w) + \langle \mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h, w \rangle \\ &\le \|\nabla u_1^h - \mathbf{q}\| \|\nabla w\| + \|\operatorname{div} \mathbf{q}\| \underbrace{\|w\|}_{\le C_\mathrm{F} \|\nabla w\|} + \|\mathbf{q} \cdot \mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_\mathrm{N}} \underbrace{\|w\|_{0,\Gamma_\mathrm{N}}}_{\le C_\mathrm{T} \|\nabla w\|} \end{aligned}$$

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Proof:

$$\begin{aligned}\|\nabla w\|^2 &= (\nabla u_1^h, \nabla w) - \langle \lambda_1^h u_1^h, w\rangle - (\mathbf{q}, \nabla w) - (\operatorname{div}\mathbf{q}, w) + \langle \mathbf{q}\cdot\mathbf{n}, w\rangle \\ &= (\nabla u_1^h - \mathbf{q}, \nabla w) - (\operatorname{div}\mathbf{q}, w) + \langle \mathbf{q}\cdot\mathbf{n} - \lambda_1^h u_1^h, w\rangle \\ &\leq \|\nabla u_1^h - \mathbf{q}\| \|\nabla w\| + \|\operatorname{div}\mathbf{q}\| \underbrace{\|w\|}_{\leq C_\mathrm{F}\|\nabla w\|} + \|\mathbf{q}\cdot\mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_\mathrm{N}} \underbrace{\|w\|_{0,\Gamma_\mathrm{N}}}_{\leq C_\mathrm{T}\|\nabla w\|}\end{aligned}$$

Application to C_T

Steklov eigenproblem: $u_i \in V: \quad (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v\rangle_{\Gamma_\mathrm{N}} \quad \forall v \in V$

Rayleigh–Ritz approximation: $\lambda_1^h \in \mathbb{R}$, $u_1^h \in V$, $\|u_1^h\|_{0,\Gamma_\mathrm{N}} = 1$

Residual: $w \in V: \quad (\nabla w, \nabla v)_\Omega = (\nabla u_1^h, \nabla v)_\Omega - \lambda_1^h \langle u_1^h, v\rangle_{\Gamma_\mathrm{N}} \quad \forall v \in V$

Theorem $\Rightarrow$ $\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \le \|w\|_{0,\Gamma_\mathrm{N}} \le C_\mathrm{T} \|\nabla w\|_{0,\Omega}$

Lemma: $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$

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&= (\nabla u_1^h - \mathbf{q}, \nabla w) - (\operatorname{div}\mathbf{q}, w) + \langle \mathbf{q}\cdot\mathbf{n} - \lambda_1^h u_1^h, w\rangle \\
&\le \|\nabla u_1^h - \mathbf{q}\| \|\nabla w\| + \|\operatorname{div}\mathbf{q}\| \underbrace{\|w\|}_{\le C_\mathrm{F}\|\nabla w\|} + \|\mathbf{q}\cdot\mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_\mathrm{N}} \underbrace{\|w\|_{0,\Gamma_\mathrm{N}}}_{\le C_\mathrm{T}\|\nabla w\|}
\end{aligned}$$

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Theorem $\Rightarrow$ $\min_i \left| \dfrac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_\mathrm{N}} \leq C_\mathrm{T} \|\nabla w\|_{0,\Omega} \leq C_\mathrm{T}\alpha + C_\mathrm{T}^2 \beta$

Lemma: $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\|\nabla w\|_{0,\Omega} \leq \underbrace{\|\nabla u_1^h - \mathbf{q}\|_{0,\Omega} + C_\mathrm{F} \|\operatorname{div} \mathbf{q}\|_{0,\Omega}}_{\alpha} + C_\mathrm{T} \underbrace{\|\mathbf{q}\cdot\mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_\mathrm{N}}}_{\beta}$$

Application to C_T

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Theorem $\Rightarrow$ $\displaystyle \min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_{\mathrm{N}}} \leq C_{\mathrm{T}} \|\nabla w\|_{0,\Omega} \leq C_{\mathrm{T}}\alpha + C_{\mathrm{T}}^2 \beta$

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Assume: λ_1^h is closest to λ_1. Use: $C_{\mathrm{T}} = 1/\sqrt{\lambda_1}$

$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}}\alpha + \frac{1}{\lambda_1}\beta$$

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$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}}\alpha + \frac{1}{\lambda_1}\beta$$

$\Leftrightarrow$ $X_2^2 \leq \lambda_1$, where $X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \alpha\right)/2$

Application to C_T

Steklov eigenproblem: $u_i \in V: \quad (\nabla u_i, \nabla v)_\Omega = \lambda_i \langle u_i, v\rangle_{\Gamma_\mathrm{N}} \quad \forall v \in V$

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Theorem $\Rightarrow$ $\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \|w\|_{0,\Gamma_\mathrm{N}} \leq C_\mathrm{T} \|\nabla w\|_{0,\Omega} \leq C_\mathrm{T}\alpha + C_\mathrm{T}^2 \beta$

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$\Leftrightarrow$ $C_\mathrm{T} \leq 1/X_2$

Computing $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$

Approximately minimize: $\eta(\mathbf{q}) \approx \alpha + C_{\mathrm{T}}\beta \qquad C_{\mathrm{T}} \approx C_{\mathrm{T},h} = (\lambda_1^h)^{-1/2}$

$C_{\mathrm{F}} \approx C_{\mathrm{F},h}$

$$\eta^2(\mathbf{q}) = \Big(\underbrace{\|\nabla u_1^h - \mathbf{q}\|_{0,\Omega}}_{A} + \underbrace{C_{\mathrm{F},h}\|\operatorname{div}\mathbf{q}\|_{0,\Omega}}_{B} + \underbrace{C_{\mathrm{T},h}\|\mathbf{q}\cdot\mathbf{n} - \lambda_1^h u_1^h\|_{0,\Gamma_{\mathrm{N}}}}_{C} \Big)^2$$

$$\leq \underbrace{(1+\varrho^{-1})(1+\sigma^{-1})A^2 + (1+\varrho^{-1})(1+\sigma)B^2 + (1+\varrho)C^2}_{\widetilde{\eta}^2(\varrho,\sigma,\mathbf{q})} \qquad \forall \varrho > 0, \sigma > 0$$

Minimize $\widetilde{\eta}(\varrho,\sigma,\mathbf{q})^2$ over $W_h \subset \mathbf{H}(\mathrm{div},\Omega)$: Find $\mathbf{q}_h \in W_h$:

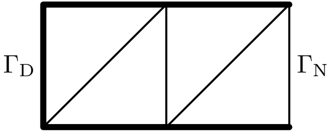
$$\frac{C_{\mathrm{F},h}^2 \lambda_1^h}{\varrho}(\operatorname{div}\mathbf{q}_h, \operatorname{div}\boldsymbol{\psi}_h)_\Omega + \frac{\lambda_1^h}{\varrho\sigma}(\mathbf{q}_h, \boldsymbol{\psi}_h)_\Omega + \frac{1}{1+\sigma}\langle \mathbf{q}_h\cdot\mathbf{n}, \boldsymbol{\psi}_h\cdot\mathbf{n}\rangle_{\Gamma_{\mathrm{N}}}$$

$$= \frac{\lambda_1^h}{\varrho\sigma}(\nabla u_1^h, \boldsymbol{\psi}_h)_\Omega + \frac{1}{1+\sigma}\langle \lambda_1^h u_1^h, \boldsymbol{\psi}_h\cdot\mathbf{n}\rangle_{\Gamma_{\mathrm{N}}} \qquad \forall \boldsymbol{\psi} \in W_h$$

Solve by standard Raviart-Thomas finite elements.

Example 1

$$\Omega = (0,2) \times (0,1)$$
$$\Gamma_{\mathrm{D}} = \{(x,y) \in \partial\Omega : x < 2\}$$
$$\Gamma_{\mathrm{N}} = \{(x,y) \in \partial\Omega : x = 2\}$$



$$\|v\|_{0,\Gamma_{\mathrm{N}}} \leq C_{\mathrm{T}} \|\nabla v\|_{0,\Omega} \qquad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

$$C_{\mathrm{T}} = (\pi \coth 2\pi)^{-1/2}$$

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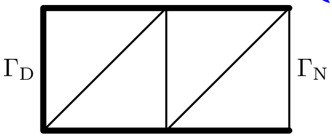
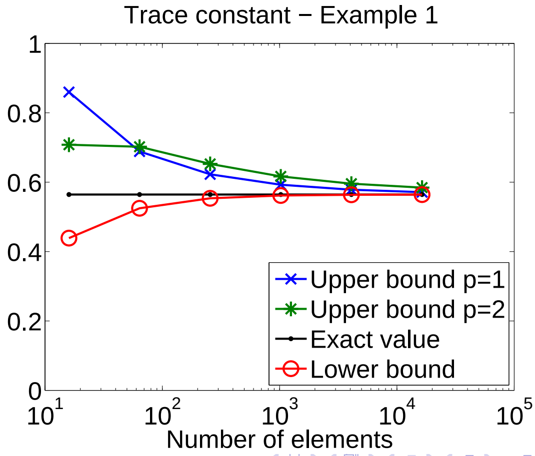


$$\|v\|_{0,\Gamma_{\mathrm{N}}} \leq C_{\mathrm{T}} \|\nabla v\|_{0,\Omega} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

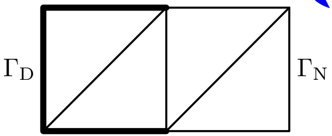
$$C_{\mathrm{T}} = (\pi \coth 2\pi)^{-1/2}$$

$$C_{\mathrm{T}}^{\mathrm{up}} = 0.571\,154$$
$$C_{\mathrm{T}} \approx 0.564\,188$$
$$C_{\mathrm{T}}^{\mathrm{low}} = 0.564\,017$$



Example 2

$$\Omega = (0,2) \times (0,1)$$
$$\Gamma_{\mathrm{D}} = \{(x,y) \in \partial\Omega : x < 1\}$$
$$\Gamma_{\mathrm{N}} = \{(x,y) \in \partial\Omega : x > 1\}$$



$$\|v\|_{0,\Gamma_{\mathrm{N}}} \leq C_{\mathrm{T}} \|\nabla v\|_{0,\Omega} \qquad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

$$C_{\mathrm{T}} = ?$$

Example 2

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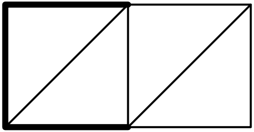
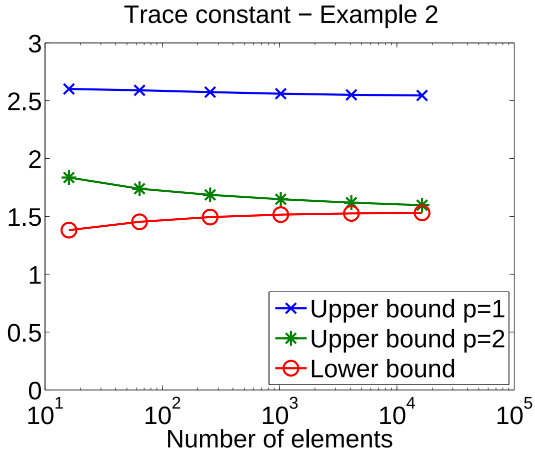


$$\|v\|_{0,\Gamma_{\mathrm{N}}} \le C_{\mathrm{T}} \|\nabla v\|_{0,\Omega} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

$C_{\mathrm{T}} = ?$

$C_{\mathrm{T}}^{\mathrm{up}} = 1.596\,98$
$C_{\mathrm{T}}^{\mathrm{low}} = 1.530\,59$



Conclusions

- Practical method
- Upper bound
- General: trace, Friedrichs', Poincaré, Korn's, ... constants
- Computationally demanding
- Exact representation of the domain Ω

Thank you for your attention

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