THE VITALI CONVERGENCE THEOREM FOR THE VECTOR-VALUED MCSHANE INTEGRAL

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Abstract. The classical Vitali convergence theorem gives necessary and sufficient conditions for norm convergence in the space of Lebesgue integrable functions. Although there are versions of the Vitali convergence theorem for the vector valued McShane and Pettis integrals given by Fremlin and Mendoza, these results do not involve norm convergence in the respective spaces. There is a version of the Vitali convergence theorem for scalar valued functions defined on compact intervals in $\mathbb{R}^n$ given by Kurzweil and Schwabik, but again this version does not consider norm convergence in the space of integrable functions. In this paper we give a version of the Vitali convergence theorem for norm convergence in the space of vector-valued McShane integrable functions.

Keywords: vector-valued McShane integral, Vitali theorem

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We begin by fixing the notation and preliminary definitions of the McShane integral used in the sequel. Let $\mathbb{R}^*$ be the extended reals (with $\pm\infty$ adjoined to $\mathbb{R}$). Let X be a (real) Banach space and let I be a left closed interval in $\mathbb{R}^*$. Any function $f\colon I \to X$ is assumed to be extended to $\mathbb{R}^*$ setting $f(\pm\infty) = 0$.

A gauge on I is a function γ which associates to each $t \in I$ an open neighborhood $\gamma(t)$ of t [a neighborhood of ∞ is an interval of the form $(a, \infty]$; similarly for $-\infty$.]. A partition of I is a finite collection of left closed, pairwise disjoint intervals I_i, $i = 1, \dots, n$, such that $I = \bigcup\limits_{i=1}^{n} I_i$ (here we agree that $(-\infty, a)$ is left closed). A tagged partition of I is a finite collection of ordered pairs $\{(I_i, t_i)\colon 1 \leqslant i \leqslant n\}$ such that $\{I_i\}$ is a partition of I and $t_i \in I$; t_i is called the tag associated with the interval I_i. Note that it is not required that the tag t_i belong the to the interval I_i; this requirement is what distinguishes the McShane integral from the Henstock-Kurzweil integral ([9], [16]).

If γ is a gauge on I, a tagged partition $\{(I_i,t_i)\colon\ 1\leqslant i\leqslant n\}$ is said to be γ-fine if $I_i\subset\gamma(t_i)$ for every i. If J is an interval in $\mathbb{R}^*$, we write $m(J)$ for its length and make the usual agreement that $0\cdot\infty=0$.

If $\mathcal{D}=\{(I_i,t_i)\colon\ 1\leqslant i\leqslant n\}$ is a tagged partition and $f\colon\ I\to X$, we write $S(f,\mathcal{D})=\sum\limits_{i=1}^{n} f(t_i)m(I_i)$ for the Riemann sum of f with respect to $\mathcal{D}$.

Definition 1. A function $f\colon\ I\to X$ is (McShane) integrable (M-integrable) over I if there exists $v\in X$ such that for every $\varepsilon>0$ there exists a gauge γ on I such that $\|S(f,\mathcal{D})-v\|<\varepsilon$ for every γ-fine tagged partition $\mathcal{D}$ of I.

The vector v is called the (McShane) integral of f over I and is denoted by $\int_I f$.

We refer the reader to [9], [4], [5], [16] for basic properties of the McShane integral.

For later use we record one important result for the McShane integral usually referred to as Henstock's Lemma.

If $(I_i,t_i), i=1,\dots,n$ is any pairwise disjoint collection I_i of left closed intervals of I and $t_i\in I$, the collection $\{(I_i,t_i)\colon\ i=1,\dots,n\}$ is called a partial tagged partition of I (it is not required that $I=\bigcup\limits_{i=1}^{n} I_i$) and such a collection is called γ-fine if $I_i\subset\gamma(t_i)$ for every i. We employ the same notation for Riemann sums with respect to partial tagged partitions.

Lemma 2 (Henstock). *Let $f\colon\ I\to X$ be M-integrable and let $\varepsilon>0$. Suppose the gauge γ is such that $\|S(f,\mathcal{D})-\int_I f\|<\varepsilon$ for every γ-fine tagged partition $\mathcal{D}$ of I.*

If $\{(I_i,t_i)\colon\ 1\leqslant i\leqslant n\}$ is any γ-fine partial tagged partition of I, then

$$\Big\|\sum_{i=1}^{n}\Big\{f(t_i)m(I_i)-\int_{I_i} f\Big\}\Big\|\leqslant\varepsilon.$$

Our proof of the Vitali convergence theorem will require that we are able to integrate over measurable subsets so we begin by showing that this is possible.

Lemma 3. *Let $f\colon\ I\to X$ be M-integrable and let $\varepsilon>0$. If γ is a gauge such that $\|S(f,\mathcal{D})-\int_I f\|<\varepsilon$ whenever $\mathcal{D}$ is a γ-fine tagged partition of I, then $\|S(f,\mathcal{D})-S(f,\mathcal{E})\|<2\varepsilon$ whenever $\mathcal{D}=\{(I_i,t_i)\colon\ i=1,\dots,n\}$ and $\mathcal{E}=\{(J_j,s_j)\colon\ j=1,\dots,m\}$ are two γ-fine partial tagged partitions with $\bigcup\limits_{i=1}^{n} I_i=\bigcup\limits_{j=1}^{m} J_j=K$.*

Proof. $I\setminus K$ is a finite disjoint union of subintervals of I so there exists a γ-fine partial tagged partition $\mathcal{P}$ of I, such that the union of the subintervals in $\mathcal{P}$

is exactly $I \setminus K$. Then $\mathcal{D} \cup \mathcal{P}$ and $\mathcal{E} \cup \mathcal{P}$ are γ-fine tagged partitions of I so

$$\begin{aligned}\|S(f,\mathcal{D}) - S(f,\mathcal{E})\| &= \|S(f,\mathcal{D}\cup\mathcal{P}) - S(f,\mathcal{E}\cup\mathcal{P})\| \\ &\leqslant \left\|S(f,\mathcal{D}\cup\mathcal{P}) - \int_I f\right\| + \left\|S(f,\mathcal{E}\cup\mathcal{P}) - \int_I f\right\| < 2\varepsilon.\end{aligned}$$

□

If $f\colon I \to X$ is M-integrable and $E \subset I$, then we say that f is M-integrable over E if $C_E f$ is M-integrable over I and we set $\int_E f = \int_I C_E f$, where C_E is the characteristic function of E.

Theorem 4. *If $f\colon I \to X$ is M-integrable over I, then f is M-integrable over every measurable subset E of I.*

Proof. Let $\varepsilon > 0$. There is a gauge γ such that $\|S(f,\mathcal{P}) - \int_I f\| < \varepsilon$ when $\mathcal{P}$ is a γ-fine tagged partition of I. Pick open $O_k \supset E$ and closed $F_k \subset E$ such that $m(O_k \setminus F_k) < \varepsilon/k2^k$, where m is Lebesgue measure. Define a gauge γ' on I by $\gamma'(t) = O_k \cap \gamma(t)$ when $t \in E$ and $k-1 \leqslant \|f(t)\| < k$ and $\gamma'(t) = \gamma(t) \setminus F_k$ when $t \notin E$ and $k-1 \leqslant \|f(t)\| < k$.

Suppose $\mathcal{P} = \{(I_i,t_i)\colon 1 \leqslant i \leqslant m\}$ and $\mathcal{Q} = \{(J_j,s_j)\colon 1 \leqslant j \leqslant n\}$ are γ'-fine tagged partitions of I. Then $\mathcal{P}' = \{(I_i \cap J_j, t_i)\colon 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ and $\mathcal{Q}' = \{(I_i \cap J_j, s_j)\colon 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ are γ'-fine tagged partitions of I and have the same Riemann sums as $\mathcal{P}$ and $\mathcal{Q}$, respectively.

Note that $\mathcal{P}'$ and $\mathcal{Q}'$ have the same subintervals but different tags. To avoid multiple subscripts assume that $\mathcal{P}'$ and $\mathcal{Q}'$ are relabelled $\mathcal{P}'' = \{(B_k,t'_k)\colon 1 \leqslant k \leqslant N\}$ and $\mathcal{Q}'' = \{(B_k,s'_k)\colon 1 \leqslant k \leqslant N\}$. Then

$$\begin{aligned}(1)\quad &\|S(C_E f,\mathcal{P}) - \mathcal{S}(C_E f,\mathcal{Q})\| \\ &= \|S(C_E f,\mathcal{P}'') - \mathcal{S}(\mathcal{C}_E f,\mathcal{Q}'')\| = \left\|\sum_{t'_k\in E} f(t'_k)m(B_k) - \sum_{s'_k\in E} f(s'_k)m(B_k)\right\| \\ &\leqslant \left\|\sum_{t'_k\in E, s'_k\in E} (f(t'_k) - f(s'_k))m(B_k)\right\| + \left\|\sum_{t'_k\in E, s'_k\notin E} f(t'_k)m(B_k)\right\| \\ &\quad + \left\|\sum_{s'_k\in E, t'_k\notin E} f(s'_k)m(B_k)\right\| = R_1 + R_2 + R_3.\end{aligned}$$

We estimate the R_i.

First, $R_1 < 2\varepsilon$ by Lemma 3.

For R_2 put

$$\sigma_l = \{k\colon\ t'_k \in E, s'_k \notin E, l-1 \leqslant \|f(t'_k)\| < l\}.$$

If $k \in \sigma_l$, $B_k \subset \gamma'(t'_k) = O_l \cap \gamma(t'_k)$ and $B_k \subset \gamma'(s'_k) = \gamma(s'_k) \setminus F_l$ so

$$\bigcup_{k\in\sigma_l} B_k \subset O_l \setminus F_l$$

and

$$m\Bigl(\bigcup_{k\in\sigma_l} B_k\Bigr) \leqslant m(O_l \setminus F_l) < \varepsilon/l2^l.$$

Therefore,

$$R_2 \leqslant \sum_{l=1}^{\infty}\sum_{k\in\sigma_l} \|f(t'_k)\| m(B_k) \leqslant \sum_{l=1}^{\infty}\sum_{k\in\sigma_l} l m(B_l) \leqslant \sum_{l=1}^{\infty} \varepsilon/2^l = \varepsilon.$$

Similarly, $R_3 \leqslant \varepsilon$. So, the left hand side of (1) is less than 4ε and $C_E f$ satisfies the Cauchy criterion for M-integrability. $\square$

This result is established in [2], Theorem 2E, but their proof uses the fact that McShane integrable functions are Pettis integrable and uses properties of the Pettis integral. In fact, Theorem 4 implies that a McShane integrable function is Pettis integrable. For if $f\colon I \to X$ is McShane integrable, then $x'f$ is McShane integrable and, therefore Lebesgue integrable for every $x' \in X'$ ([4], Theorem 8); that is f is scalarly integrable ([10], 4.1). But, from Theorem 4, for every measurable E, $\int_E f \in X$ and $x'\int_E f = \int_E x'f$ follows easily from the definition of the McShane integral, so f is Pettis integrable with value $\int_E f$ for every measurable E ([10], 4.2). Another proof of Theorem 4 is given in [7] Theorem 7.

Corollary 5. *If $f\colon I \to X$ is McShane integrable, then f is Pettis integrable and the two integrals agree.*

In particular, it follows from Corollary 5 that the indefinite McShane integral is countably additive and is absolutely continuous with respect to Lebesgue measure ([10], 4.1); see also Theorem 12 where this result is established using only properties of the McShane integral.

For the proof of our version of the Vitali convergence theorem, it will be necessary to show that the McShane integral can be developed by using measurable sets in the partitions employed in the definition of the integral. We begin by considering the case of compact intervals. This was carried out by R. Gordon in [5] for scalar functions and by Kurzweil and Schwabik in [7] for X-valued functions defined on compact intervals in $\mathbb{R}^n$.

If $E \subset \mathbb{R}$ is a measurable set, a measurable partition of E is a finite collection of pairwise disjoint measurable sets $\{E_i\colon 1 \leqslant i \leqslant n\}$ such that $E = \bigcup_{i=1}^{n} E_i$; a tagged measurable partition of E is a finite collection of pairs $\{(E_i, t_i)\colon\ 1 \leqslant i \leqslant n\}$ such that $\{E_i\colon\ 1 \leqslant i \leqslant n\}$ is a partition of E and $t_i \in E$; we use similar terminology for partial tagged measurable partitions. Again the t_i are called the tags associated with the E_i.

If γ is a gauge on E, a partial tagged measurable partition $\{(E_i, t_i)\colon\ 1 \leqslant i \leqslant n\}$ is said to be γ-fine if $E_i \subset \gamma(t_i)$ for each i. If $f\colon\ E \to X$, the Riemann sum of f with respect to a partial tagged measurable partition $\mathcal{P} = \{(E_i, t_i)\colon\ 1 \leqslant i \leqslant n\}$ is defined to be

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} f(t_i) m(E_i).$$

In what follows let $\mathcal{A}$ be the algebra of subsets generated by the left closed subintervals in $\mathbb{R}$. Thus, the members of $\mathcal{A}$ are finite pairwise disjoint unions of left closed subintervals ([13], 2.1.11).

Lemma 6. *Let K be a bounded interval and γ a gauge on K. Let $\{x_i\colon\ 1 \leqslant i \leqslant n\}$ be distinct points from K and $\{H_i\colon\ 1 \leqslant i \leqslant n\}$ closed pairwise disjoint subsets of K such that $H_i \subset \gamma(x_i)$.*

Let $\beta > 0$. There exist pairwise disjoint $\{V_i\colon\ 1 \leqslant i \leqslant n\}$ from $\mathcal{A}$ such that $V_i \subset \gamma(x_i)$ and $m(V_i \triangle H_i) < \beta/n$ for $1 \leqslant i \leqslant n$.

Proof. For every i, pick an open O_i such that $H_i \subset O_i \subset \gamma(x_i)$ with the $\{O_i\}$ pairwise disjoint. [This is possible since the $\{H_i\}$ are a positive distance apart.]

For each i, pick $V_i \in \mathcal{A}$ such that $V_i \subset O_i$ and $m(V_i \triangle H_i) < \beta/n$ ([12], 3.3.15). $\square$

Theorem 7. *Let $K = [a, b]$ be a compact interval, $f\colon\ K \to X$ M-integrable and $\varepsilon > 0$.*

If γ is a gauge on K such that $\|S(f, \mathcal{P}) - \int_K f\| < \varepsilon$ whenever $\mathcal{P}$ is a γ-fine tagged partition of K, then $\|S(f, \mathcal{P}) - \int_K f\| \leqslant 2\varepsilon$ whenever $\mathcal{P}$ is a γ-fine tagged measurable partition of K.

Proof. The indefinite integral $F(t) = \int_a^t f$ is absolutely continuous on K so there exists $0 < \eta < \varepsilon$ such that $\|\int_V f\| < \varepsilon$ when $V \in \mathcal{A}$ and $m(V) < \eta$ ([14], Theorem 10). Let $\mathcal{P} = \{(E_i, x_i)\colon\ 1 \leqslant i \leqslant n\}$ be a γ-fine tagged measurable partition of K.

Put $M = 1 + \max\{\|f(x_i)\|\colon\ 1 \leqslant i \leqslant n\}$. For each i there exists closed $H_i \subset E_i$ such that $m(E_i \setminus H_i) < \eta/2nM$.

The $\{H_i\}$ are pairwise disjoint so by Lemma 6 there exist $V_i \in \mathcal{A}$ such that $\{V_i\colon\ 1 \leqslant i \leqslant n\}$ are pairwise disjoint, $\{(V_i, x_i)\colon\ 1 \leqslant i \leqslant n\}$ is γ-fine and $m(V_i \triangle H_i) < \eta/2nM$. There exists $V_0 \in \mathcal{A}$ such that $\{V_i\colon\ 0 \leqslant i \leqslant n\}$ forms a partition of K. Now

$$E_i \triangle V_i \subset (E_i \triangle H_i) \cup (V_i \triangle H_i),$$

so $m(E_i \triangle V_i) \leqslant \eta/nM$ and, therefore,

$$m(V_0) = m\biggl(\bigcup_{i=1}^{n} E_i \setminus \bigcup_{i=1}^{n} V_i\biggr) \leqslant \sum_{i=1}^{n} m(E_i \triangle V_i) < \eta/M.$$

Thus,

$$\begin{aligned}(2) \qquad \biggl\| \sum_{i=1}^{n} f(x_i)m(E_i) - \int_K f \biggr\| &\leqslant \biggl\| \sum_{i=1}^{n} f(x_i)(m(E_i) - m(V_i)) \biggr\| \\ &\quad + \biggl\| \sum_{i=1}^{n} \biggl(f(x_i)m(V_i) - \int_{V_i} f \biggr) \biggr\| + \biggl\| \int_{V_0} f \biggr\| \\ &= R_1 + R_2 + R_3.\end{aligned}$$

We estimate each R_i.

First,

$$R_1 \leqslant \sum_{i=1}^{n} \|f(x_i)\| m(E_i \triangle V_i) \leqslant M \sum_{i=1}^{n} m(E_i \triangle V_i) < \eta.$$

Next, $R_2 \leqslant \varepsilon$ by Henstock's Lemma 2.

Last, $R_3 < \varepsilon$ by the choice of η since $m(V_0) < \eta$.

Thus, (2) is less than $\eta + 2\varepsilon$ and since $\eta > 0$ can be taken arbitrarily small, the result follows. □

This result will be extended to arbitrary intervals later in Theorem 11.

Motivated by the result in Theorem 7 we give a definition of the (generalized) McShane integral using measurable partitions.

Definition 8. Let $f\colon I \to X$. We say that f is *generalized McShane integrable over* I if there exists $v \in X$ such that for every $\varepsilon > 0$ there is a gauge γ such that $\|S(f, \mathcal{P}) - v\| < \varepsilon$ when $\mathcal{P}$ is a γ-fine tagged measurable partition of I.

We call v the generalized McShane integral of f over I and use the notation $\int_I f$ as before; this should cause no difficulty as it should be clear from the context whether we are referring to the McShane or generalized McShane integral. We will eventually show in Theorem 11 that the two integrals are the same; that this is the case for bounded intervals follows from Theorem 7. (See also [4] and [7].)

At this point in order to distinguish between the two integrals, if $f\colon I \to X$ is generalized McShane integrable, we will say that f is $\mathcal{M}$-integrable.

If $f\colon I \to X$ is $\mathcal{M}$-integrable over I and $A \subset I$ is measurable, we say that f is $\mathcal{M}$-integrable over A if $C_A f$ is $\mathcal{M}$-integrable over I and set $\int_A f = \int_I C_A f$.

The proof of Theorem 4 is easily adapted to show that if f is $\mathcal{M}$-integrable, then f is $\mathcal{M}$-integrable over every measurable subset of I.

The integral defined above is very similar to the version of the McShane integral defined by Fremlin in [3] in a much more general setting.

The basic properties of the generalized McShane integral such as linearity and the existence of a Cauchy condition are easily established, and we use them freely. We now establish a version of Henstock's Lemma for the generalized McShane integral for later use.

Our next goal is to establish the equivalence of the McShane and generalized McShane integrals for unbounded intervals. To keep the presentation as simple as possible we will state and prove the results for the unbounded interval $I = [0,\infty)$; this will allow us to consider only one point at ∞ and keep the presentation simpler.

Lemma 9. *Let $f\colon I \to X$ be $\mathcal{M}$-integrable, $\eta > 0$ and $E \subset I$ measurable. There exists a gauge γ' on I such that if $\mathcal{P} = \{(A_i,t_i)\colon 1 \leqslant i \leqslant n\}$ is a γ'-fine partial tagged measurable partition with $\bigcup\limits_{i=1}^{n} A_i = E$, then $\|S(f,\mathcal{P}) - \int_E f\| < \eta$.*

Proof. Since f is $\mathcal{M}$-integrable over E there is a gauge γ on I such that $\|S(f,\mathcal{D}) - \int_I C_E f\| < \eta$ whenever $\mathcal{D}$ is a γ-fine tagged measurable partition of I.

Pick a closed subset $F_k \subset E$ such that $m(E \setminus F_k) < \eta/k2^k$. Define the gauge γ' by $\gamma'(t) = \gamma(t)$ if $t \in E$, $\gamma'(t) = \gamma(t) \setminus F_k$ if $t \notin E$ and $k-1 \leqslant \|f(t)\| < k$ and $\gamma'(\infty) = \gamma(\infty)$.

Suppose that $\mathcal{P} = \{(A_i,t_i)\colon 1 \leqslant i \leqslant n\}$ is a γ'-fine partial tagged measurable partition with $E = \bigcup\limits_{i=1}^{n} A_i$. Let $\gamma(\infty) = (b,\infty]$ and set $E_1 = E \cap [0,b]$, $E_2 = E \cap (b,\infty]$. By intersecting each A_i with E_1 and E_2, if necessary, we may assume that each A_i is contained in either E_1 or E_2.

Let $\mathcal{D}$ be a γ'-fine tagged measurable partition of $I \setminus E$ having tags in $I \setminus E$ so $\mathcal{P} \cup \mathcal{D}$ is a γ'-fine tagged measurable partition of I. We have

$$
\begin{aligned}
(3)\quad \Bigl\|S(f,\mathcal{P}) - \int_I C_E f\Bigr\| &\leqslant \|S(f,\mathcal{P}) - S(C_E f, \mathcal{P}\cup\mathcal{D})\| \\
&\quad + \Bigl\|S(C_E f,\mathcal{P}\cup\mathcal{D}) - \int_I C_E f\Bigr\| \leqslant \|S(f,\mathcal{P}) - S(C_E f,\mathcal{P}\cup\mathcal{D})\| + \eta \\
&\leqslant \Bigl\|\sum_{t_i\notin E, A_i\subset E_1} f(t_i)m(A_i)\Bigr\| + \Bigl\|\sum_{t_i\notin E, A_i\subset E_2} f(t_i)m(A_i)\Bigr\| + \eta = R_1 + R_2 + \eta.
\end{aligned}
$$

We estimate the R_i.

First, for R_1 let

$$\sigma_l = \{i\colon\ t_i \notin E, A_i \subset E_1, l-1 \leqslant \|f(t_i)\| < l\}.$$

If $i \in \sigma_l$, $A_i \subset \gamma'(t_i) \subset E_1 \setminus F_l$ so $\bigcup_{i\in\sigma_l} A_i \subset E \setminus F_l$ and

$$m\biggl(\bigcup_{i\in\sigma_l} A_i\biggr) \leqslant m(E \setminus F_l) < \eta/l2^l.$$

Therefore,

$$R_1 = \biggl\|\sum_{l=1}^{\infty}\sum_{i\in\sigma_l} f(t_i)m(A_i)\biggr\| \leqslant \sum_{l=1}^{\infty} l\eta/l2^l = \eta.$$

For R_2 set

$$\sigma = \{i\colon\ A_i \subset E_2, t_i \notin E\}.$$

If $i \in \sigma$, $A_i \subset E_2 \subset \gamma'(\infty)$ so $\mathcal{E} = \{(A_i, \infty)\colon\ i \in \sigma\}$ and $\mathcal{E}' = \{(A_i, t_i)\colon\ i \in \sigma\}$ are γ'-fine partial tagged measurable partitions of I.

The analogue of Lemma 3 for measurable partitions gives

$$\|S(f,\mathcal{E}) - S(f,\mathcal{E}')\| = \|S(f,\mathcal{E})\| = R_2 \leqslant 2\eta.$$

Thus, the left hand side of (3) is less than 4η. □

Lemma 10 (Henstock)**.** *Let $f\colon\ I \to X$ be $\mathcal{M}$-integrable and $\varepsilon > 0$.*

If γ is a gauge on I such that $\|S(f,\mathcal{P}) - \int_I f\| < \varepsilon$ for every γ-fine tagged measurable partition of I, then

$$\biggl\|\sum_{i=1}^{n}\biggl\{f(t_i)m(A_i) - \int_{A_i} f\biggr\}\biggr\| \leqslant \varepsilon$$

whenever $\mathcal{P} = \{(A_i, t_i)\colon\ 1 \leqslant i \leqslant n\}$ is a γ-fine partial tagged measurable partition of I.

Proof. Set $E = \bigcup_{i=1}^{n} A_i$. Let γ' be the gauge in Lemma 9 relative to $I \setminus E$ and $\eta > 0$ where we may assume that $\gamma' \subset \gamma$.

Pick a γ'-fine tagged measurable partition $\mathcal{D}$ of $I \setminus E$. Then $\mathcal{P} \cup \mathcal{D}$ is a γ'-fine tagged measurable partition of I. By the conclusion of Lemma 9, we have

$$\biggl\|S(f,\mathcal{P}) - \int_E f\biggr\| \leqslant \biggl\|S(f,\mathcal{P}\cup\mathcal{D} - \int_I f\biggr\| + \biggl\|\int_I C_{I\setminus E} f - S(f,\mathcal{D})\biggr\| \leqslant \varepsilon + \eta$$

and since $\eta > 0$ is arbitrary, this gives the conclusion. □

We now show the equivalence of the McShane and generalized McShane integrals for unbounded intervals.

Theorem 11. *Let $f\colon I \to X$ be M-integrable.*

If γ is a gauge such that $\|S(f,\mathcal{P}) - \int_I f\| < \varepsilon$ for every γ-fine tagged partition of I, then

$$\left\| \sum_{i=1}^{n} \left\{ f(t_i)m(A_i) - \int_{A_i} f \right\} \right\| < 3\varepsilon$$

whenever $\mathcal{P} = \{(A_i, t_i)\colon 1 \leqslant i \leqslant n\}$ is a γ-fine tagged measurable partition of I.

Proof. There exists $b > 0$ such that

$$\|C_{[b,\infty)} f\|_1 = \sup_{\|x'\| \leqslant 1} \int_b^\infty |x' f| < \varepsilon$$

([14], Theorem 11).

Suppose $\mathcal{P} = \{(A_i, t_i)\colon 1 \leqslant i \leqslant n\}$ is a γ-fine tagged measurable partition of I; assume that $t_n = \infty$ and $t_i \neq \infty$ for $i = 1, \ldots, n-1$.

Pick $a > \max\{b, \sup \gamma(t_i), |t_i| : 1 \leqslant i \leqslant n-1\}$. Then

$$(4) \qquad \left\| S(f,\mathcal{P}) - \int_I f \right\| \leqslant \left\| \sum_{i=1}^{n-1} \left\{ f(t_i)m(A_i) - \int_{A_i} f \right\} \right\| + \left\| \int_{A_n} f \right\| = R_1 + R_2.$$

We estimate the R_i.

First, for R_1, $\bigcup\limits_{i=1}^{n-1} \subset [0,a]$ and f is $\mathcal{M}$-integrable over $[0,a]$. By Theorem 7 f is M-integrable over $[0,a]$ so Henstock's Lemma and Theorem 7 are applicable and $R_1 \leqslant 2\varepsilon$.

Next, $A_n \subset [b, \infty]$ so

$$R_2 = \sup_{\|x'\| \leqslant 1} \left| x' \int_{A_n} f \right| \leqslant \sup_{\|x'\| \leqslant 1} \int_b^\infty |x' f| = \|C_{[b,\infty]} f\|_1 < \varepsilon.$$

Thus, the left hand side of (4) is less than 3ε. $\square$

It follows from Theorems 7 and 11 that the McShane and generalized McShane integrals coincide and we will henceforth refer to M-integrability.

For later use we establish an absolute continuity result for the McShane integral.

Theorem 12. *Let $f\colon I \to X$ be M-integrable. Then (for $A \subset I$ measurable)*

$$\lim_{m(A)\to 0} \left\| \int_A f \right\| = 0,$$

i.e., $\int_{\cdot} f$ is m-continuous.

Proof. Let $\varepsilon > 0$. There is a gauge γ such that $\|S(f,\mathcal{P}) - \int_I f\| < \varepsilon$ whenever $\mathcal{P} = \{(I_i, t_i)\colon 1 \leqslant i \leqslant m\}$ is a γ-fine tagged partition of I.

Fix such a partition and set

$$M = \max\{\|f(t_i)\|\colon 1 \leqslant i \leqslant m\} + 1.$$

Set $\delta = \varepsilon/M$. Suppose $m(A) < \delta$. Then

$$\mathcal{P}_A = \{(A \cap I_i, t_i)\colon i \leqslant i \leqslant m\}$$

is a γ-fine partial tagged measurable partition of I. By Henstock's Lemma

$$\left\| \sum_{i=1}^{m} f(t_i) m(A \cap I_i) - \int_A f \right\| \leqslant \varepsilon$$

so

$$\begin{aligned} \left\| \int_A f \right\| \leqslant \varepsilon + \left\| \sum_{i=1}^{m} f(t_i) m(A \cap I_i) \right\| &\leqslant \varepsilon + M \sum_{i=1}^{m} m(A \cap I_i) \\ &= \varepsilon + M m(A) < 2\varepsilon. \end{aligned}$$

□

It follows from Theorem 12 that the indefinite McShane integral $\int_{\cdot} f$ is countably additive in the norm topology of X. This follows from the fact that a McShane integrable function is Pettis integrable, but the proof above only uses results from the McShane integral.

We are now in a position to establish our version of the Vitali convergence theorem for the McShane integral.

Let Σ be the σ-algebra of Lebesgue measurable subsets. A family Γ of X-valued countably additive set functions on Σ is uniformly m-continuous if $\lim\limits_{m(E)\to 0} \mu(E) = 0$ uniformly for $\mu \in \Gamma$.

If I is a left closed interval and $\mathcal{F}$ is a family of M-integrable functions on I, then $\mathcal{F}$ is uniformly M-integrable if for every $\varepsilon > 0$ there exists a gauge γ such that

$\|S(f,\mathcal{P}) - \int_I f\| < \varepsilon$ for every $f \in \mathcal{F}$ whenever $\mathcal{P}$ is a γ-fine tagged partition of I. In [8], uniform M-integrability is referred to as equi-integrability.

If I is a left closed interval, let $M(I,X)$ be the space of all X-valued M-integrable functions defined on I.

There are two equivalent semi-norms on $M(I,X)$ given by

$$\|f\|_1 = \sup\left\{ \int_I |x'f| : \ x' \in X', \|x'\| \leqslant 1\right\}$$

and

$$\|f\|_1' = \sup\left\{ \left\| \int_E f \right\| : \ E \in \Sigma\right\}.$$

It is easily checked that

$$\|f\|_1' \leqslant \|f\|_1 \leqslant 2\|f\|_1'.$$

A slightly different but equivalent semi-norm was defined in [14] where the supremum in the definition of $\|\cdot\|_1'$ is computed over the algebra generated by the left closed subintervals of I.

It is shown in [15] that if $\{f_k\}$ is a sequence in $M(I,X)$ which converges pointwise to a function $f\colon I \to X$ and $\{f_k\}$ is uniformly M-integrable, then $\|f_k - f\|_1 \to 0$.

We give a sufficient condition for uniform M-integrability which will be employed in the Vitali convergence theorem.

Theorem 13. *Let $f_k\colon I \to X$ be M-integrable and suppose $f_k \to f$ pointwise on I. If*

(a) $\{f_k\colon k\}$ *is uniformly m-continuous*

and

(b) $\lim\limits_{b\to\infty} \|C_{[b,\infty]} f_k\|_1 = 0$ *uniformly for $k \in \mathbb{N}$,*

then $\{f_k\}$ is uniformly M-integrable.

Proof. To keep the argument somewhat clear from technical details, first assume that the $\{f_k\}$ are strongly measurable.

Let $1 > \varepsilon > 0$. Pick $h\colon I \to (0,1)$ to be M-integrable and such that $0 < \int_I h < 1$. Set

$$r(t) = \min\left\{ n\colon \sup_{i\geqslant j\geqslant n} \|f_i(t) - f_j(t)\| < \varepsilon h(t), \|f(t)\| \leqslant n, \varepsilon h(t) \geqslant \frac{1}{n+1}\right\}$$

and $A_k = \{t\colon r(t) = k\}$.

Note that since r is measurable, each A_k is measurable, $\{A_k\}$ are pairwise disjoint and $I = \bigcup\limits_{k=1}^{\infty} A_k$.

If $t \in A_k$ and $m \geqslant k$, then

$$\|f_m(t)\| \leqslant \|f_m(t) - f(t)\| + \|f(t)\| \leqslant \varepsilon h(t) + k \leqslant 1 + k. \tag{5}$$

Also, if $H \subset A_k$ is measurable, then

$$\left\| \int_H f_n - \int_H f_m \right\| \leqslant \int_H \varepsilon h \quad \text{for } m, n \geqslant k. \tag{6}$$

By condition (a), for every k choose $\delta_k > 0$ such that

$$\left\| \int_E f_n \right\| < \frac{\varepsilon}{(k+1)2^k} = \varepsilon_k$$

for all n when $m(E) < \delta_k$.

Pick G_k open, $A_k \subset G_k$, such that $m(G_k \setminus A_k) < \min\{\varepsilon_k, \delta_k\}$.

Let γ_k be a gauge such that $\|S(f_k, \mathcal{P}) - \int_I f_k\| < \varepsilon_k$ when $\mathcal{P}$ is a γ_k-fine tagged partition of I, and let γ' be a gauge such that

$$\left| S(h, \mathcal{P}) - \int_I h \right| < 1$$

when $\mathcal{P}$ is a γ'-fine tagged partition of I. Thus,

$$|S(\varepsilon h, \mathcal{P})| \leqslant 2\varepsilon \tag{7}$$

when $\mathcal{P}$ is a γ'-fine partial tagged partition of I by Henstock's Lemma.

From (b) pick B such that $\|C_{[B,\infty]} f_k\|_1' < \varepsilon$ for all k.

Define a gauge γ by

$$\gamma(t) = \bigcap_{j=1}^{k} \gamma_j(t) \cap \gamma'(t) \cap G_k$$

when $t \in A_k$ and $\gamma(\infty) = (B, \infty]$.

Suppose $\mathcal{P} = \{(E_i, t_i)\colon\ 1 \leqslant i \leqslant N\}$ is a γ-fine tagged partition of I.

Set $I_\infty = \{i\colon\ t_i = \infty\}$ and $\mathcal{P}_\infty = \{(E_i, t_i)\colon\ i \in I_\infty\}$. Then

$$\left\| S(f_k, \mathcal{P}_\infty) - \int_{\bigcup\limits_{i \in I_\infty} E_i} f_k \right\| = \left\| \int_{\bigcup\limits_{i \in I_\infty} E_i} f_k \right\| \leqslant \|C_{[B,\infty]} f_k\|_1' < \varepsilon. \tag{8}$$

Set $I_k = \{i\colon\ t_i \in A_k\}$ and if $i \in I_k$, set $E_i^k = E_i \cap A_k$. If $i \in I_k$, then $t_i \in A_k$ so $E_i \subset \gamma(t_i) \subset G_k$ which implies

$$\begin{aligned}
\sum_{i \in I_k} m(E_i \setminus E_i^k) &\leqslant \sum_{i \in I_k} m(E_i \setminus A_k) \\
&= m\left(\bigcup_{i \in I_k} (E_i \setminus A_k) \right) \leqslant m(G_k \setminus A_k) < \min\{\varepsilon_k, \delta_k\}.
\end{aligned}$$

Therefore, if $m \geqslant k$, by (5)

$$\sum_{i \in I_k} \|f_m(t_i)\| m(E_i \setminus E_i^k) \leqslant (k+1)\varepsilon_k = \varepsilon/2^k \tag{9}$$

and $m(\bigcup_{i \in I_k} (E_i \setminus E_i^k)) < \delta_k$ implies

$$\Big\| \int_{\bigcup_{i \in I_k} (E_i \setminus E_i^k)} f_m \Big\| = \Big\| \sum_{i \in I_k} \int_{E_i \setminus E_i^k} f_m \Big\| < \varepsilon_k. \tag{10}$$

Now by (8),

$$\begin{aligned} &\Big\| \int_I f_m - \sum_{i=1}^{N} f(t_i) m(E_i) \Big\| \\ &\quad\leqslant \Big\| \sum_{k=1}^{\infty} \sum_{i \in I_k} \Big\{ \int_{E_i} f_m - f(t_i) m(E_i) \Big\} \Big\| + \Big\| S(f_m, \mathcal{P}_\infty) - \int_{\bigcup_{i \in I_\infty} E_i} f_m \Big\| \\ &\quad\leqslant \Big\| \sum_{k=1}^{m-1} \sum_{i \in I_k} \Big\{ \int_{E_i} f_m - f_m(t_i) m(E_i) \Big\} \Big\| \\ &\qquad + \Big\| \sum_{k=m}^{\infty} \sum_{i \in I_k} \Big\{ \int_{E_i} f_m - f_m(t_i) m(E_i) \Big\} \Big\| + \varepsilon = R_1 + R_2 + \varepsilon. \end{aligned} \tag{11}$$

Since $\{(E_i, t_i)\colon\ t_i \in A_k, k \geqslant m\}$ is γ_m-fine, $R_2 \leqslant \varepsilon_m < \varepsilon$ by Henstock's Lemma. To estimate R_1, we have

$$\begin{aligned} R_1 \leqslant \Big\| \sum_{k=1}^{m-1} \sum_{i \in I_k} \int_{E_i \setminus E_i^k} f_m \Big\| + \Big\| \sum_{k=1}^{m-1} \sum_{i \in I_k} \Big\{ \int_{E_i^k} f_m - f_m(t_i) m(E_i^k) \Big\} \Big\| \\ + \Big\| \sum_{k=1}^{m-1} \sum_{i \in I_k} f_m(t_i) \{ m(E_i^k) - m(E_i) \} \Big\| = T_1 + T_2 + T_3. \end{aligned}$$

Now, by (10),

$$T_1 \leqslant \sum_{k=1}^{m-1} \Big\| \sum_{i \in I_k} \int_{E_i \setminus E_i^k} f_m \Big\| \leqslant \sum_{k=1}^{m-1} \varepsilon_k < \varepsilon.$$

By (9),

$$T_3 \leqslant \sum_{k=1}^{m-1} \sum_{i \in I_k} \|f_m(t_i)\| m(E_i \setminus E_i^k) \leqslant \sum_{k=1}^{m-1} \varepsilon/2^k < \varepsilon.$$

For T_2, we have

$$T_2 \leqslant \left\| \sum_{k=1}^{m-1} \sum_{i \in I_k} \int_{E_i^k} (f_m - f_k) \right\| + \left\| \sum_{k=1}^{m-1} \sum_{i \in I_k} \left\{ \int_{E_i^k} f_k - f_k(t_i) m(E_i^k) \right\} \right\|$$
$$+ \left\| \sum_{k=1}^{m-1} \sum_{i \in I_k} \{f_k(t_i) - f_m(t_i)\} m(E_i^k) \right\| = S_1 + S_2 + S_3$$

By (6), since $E_i^k \subset A_k$,

$$S_1 \leqslant \sum_{k=1}^{m-1} \sum_{i \in I_k} \left\| \int_{E_i^k} (f_m - f_k) \right\| \leqslant \sum_{k=1}^{m-1} \sum_{i \in I_k} \int_{E_i^k} \varepsilon h \leqslant \int_I \varepsilon h < \varepsilon.$$

Since $\{(E_i^k, t_i)\colon\ i \in I_k\}$ is γ_k-fine,

$$\left\| \sum_{i \in I_k} \left\{ \int_{E_i^k} f_k - f_k(t_i) m(E_i^k) \right\} \right\| \leqslant \varepsilon/2^k$$

by Henstock's Lemma and

$$S_2 \leqslant \sum_{k=1}^{m-1} \varepsilon/2^k < \varepsilon.$$

Finally by (7),

$$S_3 \leqslant \sum_{k=1}^{m-1} \sum_{i \in I_k} \|f_k(t_i) - f_m(t_i)\| m(E_i^k) \leqslant \sum_{k=1}^{m-1} \sum_{i \in I_k} \varepsilon h(t_i) m(E_i^k) \leqslant \sum_{i \in I_k} \varepsilon h(t_i) m(E_i)$$
$$= S(\varepsilon h, \{(E_i, t_i)\colon\ i \in I_k\}) \leqslant 2\varepsilon$$

since $\{(E_i, t_i)\colon\ i \in I_k\}$ is γ'-fine.

Thus, $T_2 \leqslant 4\varepsilon$ and $R_1 \leqslant 6\varepsilon$ and the left hand side of (11) is less than 8ε.

To remove the strong measurability assumption, define r and A_k as before, but r may not be measurable so the A_k may not be measurable. Note $m^*(A_k) < \infty$ since $A_k \subset \{t\colon\ \varepsilon h(t) \geqslant 1/(k+1)\}$.

For each k, pick measurable $V_k \supset A_k$ such that $m(V_k) = m^*(A_k)$. Then (5) still holds.

Condition (6) is replaced by:

(12) $\quad H \subset V_k$ measurable implies $\displaystyle \left\| \int_H f_n - \int_H f_m \right\| \leqslant \int_H \varepsilon h$ when $m, n \geqslant k$.

To see this, first note $m(H) = m^*(A_k \cap H)$.

[If $m^*(H \cap A_k) = m(H) - \eta$ with $\eta > 0$, there exists measurable $B \supset H \cap A_k$ such that $m(B) = m^*(H \cap A_k) = m(H) - \eta$. Now

$$A_k \subset (V_k \setminus H) \cup (H \cap A_k) \subset (V_k \setminus H) \cup B$$

so

$$m^*(A_k) = m(V_k) \leqslant m(V_k \setminus H) + m(B) = m(V_k \setminus H) + m(H) - \eta$$

which implies $\eta = 0$.]

For $x' \in X'$, $\|x'\| \leqslant 1$, let

$$A_k(x') = \{t\colon\ |x'(f_n - f_m)(t)| \leqslant \varepsilon h(t), n, m \geqslant k\} \subset A_k;$$

each $A_k(x')$ is measurable since $x'(f_n - f_m)$ is Lebesgue integrable.

Note

$$m(H \setminus A_k(x')) = m(H) - m(A_k(x') \cap H) \leqslant m(H) - m^*(A_k \cap H) = 0$$

by the observation above. Thus,

$$\begin{aligned}\left\| \int_H f_n - \int_H f_m \right\| &= \sup_{\|x'\| \leqslant 1} \left| \int_H x'(f_n - f_m) \right| \\ &\leqslant \sup_{\|x'\| \leqslant 1} \int_{H \cap A_k(x')} |x'(f_n - f_m)| \leqslant \sup_{\|x'\| \leqslant 1} \int_{H \cap A_k(x')} \varepsilon h \leqslant \int_H \varepsilon h.\end{aligned}$$

Now choose open $G_k \supset V_k$ such that $m(G_k \setminus V_k) < \min\{\varepsilon_k, \delta_k\}$ and define γ as before. The argument then carries through. □

We now give our version of the Vitali convergence theorem for the McShane integral.

Theorem 14 (Vitali)**.** *Let $f_k\colon I \to X$ be M-integrable and $f_k \to f$ pointwise. The following are equivalent:*

(i) *conditions* (a) *and* (b) *of Theorem* 13*,*

(ii) *$\{f_k\}$ is uniformly M-integrable,*

(iii) *f is M-integrable and $\|f_k - f\|_1 \to 0$,*

(iv) *$\{f_k\}$ is $\|\cdot\|_1$-Cauchy.*

Proof. That (i) implies (ii) is Theorem 13; (ii) implies (iii) is Theorem 4 of [14]. Clearly (iii) implies (iv).

Suppose (iv) holds and let $\varepsilon > 0$. There exists N such that $\|f_i - f_j\|_1' < \varepsilon$ when $i, j \geqslant N$.

By Theorem 12 there exists $\delta > 0$ such that $\| \int_A f_j \| < \varepsilon$ for $1 \leqslant j \leqslant N$ and $m(A) < \delta$. If $j > N$ and $m(A) < \delta$, then $\| \int_A f_j \| \leqslant \| \int_A (f_j - f_N) \| + \| \int_A f_N \| \leqslant \|f_j - f_N\|_1' + \varepsilon < 2\varepsilon$ so (a) holds.

By Theorem 11 of [14], there exists K such that $k \geqslant K$ implies $\|C_{[k,\infty]} f_j\|_1' < \varepsilon$ for $1 \leqslant j \leqslant K$. If $j > N$ and $k \geqslant K$, then

$$\|C_{[k,\infty]} f_j\|_1' \leqslant \|C_{[k,\infty]}(f_j - f_N)\|_1' + \|C_{[k,\infty]} f_N\|_1' \leqslant \|f_j - f_N\|_1' + \varepsilon < 2\varepsilon$$

so (b) holds and (i) follows. □

For the case of a sequence of real valued functions defined on a compact interval the equivalence of conditions (i) and (ii) from Theorem 14 are given in Theorem 17 of [8].

In the classical Vitali convergence theorem for the Lebesgue integral the sequence $\{f_k\}$ is assumed to be convergent in measure and in abstract versions of the Vitali convergence theorem condition (b) is replaced by a more general condition ([1], III. 3.6, [13], 6.1.17). We do not have this condition for the McShane integral and instead impose a pointwise convergent hypothesis (see, however, [1], III. 6.5 and [6]).

Of course, if I is a bounded interval, then condition (b) from Theorem 13 is superfluous.

As an application of Vitali's Theorem we can obtain the version of the Beppo Levi theorem for the McShane integral given in Theorem 8 of [14].

Theorem 15. *Let $f, f_k \colon I \to X$. Suppose each f_k is M-integrable and $f = \sum\limits_{k=1}^{\infty} f_k$ pointwise with $\sum\limits_{k=1}^{\infty} \|f_k\|_1 < \infty$. Then f is M-integrable and*

$$\Big\| \sum_{k=1}^{n} f_k - f \Big\|_1 \to 0$$

as $n \to \infty$.

Proof. Let $s_n = \sum\limits_{k=1}^{n} f_k$.

We show that $\{s_n\}$ satisfies conditions (a) and (b) of Theorem 13; this will give the result.

Let $\varepsilon > 0$. Choose N such that $\sum\limits_{k=N}^{\infty} \|f_k\|_1' < \varepsilon$. Choose $\delta > 0$ such that $m(A) < \delta$ implies $\sum\limits_{k=1}^{N} \| \int_A f_k \| < \varepsilon$ (Theorem 12).

If $n > N$ and $m(A) < \delta$, then

$$\left\| \int_A s_n \right\| \leqslant \sum_{k=1}^{N} \left\| \int_A f_k \right\| + \sum_{k=N+1}^{n} \left\| \int_A f_k \right\| < \varepsilon + \sum_{k=N+1}^{\infty} \|f_k\|_1' < 2\varepsilon$$

so condition (a) is satisfied.

Choose B such that $\sum\limits_{k=1}^{N} \|C_{[b,\infty]} f_k\|_1 < \varepsilon$ for $b \geqslant B$ (Theorem 11 of [14]).

If $n > N$ and $b \geqslant B$, then

$$\|C_{[b,\infty]} s_n\|_1 \leqslant \sum_{k=1}^{N} \|C_{[b,\infty]} f_k\|_1 + \sum_{k=N+1}^{n} \|C_{[b,\infty]} f_k\|_1 < \varepsilon + \sum_{k=N+1}^{\infty} 2\|f_k\|_1' < 3\varepsilon$$

so that condition (b) holds. □

A dominated convergence theorem for the McShane integral also follows from Theorem 14.

Theorem 16. *Let $f_k, f\colon I \to X$. Suppose each f_k is M-integrable and $f_k \to f$ pointwise. If there exists an M-integrable scalar function $g\colon I \to \mathbb{R}$ such that $\|f_k(\cdot)\| \leqslant g$ for every k, then f is M-integrable and $\|f_k - f\|_1 \to 0$.*

P r o o f. If $E \subset I$ is measurable, then

$$\left\| \int_E f_k \right\| \leqslant \sup \left\{ \int_E |x' f_k| : \|x'\| \leqslant 1 \right\} \leqslant \int_E g$$

so conditions (a) and (b) of Theorem 14 both follow. □

The reference [11] also contains a version of the Vitali convergence theorem for an abstract vector-valued McShane integral like that defined by Fremlin in [3]. The presentation given here for Euclidean space is much less technical and perhaps more useful.

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