DOMINATION WITH RESPECT TO NONDEGENERATE AND HEREDITARY PROPERTIES

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(Received October 30, 2006)

Abstract. For a graphical property $\mathcal{P}$ and a graph G, a subset S of vertices of G is a $\mathcal{P}$-set if the subgraph induced by S has the property $\mathcal{P}$. The domination number with respect to the property $\mathcal{P}$, is the minimum cardinality of a dominating $\mathcal{P}$-set. In this paper we present results on changing and unchanging of the domination number with respect to the nondegenerate and hereditary properties when a graph is modified by adding an edge or deleting a vertex.

Keywords: domination, independent domination, acyclic domination, good vertex, bad vertex, fixed vertex, free vertex, hereditary graph property, induced-hereditary graph property, nondegenerate graph property, additive graph property

MSC 2000: 05C69

1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [8]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. The complement of a graph G is denoted by $\overline{G}$. For a vertex x of G, $N(x,G)$ denotes the set of all neighbors of x in G and $N[x,G] = N(x,G) \cup \{x\}$. The complete graph on m vertices is denoted by K_m.

For a graph G, let $x \in X \subseteq V(G)$. A vertex y is a *private neighbor of x with respect to X* if $N[y,G] \cap X = \{x\}$. The *private neighbor set of x with respect to X* is $\mathrm{pn}_G[x,X] = \{y\colon N[y,G] \cap X = \{x\}\}$.

Let $\mathcal{G}$ denote the set of all mutually nonisomorphic graphs. A *graph property* is any non-empty subset of $\mathcal{G}$. We say that a *graph G has the property $\mathcal{P}$* whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to G. For example, we list some graph properties:

- $\mathcal{I} = \{H \in \mathcal{G}\colon\ H$ is totally disconnected$\}$;
- $\mathcal{C} = \{H \in \mathcal{G}\colon\ H$ is connected$\}$;
- $\mathcal{T} = \{H \in \mathcal{G}\colon\ H$ is without isolates$\}$;
- $\mathcal{F} = \{H \in \mathcal{G}\colon\ H$ is a forest$\}$;
- $\mathcal{UK} = \{H \in \mathcal{G}$: each component of H is complete$\}$.

A graph property $\mathcal{P}$ is called *hereditary* (*induced-hereditary*), if from the fact that a graph G has the property $\mathcal{P}$, it follows that all subgraphs (induced subgraphs) of G also belong to $\mathcal{P}$. A property is called *additive* if it is closed under taking disjoint unions of graphs. A property $\mathcal{P}$ is called *nondegenerate* if $\mathcal{I} \subseteq \mathcal{P}$. Note that: (a) $\mathcal{I}$ and $\mathcal{F}$ are nondegenerate, additive and hereditary properties; (b) $\mathcal{UK}$ is nondegenerate, additive, induced-hereditary and is not hereditary; (c) $\mathcal{C}$ is neither additive nor induced-hereditary nor nondegenerate; (d) $\mathcal{T}$ is additive but neither induced-hereditary nor nondegenerate. Further, an additive and induced-hereditary property is always nondegenerate.

A *dominating set* for a graph G is a set of vertices $D \subseteq V(G)$ such that every vertex of G is either in D or is adjacent to an element of D. A dominating set D is a *minimal dominating set* if no set $D' \subsetneq D$ is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $\mathrm{MDS}(G)$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. The *upper domination number* $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of G.

Any set $S \subseteq V(G)$ such that the subgraph $\langle S, G\rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set. The concept of domination with respect to any property $\mathcal{P}$ was introduced by Goddard et al. [7]. The *domination number with respect to the property* $\mathcal{P}$, denoted by $\gamma_{\mathcal{P}}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of G. Note that there may be no dominating $\mathcal{P}$-set of G at all. For example, all graphs having at least two isolated vertices are without dominating $\mathcal{P}$-sets, where $\mathcal{P} \in \{\mathcal{C}, \mathcal{T}\}$. On the other hand, if a property $\mathcal{P}$ is nondegenerate then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_{\mathcal{P}}(G)$ exists. Let S be a dominating $\mathcal{P}$-set of a graph G. Then S is a *minimal dominating $\mathcal{P}$-set* if no set $S' \subsetneq S$ is a dominating $\mathcal{P}$-set. The set of all minimal dominating $\mathcal{P}$-sets of a graph G is denoted by $\mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G)$. The *upper domination number with respect to the property* $\mathcal{P}$, denoted by $\Gamma_{\mathcal{P}}(G)$, is the maximum cardinality of a minimal dominating $\mathcal{P}$-set of G. Michalak [12] has considered these parameters when the property is additive and induced-hereditary. Note that:

(a) in the case $\mathcal{P} = \mathcal{G}$ we have $\mathrm{MD}_{\mathcal{G}}\,\mathrm{S}(G) = \mathrm{MDS}(G)$, $\gamma_{\mathcal{G}}(G) = \gamma(G)$ and $\Gamma_{\mathcal{G}}(G) = \Gamma(G)$;

(b) in the case $\mathcal{P} = \mathcal{I}$, every element of $\mathrm{MD}_{\mathcal{I}}\,\mathrm{S}(G)$ is an independent dominating set and the numbers $\gamma_{\mathcal{I}}(G)$ and $\Gamma_{\mathcal{I}}(G)$ are well known as the *independent domination number* $i(G)$ and the *independence number* $\beta_0(G)$;

(c) in the case $\mathcal{P} = \mathcal{C}$, every element of $\mathrm{MD}_{\mathcal{C}}\,\mathrm{S}(G)$ is a connected dominating set of G, $\gamma_{\mathcal{C}}(G)$ $(\Gamma_{\mathcal{C}}(G))$ is denoted by $\gamma_c(G)$ $(\Gamma_c(G))$ and is called the *connected (upper connected) domination number*;

(d) in the case $\mathcal{P} = \mathcal{T}$, every element of $\mathrm{MD}_{\mathcal{T}}\,\mathrm{S}(G)$ is a total dominating set of G, $\gamma_{\mathcal{T}}(G)$ $(\Gamma_{\mathcal{T}}(G))$ is denoted by $\gamma_t(G)$ $(\Gamma_t(G))$ and is called the *total (upper total) domination number*;

(e) in the case $\mathcal{P} = \mathcal{F}$, every element of $\mathrm{MD}_{\mathcal{F}}\,\mathrm{S}(G)$ is an acyclic and dominating set of G, $\gamma_{\mathcal{F}}(G)$ $(\Gamma_{\mathcal{F}}(G))$ is denoted by $\gamma_a(G)$ $(\Gamma_a(G))$ and is called the *acyclic (upper acyclic) domination number*. The concept of acyclic domination in graphs was introduced by Hedetniemi et al. [10].

From the above definitions we immediately have

Observation 1.1. *Let $\mathcal{I} \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_1 \subseteq \mathcal{G}$ and let G be a graph. Then*

(1) [7] $\gamma(G) \leqslant \gamma_{\mathcal{P}_1}(G) \leqslant \gamma_{\mathcal{P}_2}(G) \leqslant i(G)$;

(2) [7] $\Gamma(G) \geqslant \Gamma_{\mathcal{P}_1}(G) \geqslant \Gamma_{\mathcal{P}_2}(G) \geqslant \beta_0(G)$.

Observation 1.2. *Let G be a graph, $\mathcal{P} \subseteq \mathcal{G}$ and $\mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G) \neq \emptyset$. A dominating $\mathcal{P}$-set $S \subseteq V(G)$ is a minimal dominating $\mathcal{P}$-set if and only if for each nonempty subset $U \subsetneq S$ at least one of the following holds:*

(a) *there is a vertex $v \in (V(G) - S) \cup U$ with $\emptyset \neq N[v, G] \cap S \subseteq U$;*

(b) *$S - U$ is no $\mathcal{P}$-set.*

Proof. Assume first that $S \in \mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G)$, $\emptyset \neq U \subsetneq S$ and $S_U = S - U$ is a $\mathcal{P}$-set of G. Hence some vertex v in $V(G) - S_U$ has no neighbors in S_U. If $v \in U$ then $\emptyset \neq N[v, G] \cap S \subseteq U$. Let $v \in V(G) - S$. Since v is not dominated by S_U but is dominated by S it follows that $\emptyset \neq N[v, G] \cap S \subseteq U$. In both cases, condition (a) holds.

For the converse, suppose S is a dominating $\mathcal{P}$-set of G and for each U, $\emptyset \neq U \subsetneq S$ one of the two above stated conditions holds. Suppose to the contrary that $S \notin \mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G)$. Then there exists a set U, $\emptyset \neq U \subsetneq S$ such that $S_U = S - U$ is a dominating $\mathcal{P}$-set. Since S_U is a $\mathcal{P}$-set, condition (b) does not hold. Since S_U is a dominating set it follows that every vertex of $V(G) - S_U$ has at least one neighbor in S_U, that is, condition (a) does not hold. Thus in all cases we have a contradiction. $\square$

Corollary 1.3. *Let G be a graph, $\mathcal{P} \subseteq \mathcal{G}$ be an induced-hereditary property and $\mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G) \neq \emptyset$. A dominating $\mathcal{P}$-set $S \subseteq V(G)$ is a minimal dominating $\mathcal{P}$-set if and only if $\mathrm{pn}_G[u, S] \neq \emptyset$ for each vertex $u \in S$.*

This result when $\mathcal{P} = \mathcal{G}$ was proved by Ore [13].

We shall use the therm $\gamma_{\mathcal{P}}$-set for a minimal dominating $\mathcal{P}$-set of cardinality $\gamma_{\mathcal{P}}(G)$. Let G be a graph and $v \in V(G)$. Fricke et al. [5] defined a vertex v to be

(f) $\gamma_{\mathcal{P}}$-*good*, if v belongs to some $\gamma_{\mathcal{P}}$-set of G;

(g) $\gamma_{\mathcal{P}}$-*bad*, if v belongs to no $\gamma_{\mathcal{P}}$-set of G;

Sampathkumar and Neerlagi [16] defined a $\gamma_{\mathcal{P}}$-good vertex v to be

(h) $\gamma_{\mathcal{P}}$-*fixed* if v belongs to every $\gamma_{\mathcal{P}}$-set;

(i) $\gamma_{\mathcal{P}}$-*free* if v belongs to some $\gamma_{\mathcal{P}}$-set but not to all $\gamma_{\mathcal{P}}$-sets.

For a graph G and a property $\mathcal{P} \subseteq \mathcal{G}$ such that $\mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G) \neq \emptyset$ we define:

$\mathbf{G}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-good}\}$;

$\mathbf{B}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-bad}\}$;

$\mathbf{Fi}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-fixed}\}$;

$\mathbf{Fr}_{\mathcal{P}}(G) = \{x \in V(G)\colon\ x \text{ is } \gamma_{\mathcal{P}}\text{-free}\}$.

Clearly $\{\mathbf{G}_{\mathcal{P}}(G), \mathbf{B}_{\mathcal{P}}(G)\}$ is a partition of $V(G)$, and $\{\mathbf{Fi}_{\mathcal{P}}(G), \mathbf{Fr}_{\mathcal{P}}(G)\}$ is a partition of $\mathbf{G}_{\mathcal{P}}(G)$. If additionally $\mathrm{MD}_{\mathcal{P}}\,\mathrm{S}(G - v) \neq \emptyset$ for each vertex $v \in V(G)$, then we define:

$\mathbf{V}^0_{\mathcal{P}}(G) = \{x \in V(G)\colon\ \gamma_{\mathcal{P}}(G - x) = \gamma_{\mathcal{P}}(G)\}$;

$\mathbf{V}^-_{\mathcal{P}}(G) = \{x \in V(G)\colon\ \gamma_{\mathcal{P}}(G - x) < \gamma_{\mathcal{P}}(G)\}$;

$\mathbf{V}^+_{\mathcal{P}}(G) = \{x \in V(G)\colon\ \gamma_{\mathcal{P}}(G - x) > \gamma_{\mathcal{P}}(G)\}$.

In this case $\{\mathbf{V}^-_{\mathcal{P}}(G), \mathbf{V}^0_{\mathcal{P}}(G), \mathbf{V}^+_{\mathcal{P}}(G)\}$ is a partition of $V(G)$.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In this connection, in this paper we consider this question in the case $\gamma_{\mathcal{P}}(G)$ when a vertex is deleted from G or an edge from $\overline{G}$ is added to G.

2. Vertex deletion

In this section we examine the effects on $\gamma_{\mathcal{P}}$ when a graph is modified by deleting a vertex.

Theorem 2.1. *Let G be a graph, $u, v \in V(G)$, $u \neq v$ and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under union with K_1.*

(i) *Let $v \in \mathbf{V}^-_{\mathcal{H}}(G)$.*

(i.1) *If $uv \in E(G)$ then u is a $\gamma_{\mathcal{H}}$-bad vertex of $G - v$;*

(i.2) *if* M *is a* $\gamma_{\mathcal{H}}$*-set of* $G-v$ *then* $M\cup\{v\}$ *is a* $\gamma_{\mathcal{H}}$*-set of* G *and* $\{v\}=\mathrm{pn}_G[v,M\cup\{v\}]$;

(i.3) $\gamma_{\mathcal{H}}(G-v)=\gamma_{\mathcal{H}}(G)-1$;

(ii) *let* $v\in\mathbf{V}^+_{\mathcal{H}}(G)$. *Then* v *is a* $\gamma_{\mathcal{H}}$*-fixed vertex of* G;

(iii) *if* $v\in\mathbf{V}^-_{\mathcal{H}}(G)$ *and* u *is a* $\gamma_{\mathcal{H}}$*-fixed vertex of* G *then* $uv\notin E(G)$;

(iv) *if* v *is a* $\gamma_{\mathcal{H}}$*-bad vertex of* G *then* $\gamma_{\mathcal{H}}(G-v)=\gamma_{\mathcal{H}}(G)$;

(v) *if* $v\in\mathbf{V}^-_{\mathcal{H}}(G)$ *and* $uv\in E(G)$ *then* $\gamma_{\mathcal{H}}(G-\{u,v\})=\gamma_{\mathcal{H}}(G)-1$.

Proof. (i.1): Let $uv\in E(G)$ and let M be a $\gamma_{\mathcal{H}}$-set of $G-v$. If $u\in M$ then M is a dominating $\mathcal{H}$-set of G with $|M|<\gamma_{\mathcal{H}}(G)$—a contradiction.

(i.2) and (i.3): Let M be a $\gamma_{\mathcal{H}}$-set of $G-v$. By (i.1), $M_1=M\cup\{v\}$ is a dominating set of G. Any vertex $u\in V(G)-M_1$ has a neighbor in M, hence v is isolated in M_1 (otherwise M would dominate G) and $\{v\}=\mathrm{pn}_G[v,M\cup\{v\}]$. Since $\mathcal{H}$ is closed under union with K_1 it follows that M_1 is a dominating $\mathcal{H}$-set of G and $|M_1|=\gamma_{\mathcal{H}}(G-v)+1\leqslant\gamma_{\mathcal{H}}(G)$. Hence M_1 is a $\gamma_{\mathcal{H}}$-set of G and $\gamma_{\mathcal{H}}(G-v)=\gamma_{\mathcal{H}}(G)-1$.

(ii): If M is a $\gamma_{\mathcal{H}}$-set of G and $v\notin M$ then M is a dominating $\mathcal{H}$-set of $G-v$. But then $\gamma_{\mathcal{H}}(G)=|M|\geqslant\gamma_{\mathcal{H}}(G-v)>\gamma_{\mathcal{H}}(G)$ and the result follows.

(iii): Let $\gamma_{\mathcal{H}}(G-v)<\gamma_{\mathcal{H}}(G)$ and let M be a $\gamma_{\mathcal{H}}$-set of $G-v$. Then by (i.2), $M\cup\{v\}$ is a $\gamma_{\mathcal{H}}$-set of G. This implies that $u\in M$ and by (i.1) we have $uv\notin E(G)$.

(iv): By (ii), $\gamma_{\mathcal{H}}(G-v)\leqslant\gamma_{\mathcal{H}}(G)$ and by (i.2), $\gamma_{\mathcal{H}}(G-v)\geqslant\gamma_{\mathcal{H}}(G)$.

(v): Immediately follows by (i) and (iv). □

Let $\mathcal{P}\subseteq\mathcal{G}$ be nondegenerate and closed under union with K_1. Since $\gamma_{\mathcal{P}}(G-v)\leqslant|V(G)|-1$ for every $v\in V(G)$ and because of Theorem 2.1 we have $\gamma_{\mathcal{P}}(G-v)=\gamma_{\mathcal{P}}(G)+p$, where $p\in\{-1,0,1,\dots,|V(G)|-2\}$. This motivated us to define for a nontrivial graph G:

$\mathbf{Fr}^-_{\mathcal{P}}(G)=\{x\in\mathbf{Fr}_{\mathcal{P}}(G)\colon\ \gamma_{\mathcal{P}}(G-x)=\gamma_{\mathcal{P}}(G)-1\}$;

$\mathbf{Fr}^0_{\mathcal{P}}(G)=\{x\in\mathbf{Fr}_{\mathcal{P}}(G)\colon\ \gamma_{\mathcal{P}}(G-x)=\gamma_{\mathcal{P}}(G)\}$;

$\mathbf{Fi}^p_{\mathcal{P}}(G)=\{x\in\mathbf{Fi}_{\mathcal{P}}(G)\colon\ \gamma_{\mathcal{P}}(G-x)=\gamma_{\mathcal{P}}(G)+p\}$, $p\in\{-1,0,1,\dots,|V(G)|-2\}$.

We will refine the definitions of the $\gamma_{\mathcal{P}}$-free vertex and the $\gamma_{\mathcal{P}}$-fixed vertex. Let G be a graph and let $\mathcal{P}\subseteq\mathcal{G}$ be nondegenerate and closed under union with K_1. A vertex $x\in V(G)$ is called

(j) $\gamma^0_{\mathcal{P}}$*-free* if $x\in\mathbf{Fr}^0_{\mathcal{P}}(G)$;

(k) $\gamma^-_{\mathcal{P}}$*-free* if $x\in\mathbf{Fr}^-_{\mathcal{P}}(G)$;

(l) $\gamma^q_{\mathcal{P}}(G)$*-fixed* if $x\in\mathbf{Fi}^q_{\mathcal{P}}(G)$, where $q\in\{-1,0,1,\dots,|V(G)|-2\}$.

Now, by Theorem 2.1 we have

Corollary 2.2. *Let G be a graph of order $n \geqslant 2$ and let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under union with K_1. Then*
(1) $\{\mathbf{Fr}^{-}_{\mathcal{H}}(G), \mathbf{Fr}^{0}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{Fr}_{\mathcal{H}}(G)$*;*
(2) $\{\mathbf{Fi}^{-1}_{\mathcal{H}}(G), \mathbf{Fi}^{0}_{\mathcal{H}}(G), \ldots, \mathbf{Fi}^{n-2}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{Fi}_{\mathcal{H}}(G)$*;*
(3) $\{\mathbf{Fi}^{-1}_{\mathcal{H}}(G), \mathbf{Fr}^{-}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{V}^{-}_{\mathcal{H}}(G)$*;*
(4) $\{\mathbf{Fi}^{0}_{\mathcal{H}}(G), \mathbf{Fr}^{0}_{\mathcal{H}}(G), \mathbf{B}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{V}^{0}_{\mathcal{H}}(G)$*;*
(5) $\{\mathbf{Fi}^{1}_{\mathcal{H}}(G), \mathbf{Fi}^{2}_{\mathcal{H}}(G), \ldots, \mathbf{Fi}^{n-2}_{\mathcal{H}}(G)\}$ *is a partition of* $\mathbf{V}^{+}_{\mathcal{H}}(G)$*.*

A vertex v of a graph G is *$\gamma_{\mathcal{P}}$-critical* if $\gamma_{\mathcal{P}}(G - v) \neq \gamma_{\mathcal{P}}(G)$. The graph G is *vertex-$\gamma_{\mathcal{P}}$-critical* if all its vertices are $\gamma_{\mathcal{P}}$-critical.

Theorem 2.3. *Let G be a graph of order $n \geqslant 2$ and let $\mathcal{H} \subseteq \mathcal{G}$ be additive and induced-hereditary. Then G is a vertex-$\gamma_{\mathcal{H}}$-critical graph if and only if $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for all $v \in V(G)$.*

Proof. Necessity is obvious. Sufficiency: Let G be a vertex-$\gamma_{\mathcal{H}}$-critical graph. Clearly, $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for every isolated vertex $v \in V(G)$. Hence if G is isomorphic to $\overline{K}_n$ then $\gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1$ for all $v \in V(G)$. So, let G have a component of order at least two, say Q. Because of Theorem 2.1 (ii), (iii) and (i.3), either $\gamma_{\mathcal{H}}(Q - v) > \gamma_{\mathcal{H}}(Q)$ for all $v \in V(Q)$, or $\gamma_{\mathcal{H}}(Q - v) = \gamma_{\mathcal{H}}(Q) - 1$ for all $v \in V(Q)$. Suppose that $\gamma_{\mathcal{H}}(Q - v) > \gamma_{\mathcal{H}}(Q)$ for all $v \in V(Q)$. But then Theorem 2.1 (ii) implies that $V(Q)$ is a $\gamma_{\mathcal{H}}$-set of Q. This is a contradiction with $\gamma_{\mathcal{H}}(Q - v) > \gamma_{\mathcal{H}}(Q)$. □

Theorem 2.3 when $\mathcal{H} \in \{\mathcal{G}, \mathcal{I}, \mathcal{F}\}$ is due to Carrington et al. [2], Ao and MacGillivray (see [9, Chapter 16]) and the present author [15], respectively. Further properties of these graphs can be found in [1], [6], [8, Chapter 5], [9, Chapter 16], [11], [14].

Now we concentrate on graphs having cut-vertices. Observe that domination and some of its variants in graphs having cut-vertices have been the topic of several studies—see for example [1], [18], [14] and [9, Chapter 16].

Let G_1 and G_2 be connected graphs, both of order at least two, and let them have a unique vertex in common, say x. Then a *coalescence* $G_1 \overset{x}{\circ} G_2$ is the graph $G_1 \cup G_2$. Clearly, x is a cut-vertex of $G_1 \overset{x}{\circ} G_2$.

Theorem 2.4. *Let $G = G_1 \overset{x}{\circ} G_2$ and let $\mathcal{H} \subseteq \mathcal{G}$ be induced-hereditary and closed under union with K_1. Then $\gamma_{\mathcal{H}}(G) \geqslant \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$.*

Proof. Since $\mathcal{H}$ is induced-hereditary and closed under union with K_1 it follows that $\mathcal{H}$ is nondegenerate. Let M be a $\gamma_{\mathcal{H}}$-set of G and $M_i = M \cap V(G_i)$, $i = 1, 2$. Since $\mathcal{H}$ is induced-hereditary it follows that M_1 and M_2 are $\mathcal{H}$-sets of G_1 and G_2, respectively. Hence there exist three possibilities:

(a) $x \notin M$ and M_i is a dominating $\mathcal{H}$-set of G_i, $i = 1, 2$;

(b) $x \notin M$ and there are i, j such that $\{i, j\} = \{1, 2\}$, M_i is a dominating $\mathcal{H}$-set of G_i and M_j is a dominating $\mathcal{H}$-set of $G_j - x$;

(c) $x \in M$ and M_i is a dominating $\mathcal{H}$-set of G_i, $i = 1, 2$.

If (a) holds, then $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| \geqslant \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2)$. If (c) holds then $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| - 1 \geqslant \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$. Finally, let (b) hold. Then $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| \geqslant \gamma_{\mathcal{H}}(G_i) + \gamma_{\mathcal{H}}(G_j - x)$. Now by Theorem 2.1 (i), $\gamma_{\mathcal{H}}(G) \geqslant \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$.

Thus, in all cases, $\gamma_{\mathcal{H}}(G) \geqslant \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$. $\square$

Theorem 2.5. *Let $G = G_1 \overset{x}{\circ} G_2$, let $\mathcal{H} \subseteq \mathcal{G}$ be additive and induced-hereditary, and $\gamma_{\mathcal{H}}(G_1 - x) < \gamma_{\mathcal{H}}(G_1)$. Then*

(a) *$\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$;*

(b) *if $\gamma_{\mathcal{H}}(G_2 - x) < \gamma_{\mathcal{H}}(G_2)$ then $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G) - 1$;*

(c) *if $\gamma_{\mathcal{H}}(G_2 - x) > \gamma_{\mathcal{H}}(G_2)$ then x is a $\gamma_{\mathcal{H}}$-fixed vertex of G;*

(d) *if x is a $\gamma_{\mathcal{H}}$-bad vertex of G_2 then x is a $\gamma_{\mathcal{H}}$-bad vertex of G.*

Proof. Since $\mathcal{H}$ is additive and induced-hereditary it follows that $\mathcal{H}$ is nondegenerate and closed under union with K_1.

(a): Let U_1 be a $\gamma_{\mathcal{H}}$-set of $G_1 - x$ and let U_2 be a $\gamma_{\mathcal{H}}$-set of G_2. Then $U = U_1 \cup U_2$ is a dominating set of G. It follows by Theorem 2.1(i.2) that $\langle U, G \rangle$ has two components, namely $\langle U_1, G \rangle$ and $\langle U_2, G \rangle$. Since $\mathcal{H}$ is additive, U is an $\mathcal{H}$-set of G. Thus U is a dominating $\mathcal{H}$-set of G. Hence $\gamma_{\mathcal{H}}(G) \leqslant |U_1 \cup U_2| = \gamma_{\mathcal{H}}(G_1 - x) + \gamma_{\mathcal{H}}(G_2) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$. Now the result follows by Theorem 2.4.

(b): By Theorem 2.1 (i.3) we have $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G_1 - x) + \gamma_{\mathcal{H}}(G_2 - x) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 2$. Hence by (a), $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G) - 1$.

(c): $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G_1 - x) + \gamma_{\mathcal{H}}(G_2 - x) = \gamma_{\mathcal{H}}(G_1) - 1 + \gamma_{\mathcal{H}}(G_2 - x) = \gamma_{\mathcal{H}}(G) + \gamma_{\mathcal{H}}(G_2 - x) - \gamma_{\mathcal{H}}(G_2) > \gamma_{\mathcal{H}}(G)$. The result now follows by Theorem 2.1 (ii).

(d): Let M be a $\gamma_{\mathcal{H}}$-set of G and $M_i = M \cap V(G_i)$, $i = 1, 2$. Suppose $x \in M$. Hence M_i is a dominating $\mathcal{H}$-set of G_i, $i = 1, 2$ and then $\gamma_{\mathcal{H}}(G_i) \leqslant |M_i|$. Since x belongs to no $\gamma_{\mathcal{H}}$-set of G_2 we have $|M_2| > \gamma_{\mathcal{H}}(G_2)$. Hence $\gamma_{\mathcal{H}}(G) = |M| = |M_1| + |M_2| - 1 \geqslant \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2)$—a contradiction with (a). $\square$

Theorem 2.6. *Let $\mathcal{H} \subseteq \mathcal{G}$ be additive and induced-hereditary and let $G = G_1 \overset{x}{\circ} G_2$, where G_1, G_2 are both vertex-$\gamma_{\mathcal{H}}$-critical. Then G is vertex-$\gamma_{\mathcal{H}}$-critical and $\gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G_1) + \gamma_{\mathcal{H}}(G_2) - 1$.*

Proof. By Theorem 2.5(b) it follows that $\gamma_{\mathcal{H}}(G) - 1 = \gamma_{\mathcal{H}}(G - x)$. Let without loss of generality $y \in V(G_2 - x)$. If $G_2 - y$ is connected then $G - y = G_1 \overset{x}{\circ} (G_2 - y)$ and by Theorem 2.5(a), $\gamma_{\mathcal{H}}(G-y)=\gamma_{\mathcal{H}}(G_1)+\gamma_{\mathcal{H}}(G_2-y)-1=\gamma_{\mathcal{H}}(G_1)+\gamma_{\mathcal{H}}(G_2)-2=\gamma_{\mathcal{H}}(G)-1$.

So, assume G_2-y is not connected and let Q be the component of G_2-y which contains x. By Theorem 2.1 (i), $V(Q)\neq\{x\}$. Now, by Theorem 2.5 (a), $\gamma_{\mathcal{H}}(G_1\overset{x}{\circ}Q)=\gamma_{\mathcal{H}}(G_1)+\gamma_{\mathcal{H}}(Q)-1$ and then $\gamma_{\mathcal{H}}(G-y)=\gamma_{\mathcal{H}}(G_1\overset{x}{\circ}Q)+\gamma_{\mathcal{H}}(G_2-(V(Q)\cup\{y\}))=\gamma_{\mathcal{H}}(G_1)+\gamma_{\mathcal{H}}(G_2-y)-1=\gamma_{\mathcal{H}}(G_1)+\gamma_{\mathcal{H}}(G_2)-2=\gamma_{\mathcal{H}}(G)-1$. □

3. Edge addition

Here we present results on changing and unchanging of $\gamma_{\mathcal{P}}(G)$ when an edge from $\overline{G}$ is added to G. Recall that if a property $\mathcal{P}$ is hereditary and closed under union with K_1 then $\mathcal{P}$ is nondegenerate and hence all graphs have a domination number with respect to $\mathcal{P}$.

Theorem 3.1. *Let x and y be two different and nonadjacent vertices in a graph G. Let $\mathcal{H}\subseteq\mathcal{G}$ be hereditary and closed under union with K_1. If $\gamma_{\mathcal{H}}(G+xy)<\gamma_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G+xy)=\gamma_{\mathcal{H}}(G)-1$. Moreover, $\gamma_{\mathcal{H}}(G+xy)=\gamma_{\mathcal{H}}(G)-1$ if and only if at least one of the following holds:*

(i) *$x\in\mathbf{V}^-_{\mathcal{H}}(G)$ and y is a $\gamma_{\mathcal{H}}$-good vertex of $G-x$;*

(ii) *x is a $\gamma_{\mathcal{H}}$-good vertex of $G-y$ and $y\in\mathbf{V}^-_{\mathcal{H}}(G)$.*

Proof. Let $\gamma_{\mathcal{H}}(G+xy)<\gamma_{\mathcal{H}}(G)$ and let M be a $\gamma_{\mathcal{H}}$-set of $G+xy$. Since $\mathcal{H}$ is hereditary, M is an $\mathcal{H}$-set of G. Further, $|\{x,y\}\cap M|=1$, otherwise M would be a dominating $\mathcal{H}$-set of G, a contradiction. Let without loss of generality $x\notin M$ and $y\in M$. Since M is an $\mathcal{H}$-set of G it follows that M is no dominating set of G, which implies $M\cap N(x,G)=\emptyset$. Hence $M_1=M\cup\{x\}$ is a dominating $\mathcal{H}$-set of G with $|M_1|=\gamma_{\mathcal{H}}(G+xy)+1$, which implies $\gamma_{\mathcal{H}}(G)=\gamma_{\mathcal{H}}(G+xy)+1$. Since M is a dominating $\mathcal{H}$-set of $G-x$ we have $\gamma_{\mathcal{H}}(G-x)\leqslant\gamma_{\mathcal{H}}(G+xy)$. Hence $\gamma_{\mathcal{H}}(G)\geqslant\gamma_{\mathcal{H}}(G-x)+1$ and Theorem 2.1 implies $\gamma_{\mathcal{H}}(G)=\gamma_{\mathcal{H}}(G-x)+1$. Thus x is in $\mathbf{V}^-_{\mathcal{H}}(G)$ and M is a $\gamma_{\mathcal{H}}$-set of $G-x$. Since $y\in M$, y is a $\gamma_{\mathcal{H}}$-good vertex of $G-x$.

For the converse let without loss of generality (i) hold. Then there is a $\gamma_{\mathcal{H}}$-set M of $G-x$ with $y\in M$. Certainly M is a dominating $\mathcal{H}$-set of $G+xy$ and consequently $\gamma_{\mathcal{H}}(G+xy)\leqslant|M|=\gamma_{\mathcal{H}}(G-x)=\gamma_{\mathcal{H}}(G)-1\leqslant\gamma_{\mathcal{H}}(G+xy)$. □

Corollary 3.2. *Let x and y be two different and nonadjacent vertices in a graph G, let $\mathcal{H}\subseteq\mathcal{G}$ be hereditary and closed under union with K_1, and let $x\in\mathbf{V}^-_{\mathcal{H}}(G)$. Then $\gamma_{\mathcal{H}}(G)-1\leqslant\gamma_{\mathcal{H}}(G+xy)\leqslant\gamma_{\mathcal{H}}(G)$.*

Proof. Let M be a $\gamma_{\mathcal{H}}$-set of $G-x$. If $y\in\mathbf{G}_{\mathcal{H}}(G-x)$ then Theorem 3.1 yields $\gamma_{\mathcal{H}}(G)-1=\gamma_{\mathcal{H}}(G+xy)$. So, let $y\in\mathbf{B}_{\mathcal{H}}(G-x)$. By Theorem 2.1, $M_1=M\cup\{x\}$

is a $\gamma_{\mathcal{H}}$-set of G and $M_1 \cap N(x,G) = \emptyset$. Hence M_1 is a dominating $\mathcal{H}$-set of $G+xy$ and $\gamma_{\mathcal{H}}(G+xy) \leqslant |M_1| = \gamma_{\mathcal{H}}(G-x)+1 = \gamma_{\mathcal{H}}(G)$. $\square$

We need the following lemma:

Lemma 3.3. *Let $\mathcal{H} \subseteq \mathcal{G}$ be nondegenerate and closed under union with K_1 and let x be a $\gamma^0_{\mathcal{H}}$-fixed vertex of a graph G. Then $N(x,G) \subseteq \mathbf{B}_{\mathcal{H}}(G-x) \cap (\mathbf{V}^0_{\mathcal{H}}(G) \cup \mathbf{Fi}^1_{\mathcal{H}}(G))$ and for each $y \in N(x,G)$, $\gamma_{\mathcal{H}}(G-\{x,y\}) = \gamma_{\mathcal{H}}(G)$.*

Proof. Let M be a $\gamma_{\mathcal{H}}$-set of $G-x$ and $y \in N(x,G)$. If $y \in M$ then M is a dominating $\mathcal{H}$-set of G of cardinality $|M| = \gamma_{\mathcal{H}}(G-x) = \gamma_{\mathcal{H}}(G)$—a contradiction with $x \in \mathbf{Fi}_{\mathcal{H}}(G)$. Thus $N(x,G) \subseteq \mathbf{B}_{\mathcal{H}}(G-x)$. Now by Theorem 2.1 (iv), $\gamma_{\mathcal{H}}(G-\{x,y\}) = \gamma_{\mathcal{H}}(G-x) = \gamma_{\mathcal{H}}(G)$. Further, Theorem 2.1(iii) implies $y \notin \mathbf{V}^-_{\mathcal{H}}(G)$. If $y \notin \mathbf{V}^0_{\mathcal{H}}(G)$, from Corollary 2.2(5) it follows that $y \in \mathbf{Fi}^p_{\mathcal{H}}(G)$ for some $p \geqslant 1$. Assume $p \geqslant 2$. Since M is a dominating $\mathcal{H}$-set of $G-x$ and $M \cap N(x,G) = \emptyset$ it follows that $M_2 = M \cup \{x\}$ is a dominating $\mathcal{H}$-set of G and $y \notin M_2$. Hence M_2 is a dominating $\mathcal{H}$-set of $G-y$. This implies $\gamma_{\mathcal{H}}(G)+p = \gamma_{\mathcal{H}}(G-y) \leqslant |M_2| = |M|+1 = \gamma_{\mathcal{H}}(G-x)+1 = \gamma_{\mathcal{H}}(G)+1$, a contradiction. $\square$

It is a well known fact that $\gamma(G+e) \leqslant \gamma(G)$ for any edge $e \in \overline{G}$. In general, for $\gamma_{\mathcal{P}}$ this is not valid.

Theorem 3.4. *Let x and y be two different and nonadjacent vertices in a graph G and let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with K_1. Then $\gamma_{\mathcal{H}}(G+xy) > \gamma_{\mathcal{H}}(G)$ if and only if no $\gamma_{\mathcal{H}}$-set of G is an $\mathcal{H}$-set of $G+xy$ and one of the following holds:*

(1) *x is a $\gamma^p_{\mathcal{H}}$-fixed vertex of G and y is a $\gamma^q_{\mathcal{H}}$-fixed vertex of G for some $p,q \geqslant 1$;*
(2) *$x \in \mathbf{Fi}^0_{\mathcal{H}}(G)$ and $y \in \mathbf{Fi}^1_{\mathcal{H}}(G) \cap \mathbf{B}_{\mathcal{H}}(G-x)$;*
(3) *$x \in \mathbf{Fi}^1_{\mathcal{H}}(G) \cap \mathbf{B}_{\mathcal{H}}(G-y)$ and $y \in \mathbf{Fi}^0_{\mathcal{H}}(G)$;*
(4) *$x,y \in \mathbf{Fi}^0_{\mathcal{H}}(G)$, $x \in \mathbf{B}_{\mathcal{H}}(G-y)$ and $y \in \mathbf{B}_{\mathcal{H}}(G-x)$.*

Proof. Let $\gamma_{\mathcal{H}}(G+xy) > \gamma_{\mathcal{H}}(G)$. By Corollary 3.2 we have $x,y \in \mathbf{V}^0_{\mathcal{H}}(G) \cup \mathbf{V}^+_{\mathcal{H}}(G)$. Assume to the contrary that (without loss of generality) $x \notin \mathbf{Fi}_{\mathcal{H}}(G)$. Hence there is a $\gamma_{\mathcal{H}}$-set M of G with $x \notin M$. But then M is a dominating $\mathcal{H}$-set of $G+xy$ and $|M| = \gamma_{\mathcal{H}}(G) < \gamma_{\mathcal{H}}(G+xy)$—a contradiction. Thus both x and y are $\gamma_{\mathcal{H}}$-fixed vertices of G. This implies that each $\gamma_{\mathcal{H}}$-set M of G is a dominating set of $G+xy$ but not an $\mathcal{H}$-set of $G+xy$.

Let x be $\gamma^p_{\mathcal{H}}$-fixed, let y be $\gamma^q_{\mathcal{H}}$-fixed and without loss of generality, $q \geqslant p \geqslant 0$. Assume (1) does not hold. Hence $p=0$. Let M_1 be a $\gamma_{\mathcal{H}}$-set of $G-x$. Then $|M_1| = \gamma_{\mathcal{H}}(G-x) = \gamma_{\mathcal{H}}(G) < \gamma_{\mathcal{H}}(G+xy)$ and $y \notin M_1$, for otherwise M_1 would be a dominating $\mathcal{H}$-set of $G+xy$; thus y is a $\gamma_{\mathcal{H}}$-bad vertex of $G-x$. By Lemma 3.3,

$N(x, G) \cap M_1 = \emptyset$. Then $M_1 \cup \{x\}$ is a dominating $\mathcal{H}$-set of $G + xy$, which implies $\gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G)+1$. Since $y \notin M_1 \cup \{x\}$ it follows that $M_1 \cup \{x\}$ is a dominating $\mathcal{H}$-set of $G - y$ and then $\gamma_{\mathcal{H}}(G) + 1 = |M_1 \cup \{x\}| \geqslant \gamma_{\mathcal{H}}(G - y) = \gamma_{\mathcal{H}}(G) + q$. So, $q \in \{0, 1\}$. If $q = 1$ then (2) holds. If $q = 0$ then, by symmetry, it follows that x is a $\gamma_{\mathcal{H}}$-bad vertex of $G - y$ and hence (4) holds.

For the converse, let no $\gamma_{\mathcal{H}}$-set of G be an $\mathcal{H}$-set of $G + xy$ and let one of the conditions (1), (2), (3) and (4) hold. Assume to the contrary that $\gamma_{\mathcal{H}}(G + xy) \leqslant \gamma_{\mathcal{H}}(G)$. By Theorem 3.1, $\gamma_{\mathcal{H}}(G + xy) = \gamma_{\mathcal{H}}(G)$. Let M_2 be a $\gamma_{\mathcal{H}}$-set of $G + xy$. Hence $|M_2 \cap \{x, y\}| = 1$—otherwise M_2 would be a $\gamma_{\mathcal{H}}$-set of G. Let without loss of generality $x \notin M_2$. Then M_2 is a dominating $\mathcal{H}$-set of $G - x$, which implies $\gamma_{\mathcal{H}}(G - x) \leqslant |M_2| = \gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G)$. Thus $\gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G)$ and then M_2 is a $\gamma_{\mathcal{H}}$-set of $G - x$. Hence x is a $\gamma^0_{\mathcal{H}}$-fixed vertex of G and y is a $\gamma_{\mathcal{H}}$-good vertex of $G - x$, which is a contradiction with each of (1)–(4). □

By Theorem 3.1 and Theorem 3.4 we immediately obtain:

Theorem 3.5. *Let x and y be two different and nonadjacent vertices in a graph G. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with K_1. Then $\gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G)$ if and only if at least one of the following holds:*

(1) *$x \in \mathbf{V}^-_{\mathcal{H}}(G) \cap \mathbf{B}_{\mathcal{H}}(G - y)$ and $y \in \mathbf{V}^-_{\mathcal{H}}(G) \cap \mathbf{B}_{\mathcal{H}}(G - x)$;*
(2) *$x \in \mathbf{V}^-_{\mathcal{H}}(G)$ and $y \in \mathbf{B}_{\mathcal{H}}(G - x) - \mathbf{V}^-_{\mathcal{H}}(G)$;*
(3) *$x \in \mathbf{B}_{\mathcal{H}}(G - y) - \mathbf{V}^-_{\mathcal{H}}(G)$ and $y \in \mathbf{V}^-_{\mathcal{H}}(G)$;*
(4) *$x, y \notin \mathbf{V}^-_{\mathcal{H}}(G)$ and $|\{x, y\} \cap \mathbf{Fi}_{\mathcal{H}}(G)| \leqslant 1$;*
(5) *$x \in \mathbf{Fi}^0_{\mathcal{H}}(G)$ and $y \in \mathbf{Fi}^s_{\mathcal{H}}(G) \cap \mathbf{G}_{\mathcal{H}}(G - x)$ for some $s \in \{0, 1\}$;*
(6) *$x \in \mathbf{Fi}^s_{\mathcal{H}}(G) \cap \mathbf{G}_{\mathcal{H}}(G - y)$ and $y \in \mathbf{Fi}^0_{\mathcal{H}}(G)$ for some $s \in \{0, 1\}$;*
(7) *$x \in \mathbf{Fi}^0_{\mathcal{H}}(G)$ and $y \in \mathbf{Fi}^q_{\mathcal{H}}(G)$ for some $q \geqslant 2$;*
(8) *$x \in \mathbf{Fi}^q_{\mathcal{H}}(G)$ and $y \in \mathbf{Fi}^0_{\mathcal{H}}(G)$ for some $q \geqslant 2$;*
(9) *there is a $\gamma_{\mathcal{H}}$-set of G which is an $\mathcal{H}$-set of $G + xy$ and one of the conditions (1), (2), (3) and (4) stated in Theorem 3.4 holds.*

Corollary 3.6. *Let x and y be two different and nonadjacent vertices in a graph G. Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with K_1. If $x \in \mathbf{B}_{\mathcal{H}}(G)$ then $\gamma_{\mathcal{H}}(G + xy) = \gamma_{\mathcal{H}}(G)$.*

Proof. By Theorem 2.1 (iv), $x \notin \mathbf{V}^-_{\mathcal{H}}(G)$. If $y \notin \mathbf{V}^-_{\mathcal{H}}(G)$ then the result follows by Theorem 3.5(4). If $y \in \mathbf{V}^-_{\mathcal{H}}(G)$ then by Theorem 2.1 (i.2) we have $x \in \mathbf{B}_{\mathcal{H}}(G-y)$ and the result now follows by Theorem 3.5(3). □

Let $\mu \in \{\gamma, \gamma_c, i\}$. A graph G is edge-μ-critical if $\mu(G + e) < \mu(G)$ for every edge e not belonging to G. These concepts were introduced by Sumner and Blitch [17], Xue-Gang Chen et al. [3] and Ao and MacGillivray [9, Chapter 16], respectively.

Here we define a graph G to be *edge-$\gamma_{\mathcal{P}}$-critical* if $\gamma_{\mathcal{P}}(G+e) \neq \gamma_{\mathcal{P}}(G)$ for every edge e of $\overline{G}$, where $\mathcal{P} \subseteq \mathcal{G}$ is hereditary and closed under union with K_1. Relating edge addition and vertex removal, Sumner and Blitch [17] and Ao and MacGillivray showed that $\mathbf{V}^+_{\mathcal{P}}(G)$ is empty for $\mathcal{P} \in \{\mathcal{G}, \mathcal{I}\}$, respectively. Furthermore, Favaron et al. [4] showed that if $\mathbf{V}^0_{\mathcal{G}}(G) \neq \emptyset$ then $\langle \mathbf{V}^0_{\mathcal{G}}(G), G \rangle$ is complete. In general, for edge-$\gamma_{\mathcal{P}}$-critical graphs the following holds.

Theorem 3.7. *Let $\mathcal{H} \subseteq \mathcal{G}$ be hereditary and closed under union with K_1 and let G be an edge-$\gamma_{\mathcal{H}}$-critical graph. Then*

(1) *$V(G) = \mathbf{Fi}^{-1}_{\mathcal{H}}(G) \cup \mathbf{Fr}_{\mathcal{H}}(G)$ and if $\mathbf{Fr}^0_{\mathcal{H}}(G) \neq \emptyset$ then $\langle \mathbf{Fr}^0_{\mathcal{H}}(G), G \rangle$ is complete;*

(2) *$\gamma_{\mathcal{H}}(G+e) < \gamma_{\mathcal{H}}(G)$ for every edge e not belonging to G.*

Proof. (1) If $x, y \in \mathbf{Fr}^0_{\mathcal{H}}(G)$ and $xy \notin E(G)$ then Theorem 3.5(4) implies $\gamma_{\mathcal{H}}(G+xy) = \gamma_{\mathcal{H}}(G)$, a contradiction. If $x \in \mathbf{B}_{\mathcal{H}}(G)$ then Corollary 3.6 implies $N[x, G] = V(G)$ and hence $\{x\}$ is a $\gamma_{\mathcal{H}}$-set of G—a contradiction. Assume $x \in \mathbf{Fi}^q_{\mathcal{H}}(G)$ for some $q \geqslant 0$. Let M be any $\gamma_{\mathcal{H}}$-set of G. By Corollary 1.3, $\operatorname{pn}_G[x, M] \neq \emptyset$. If $\operatorname{pn}_G[x, M] = \{x\}$ then $M - \{x\}$ dominates $G - x$, so $x \in \mathbf{V}^-_{\mathcal{H}}(G)$—a contradiction. Hence there is $y \in \operatorname{pn}_G[x, M] - \{x\}$. Since $\operatorname{pn}_G[x, M] \cap \mathbf{V}^-_{\mathcal{H}}(G) = \emptyset$ (by Theorem 2.1 (iii)), $\mathbf{B}_{\mathcal{H}}(G) = \emptyset$ and $y \notin M$, it follows that $y \in \mathbf{Fr}^0_{\mathcal{H}}(G)$. Let M_1 be a $\gamma_{\mathcal{H}}$-set of G and $y \in M_1$. Then there is $z \in (\operatorname{pn}_G[x, M_1] - \{x\}) \cap \mathbf{Fr}^0_{\mathcal{H}}(G)$. Hence $y, z \in \mathbf{Fr}^0_{\mathcal{H}}(G)$ and $yz \notin E(G)$—a contradiction. Thus $\mathbf{Fi}_{\mathcal{H}}(G) = \mathbf{Fi}^{-1}_{\mathcal{H}}(G)$ and the result follows.

(2) This immediately follows by (1) and Theorem 3.4. □

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