Two-sided bounds of eigenvalues with applications to trace inequalities

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Introduction

Friedrichs' inequality:

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{F}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$

$$\|v\|^2_{L^2(\Omega)} \leq C^2_{\mathrm{F}} \left(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(B)} \right) \quad \forall v \in H^1(\Omega),\ B \subset \Omega$$

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{F}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma \subset \partial\Omega$$

Poincaré inequality:

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{P}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega),\ \int_\Omega v = 0$$

Trace inequality:

$$\|\gamma v\|^2_{L^2(\partial\Omega)} \leq C^2_{\mathrm{T}} \left(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right) \quad \forall v \in H^1(\Omega)$$

$$\|\gamma v\|_{L^2(\Gamma)} \leq C_{\mathrm{T}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega),\ \Gamma \subset \partial\Omega,\ v = 0 \text{ on } \partial\Omega \setminus \Gamma$$

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Outline

- Abstract theory
  - Hilbert space setting
  - eigenvalue problem
  - abstract complementarity estimate
- Application to Friedrichs' inequality
- Application to trace inequality

Abstract setting

- V, H Hilbert spaces
- $\gamma : V \to H$ linear, continuous, compact

Eigenproblem: Find $\lambda_i \in \mathbb{R}$, $u_i \in V$, $u_i \neq 0$ such that

$$(u_i, v)_V = \lambda_i (\gamma u_i, \gamma v)_H \quad \forall v \in V$$

Properties:

- $\lambda_i > 0$ and $\gamma u_i \neq 0$

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Properties:

- $\{\lambda_i : \lambda_i \leq M\}$ is finite for all $M > 0$

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Properties:

- $\lambda_1 = \inf_{v \in V, v \neq 0} \|v\|_V^2 / \|\gamma v\|_H^2$ is the smallest eigenvalue

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Theorem (abstract inequality):
There exists $C_\gamma > 0$ such that $\|\gamma v\|_H \leq C_\gamma \|v\|_V \quad \forall v \in V$.
Moreover, $C_\gamma = \lambda_1^{-1/2}$ is optimal.

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Moreover, $C_\gamma = \lambda_1^{-1/2}$ is optimal.

Proof: $\lambda_1 \leq \dfrac{\|v\|_V^2}{\|\gamma v\|_H^2} \quad \Leftrightarrow \quad \|\gamma v\|_H \leq \dfrac{1}{\sqrt{\lambda_1}} \|v\|_V \quad \forall v \in V$ □

Upper bound on λ_1 (Galerkin method)

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$V^h \subset V$

Discrete eigenproblem:
Find $\lambda_i^h \in \mathbb{R}$, $u_i^h \in V^h$, $u_i^h \neq 0$ such that

$$(u_i^h, v^h)_V = \lambda_i^h (\gamma u_i^h, \gamma v^h)_H \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$
Proof:

$$\lambda_1 = \inf_{0 \neq v \in V} \frac{\|v\|_V^2}{\|v\|_H^2} \leq \inf_{0 \neq v^h \in V^h} \frac{\|v\|_V^2}{\|v\|_H^2} = \lambda_1^h$$

□

Lower bound on λ_1

Theorem

- $u_* \in V$, $\lambda_* \in \mathbb{R}$ arbitrary
- $w \in V: \quad (w, v)_V = (u_*, v)_V - \lambda_*(\gamma u_*, \gamma v)_H \quad \forall v \in V$
- $\left|\dfrac{\lambda_1 - \lambda_*}{\lambda_1}\right| \leq \left|\dfrac{\lambda_i - \lambda_*}{\lambda_i}\right| \quad \forall i = 1, 2, \dots.$
- $\|w\|_V \leq A + C_\gamma B, \quad B < \lambda_* \|\gamma u_*\|_H$

then

$$X_2^2 \leq \lambda_1,$$

$$X_2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4(\lambda_* - \beta)}\right), \quad \alpha = \frac{A}{\|\gamma u_*\|_H}, \quad \beta = \frac{B}{\|\gamma u_*\|_H}.$$

Application to Friedrichs' inequality

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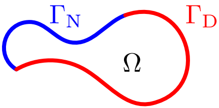
Notation and assumptions

- $a(u,v) = \int_\Omega (\nabla u)^T \mathcal{A} \nabla v \,\mathrm{d}x$
- $\mathcal{A} \in [L^\infty(\Omega)]^{d\times d}$ symmetric
- $\boldsymbol{\xi}^T \mathcal{A}(x) \boldsymbol{\xi} \geq \lambda_{\mathcal{A}} |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $x \in \Omega$
- $H^1_{\Gamma_\mathrm{D}}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_\mathrm{D}\}$
- $a(\cdot,\cdot)$ scalar product in $H^1_{\Gamma_\mathrm{D}}(\Omega)$
- $\|v\|_a^2 = a(v,v)$



Application to Friedrichs' inequality

Setting

- $V = H^1_{\Gamma_{\mathrm{D}}}(\Omega), \quad (u,v)_V = a(u,v)$
- $H = L^2(\Omega), \quad (u,v)_H = (u,v)$
- $\gamma : H^1_{\Gamma_{\mathrm{D}}}(\Omega) \to L^2(\Omega)$ identity mapping,

  compact by Rellich theorem

Conclusions

- $\exists C_{\mathrm{F}} > 0 : \quad \|v\|_{L^2(\Omega)} \le C_{\mathrm{F}} \|v\|_a \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$
- $C_{\mathrm{F}} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue:

$$\lambda_i \in \mathbb{R},\ 0 \neq u_i \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad (\mathcal{A}\nabla u, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

Friedrichs' inequality – complementarity

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Notation:

- $\mathbf{H}(\operatorname{div}, \Omega) = \{\mathbf{q} \in [L^2(\Omega)]^d : \operatorname{div} \mathbf{q} \in L^2(\Omega)\}$
- $\|\mathbf{q}\|_{\mathcal{A}}^2 = (\mathcal{A}\mathbf{q}, \mathbf{q})$ a norm in $[L^2(\Omega)]^d$

Theorem: If

- $\lambda_* \in \mathbb{R}, \quad u_* \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$
- $w \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad a(w, v) = a(u_*, v) - \lambda_*(u_*, v) \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$

Then

$$\|w\|_a \le \underbrace{\|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}}_{A} + C_{\mathrm{F}} \underbrace{\|\lambda_* u_* + \operatorname{div} \mathbf{q}\|_{L^2(\Omega)}}_{B} \quad \forall \mathbf{q} \in W_0,$$

where $W_0 = \{\mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega) : \mathbf{q} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{N}}\}$

Choice of $\mathbf{q} \in W$

- $A = A(\mathbf{q}) = \|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}$
  $B = B(\mathbf{q}) = \|\lambda_1^h u_1^h + \operatorname{div}\mathbf{q}\|_{L^2(\Omega)}$

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- Best choice: $\mathbf{q}^{\mathrm{best}} = \arg\min_{\mathbf{q}\in W_0}\left\{A(\mathbf{q}) + C_{\mathrm{F}} B(\mathbf{q})\right\}$

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- Practical: $\mathbf{q}^h = \arg\min_{\mathbf{q}\in W_0^h}\left\{(1+\varrho^{-1})A^2(\mathbf{q}) + (1+\varrho)(\lambda_1^h)^{-1}B^2(\mathbf{q})\right\}$
- $W_0^h \subset W_0$ Raviart-Thomas finite element space

Choice of $\mathbf{q} \in W$

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- $A = A(\mathbf{q}) = \|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}$
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- $W_0^h \subset W_0$ Raviart-Thomas finite element space
- Equivalent to

$$\mathbf{q}^h \in W_0^h : \quad \mathcal{B}(\mathbf{q}^h, \mathbf{w}^h) = \mathcal{F}(\mathbf{w}^h) \quad \forall \mathbf{w}^h \in W_0^h$$

where

$$\mathcal{B}(\mathbf{q}, \mathbf{w}) = (\operatorname{div}\mathbf{q}, \operatorname{div}\mathbf{w}) + \frac{\lambda_1^h}{\varrho}(\mathcal{A}^{-1}\mathbf{q}, \mathbf{w}),$$

$$\mathcal{F}(\mathbf{w}) = \frac{\lambda_1^h}{\varrho}(\nabla u_1^h, \mathbf{w}) - (\lambda_1^h u_1^h, \operatorname{div}\mathbf{w})$$

Application to trace inequality

Setting

- $\operatorname{meas}_{d-1} \Gamma_{\mathrm{N}} > 0$
- $V = H^1_{\Gamma_{\mathrm{D}}}(\Omega), \quad (u,v)_V = a(u,v)$
- $H = L^2(\Gamma_{\mathrm{N}}), \quad (u,v)_H = (u,v)_{\Gamma_{\mathrm{N}}}$
- $\gamma : H^1_{\Gamma_{\mathrm{D}}}(\Omega) \to L^2(\Gamma_{\mathrm{N}})$ trace operator,
  compact, see e.g. [Kufner, John, Fučík, 1977]

Conclusions

- $\exists C_{\mathrm{T}} > 0 : \quad \|v\|_{L^2(\Gamma_{\mathrm{N}})} \leq C_{\mathrm{T}} \|v\|_a \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$
- $C_{\mathrm{T}} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue:

$$\lambda_i \in \mathbb{R},\ 0 \neq u_i \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : \quad (\mathcal{A}\nabla u, \nabla v) = \lambda_i (u_i, v)_{\Gamma_{\mathrm{N}}} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$

Trace inequality – complementarity

$$\|w\|_a \le A(\mathbf{q}) + C_{\mathrm{T}} B(\mathbf{q}) \quad \forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$$

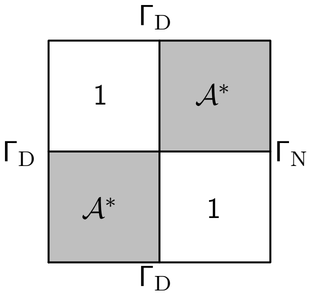
- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + C_{\mathrm{F}} \|\operatorname{div} \mathbf{q}\|_{L^2(\Omega)}$
  $B(\mathbf{q}) = \|\lambda_* u_* - \mathbf{q} \cdot \mathbf{n}\|_{L^2(\Gamma_{\mathrm{N}})}$

Variants

- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}}, \quad \forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega), \ \operatorname{div} \mathbf{q} = 0$
- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + C_{\mathrm{P}} \|\operatorname{div} \mathbf{q}\|_{L^2(\Omega)}$
  $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega), \ \int_\Omega \operatorname{div} \mathbf{q} \, \mathrm{d}x = 0$
- $A(\mathbf{q}) = \|\nabla u_* - \mathcal{A}^{-1}\mathbf{q}\|_{\mathcal{A}} + \dfrac{h}{\lambda_{\mathcal{A}} \pi} \|\operatorname{div} \mathbf{q}\|_{L^2(\Omega)}$
  $\forall \mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega) : \ \int_K \operatorname{div} \mathbf{q} \, \mathrm{d}x = 0 \ \forall K \in \mathcal{T}_h$

Friedrichs' inequality – numerical example

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}} \|\mathcal{A}^{1/2}\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_{\Gamma_{\mathrm{D}}}(\Omega)$$



- $\mathcal{A}(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 x_2 \le 0 \\ \mathcal{A}^* & \text{for } x_1 x_2 > 0 \end{cases}$
- $u_1^h \in V^h = \{v^h \in H^1_{\Gamma_{\mathrm{D}}}(\Omega) : v^h|_K \in P^1(K),\ \forall K \in \mathcal{T}_h\}$
- $\mathbf{q}^h \in W_0^h = \{\mathbf{w}_h \in W_0 : \mathbf{w}_h \in [P^2(K)]^2,\ \forall K \in \mathcal{T}_h\}$
- Adaptive algorithm
  - Driven by $\eta_K^2 = (1+\varrho^{-1})\|\nabla u_1^h - \mathcal{A}^{-1}\mathbf{q}^h\|^2_{\mathcal{A},K} + (1+\varrho)(\lambda_1^h)^{-1}\|\lambda_1^h u_1^h + \operatorname{div}\mathbf{q}^h\|^2_{L^2(K)}$
  - Stopped if $E_{\mathrm{REL}} = \dfrac{C_{\mathrm{F}}^{\mathrm{up}} - C_{\mathrm{F}}^{\mathrm{low}}}{C_{\mathrm{F}}^{\mathrm{avg}}} \le E_{\mathrm{TOL}}$

Friedrichs' inequality – numerical results

$\mathcal{A}^*$	$C_{\mathrm{F}}^{\mathrm{low}}$	$C_{\mathrm{F}}^{\mathrm{up}}$	E_{REL}	N_{DOF}
0.001	9.0086	9.0939	0.94 %	4 832
0.01	2.8697	2.8971	0.95 %	5 003
0.1	1.0035	1.0124	0.88 %	7 866
1	0.5693	0.5743	0.86 %	4 802
10	0.3173	0.3201	0.88 %	7 866
100	0.2870	0.2897	0.95 %	5 003
1000	0.2849	0.2876	0.94 %	4 832

Note: $C_{\mathrm{F}} = 4/(\pi\sqrt{5}) \approx 0.5694$ for $\mathcal{A}^* = 1$.

Trace inequality – numerical results

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$$\|v\|_{L^2(\Gamma_\mathrm{N})} \le C_\mathrm{T} \|\mathcal{A}^{1/2}\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$$

$\mathcal{A}^*$	$C_\mathrm{T}^\mathrm{low}$	C_T^up	E_REL	N_DOF
0.001	17.8110	17.9760	0.92 %	5 523
0.01	5.6490	5.7047	0.98 %	5 418
0.1	1.8433	1.8593	0.86 %	7 775
1	0.7963	0.8033	0.88 %	5 499
10	0.5829	0.5880	0.86 %	7 775
100	0.5649	0.5705	0.98 %	5 421
1000	0.5632	0.5685	0.92 %	5 523

Note: $C_\mathrm{T} = \sqrt{2/(\pi \coth \pi)} \approx 0.7964$ for $\mathcal{A}^* = 1$

Conclusions

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- General method for two-sided bounds of principal eigenvalues
- Straightforward applications
- Guaranteed bounds if
  - no round-off errors
  - all integrals evaluated exactly
  - domain Ω represented exactly
  - Galerkin method requires exact solution of matrix eigenproblem, but complementarity does not.
- Crucial assumption: $\left|\frac{\lambda_1 - \lambda_*}{\lambda_1}\right| \leq \left|\frac{\lambda_i - \lambda_*}{\lambda_i}\right| \quad \forall i = 1, 2, \dots.$

Outlook

- Local construction of $\mathbf{q}$
- Nonlinear and nonsymmetric problems
- Non-Hilbert case

Thank you for your attention

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Applications of Mathematics 2013, Prague, May 15–17, 2013
in honor of the 70th birthday of Karel Segeth