130 (2005) MATHEMATICA BOHEMICA No. 2, 203–220

GENERALIZED F-SEMIGROUPS

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(Received October 29, 2004)

Abstract. A semigroup S is called a *generalized F-semigroup* if there exists a group congruence on S such that the identity class contains a greatest element with respect to the natural partial order $\leqslant_S$ of S. Using the concept of an *anticone*, all partially ordered groups which are epimorphic images of a semigroup $(S, \cdot, \leqslant_S)$ are determined. It is shown that a semigroup S is a generalized F-semigroup if and only if S contains an anticone, which is a principal order ideal of $(S, \leqslant_S)$. Also a characterization by means of the structure of the set of idempotents or by the existence of a particular element in S is given. The generalized F-semigroups in the following classes are described: monoids, semigroups with zero, trivially ordered semigroups, regular semigroups, bands, inverse semigroups, Clifford semigroups, inflations of semigroups, and strong semilattices of monoids.

Keywords: semigroup, natural partial order, group congruence, anticone, pivot

MSC 2000: 20M10

1. Introduction

A semigroup $(S, \cdot)$ is called *F-inverse* if S is inverse and for the least group congruence σ on S, every σ-class has a greatest element with respect to the natural partial order $\leqslant_S$ of S (see [16] or [10] for a detailed treatment of this class of semigroups). This concept appeared originally in [19]. A construction of such semigroups was given in [12] by means of groups acting on semilattices with identity obeying certain axioms.

Dropping the condition that the semigroup is inverse we will call a semigroup S an *F-semigroup* if for some group congruence ϱ on S every ϱ-class of S contains a

Research supported by the Portuguese Foundation for Science and Technology through the research program POCTI.

greatest element with respect to the natural partial order $\leqslant_S$ of S. Recall that for any semigroup S, $\leqslant_S$ is defined by

$$a \leqslant_S b \text{ if and only if } \; a = xb = by, \; xa = a \text{ for some } x, y \in S^1$$

(see [13]). In this paper we will more generally study *generalized F-semigroups*, which are semigroups S for which there exists a group congruence ϱ such that the identity class (only) has a greatest element with respect to the natural partial order $\leqslant_S$ of S (equivalently, there exists a homomorphism of S onto a group G such that the preimage of the identity element of G has a greatest element with respect to $\leqslant_S$). Thus we are dealing with semigroups, which are extensions of a subsemigroup T with greatest element by a group (the semigroups of type T were first characterized in [18]). The particular case of F-semigroups will be considered in a subsequent paper.

This generalization of F-inverse semigroups is motivated by a class of partially ordered semigroups (i.e., semigroups S endowed with a partial order $\leqslant$ which is compatible with multiplication). $(S, \cdot, \leqslant)$ is called a *Dubreil-Jacotin semigroup* if there exists an isotone semigroup homomorphism of $(S, \cdot, \leqslant)$ onto a partially ordered group $(G, \cdot, \preceq)$ such that the preimage of the negative cone of G is a pricipal order ideal of $(S, \leqslant)$. This concept was introduced in [6] (see also [4], Theorem 25.3). Specializing the partial order $\leqslant$ given on S to the natural partial order $\leqslant_S$ and dropping the compatibility condition for $\leqslant_S$ (which is not satisfied, in general) it turns out that in this case the partial order $\preceq$ given on G reduces to the equality relation, so that the negative cone of G consists of the identity element of G alone. Thus we arrive at the concept of a generalized F-semigroup.

In Section 2 we determine all partially ordered groups, which are isotone semigroup-homomorphic images of an arbitrary semigroup S with S considered partially ordered by its natural partial order. In the particular case that S is inverse this question was dealt with in [3], where the greatest such partially ordered group was considered. For this purpose we use the concept of an *anticone* of S defined in [2] (see also [4]). In Section 3 we specialize the concept of an anticone to be *principal* in the sense that it is also a principal order ideal of $(S, \leqslant_S)$. In analogy with F-inverse semigroups we show that for generalized F-semigroups S the congruence ϱ appearing in the definition is the least group congruence on S. Characterizations by the existence of a principal anticone, of a particular element, and by properties of the set of all idempotents are provided. Also, generalized F-semigroups which are regular or contain an identity, are considered. The characterization of the latter allows a construction of all generalized F-inverse monoids. In Section 4 the generalized F-semigroups in the following classes are described: semigroups with zero, trivially ordered semigroups, bands, inflations of semigroups, and strong semilattices of monoids (in particular, Clifford semigroups).

2. Epimorphic partially ordered groups

Throughout the paper, S stands for an arbitrary semigroup unless specified otherwise, and $\leqslant_S$ for the natural partial order defined on S. No other partial order on S will be considered.

Since we are interested in homomorphic images of a semigroup S onto groups, we first observe that for any group G and every homomorphism $\varphi\colon S \to G$, $a \leqslant_S b$ implies $a\varphi = b\varphi$, i.e., φ is trivially isotone.

In this section we give a method for constructing all groups G and all partial orders on G such that the partial ordered group G is a semigroup- and an order-homomorphic image of S. For this purpose we follow the account given in [4, Section 24] using the concept of an anticone in a partially ordered semigroup introduced in [2]. Since the natural partial order of S need not be compatible with multiplication, the theory developed in [4] cannot be applied directly to our case. At several stages other proofs have to be given in order to establish the corresponding results needed in the sequel.

Let $X \subseteq S$ and $a, b \in S$. Define

$$X \mathbin{.\!\cdot} a = \{x \in S;\ ax \in X\} \text{ and } X \mathbin{\cdot\!.} a = \{x \in S;\ xa \in X\}.$$

It is readily seen that

$$X \mathbin{.\!\cdot} ab = (X \mathbin{.\!\cdot} a) \mathbin{.\!\cdot} b \text{ and } X \mathbin{\cdot\!.} ab = (X \mathbin{\cdot\!.} b) \mathbin{\cdot\!.} a.$$

We say that $X \neq \emptyset$ is *reflexive* if $ab \in X$ implies $ba \in X$ $(a, b \in S)$. If X is reflexive then $X \mathbin{.\!\cdot} a = X \mathbin{\cdot\!.} a$ for any $a \in S$, in which case we will use the notation $X : a$. We say that X is *neat* if X is reflexive and $X : c \neq \emptyset$ for all $c \in S$. If X is a reflexive subsemigroup of S, define

$$I_X = \{x \in S;\ X : x = X\}.$$

We call a subsemigroup H of S an *anticone* of S if $I_H \cap H \neq \emptyset$ and both H and I_H are reflexive and neat. As we will see later, this definition is equivalent to the definition given in [4] in the context of partially ordered semigroups.

A subset T of a semigroup S is called *unitary* in S if (i) t, $ta \in T$ implies that $a \in T$, and (ii) t, $at \in T$ implies that $a \in T$ (see [5]). If T is reflexive then (i) and (ii) are equivalent.

Proposition 2.1. *Let H be an anticone of S. Then I_H is a maximal unitary subsemigroup of S contained in H. In particular, $I_{I_H} = I_H$ is also an anticone of S, and $I_H = H$ if and only if H is unitary in S.*

Proof. Clearly, by the definition of an anticone, $I_H \neq \emptyset$. That I_H is a unitary subsemigroup follows easily from the fact that $H : xy = (H : x) .\cdot y = (H : y) \cdot. x$ for all $x, y \in S$. If $x \in I_H$ then $H : x = H$ and so $xH \subseteq H$. Let $k \in I_H \cap H$. Then $xk \in H$, i.e., $x \in H : k = H$. Thus $I_H \subseteq H$.

Next let us consider any unitary subsemigroup K of S such that $I_H \subseteq K \subseteq H$. Let $u \in K$. Since I_H is neat, choose $v \in S$ such that $uv \in I_H$. But K is unitary, so $v \in K$. If $z \in H : u$ then $uz \in H$, so $vuz \in H$, giving $z \in H : vu$. Since I_H is reflexive and $uv \in I_H$, we have $vu \in I_H$. Thus $H : vu = H$ and so $z \in H$. Since $H \subseteq H : u$, we get $H : u = H$, proving $u \in I_H$. Hence $K \subseteq I_H$ and so I_H is a maximal unitary subsemigroup of S contained in H. We now show that $I_{I_H} = I_H$. As I_H is unitary, $I_{I_H} \subseteq I_H$. If $x \in I_H$ and $y \in I_H : x$ then $xy \in I_H$ and so, since I_H is unitary, $y \in I_H$. Since I_H is a subsemigroup of S, it follows that $I_H : x = I_H$, that is $x \in I_{I_H}$. That I_H is an anticone is now immediate. The assertion follows and the proof is complete. $\square$

Let H be an anticone. Since H is reflexive, we can define the Dubreil equivalence R_H on S by

$$(a, b) \in R_H \iff H : a = H : b.$$

Following the proof in [4, Section 24] we obtain that S/R_H is a group whose identity is I_H. Also, the binary relation on S/R_H given by

$$aR_H \preceq bR_H \iff H : b \subseteq H : a$$

is a partial order which is compatible with multiplication. Hence $G = (S/R_H, \cdot, \preceq)$ is a partially ordered group. Moreover, following the arguments given in [4, pages 249–251], H is the pre-image, under the natural homomorphism, of the set $\{xR_H \in S/R_H | xR_H \preceq I_H\}$, called the *negative cone* of S/R_H.

Remark 2.2. 1. Notice that any anticone H of $(S, \leqslant_S)$ is an order ideal of $(S, \leqslant_S)$. In fact, if $h \in H$ and $x \in S$, then hR_H belongs to the negative cone of S/R_H and

$$\begin{aligned} x \leqslant_S h &\Longrightarrow x = th = tx \text{ for some } t \in S \\ &\Longrightarrow xR_H = tR_H \cdot hR_H = tR_H \cdot xR_H \\ &\Longrightarrow xR_H = hR_H \preceq I_H \\ &\Longrightarrow x \in H. \end{aligned}$$

2. From the observation at the beginning of this section it follows that the natural homomorphism $\varphi\colon S \to S/R_H$ is isotone.

3. Since I_H is a subsemigroup of H (Proposition 2.1) and H is an order ideal of S, the definition of an anticone that we have given is equivalent to the definition given in [4] in the context of partially ordered semigroups.

We summarize the previous results in

Theorem 2.3. *Let S be a semigroup and H an anticone. Then S/R_H, partially ordered by the relation $\preceq$ defined by $aR_H \preceq bR_H \iff H : b \subseteq H : a$, is an (isotone) homomorphic group image of S under the natural homomorphism such that H is the preimage of the negative cone of $(S/R_H, \preceq)$.*

The next result shows that every partially ordered group which is an (isotone) homomorphic image of a semigroup S arises in this way, i.e., is given by an anticone of S.

Theorem 2.4. *Let S be a semigroup, G a group with compatible partial order $\leqslant$ and $\varphi\colon S \to G$ an (isotone) epimorphism. Let $H = \{x \in S;\ x\varphi \leqslant 1_G\}$. Then H is an anticone and $\psi\colon S/R_H \to G$, given by $xR_H \longmapsto x\varphi$, is an isomorphism such that ψ and ψ^{-1} are order preserving.*

P r o o f. To justify that H is an anticone of S we can apply the arguments given in [4, Section 24] since compatibility of the partial order given on S is not used in those arguments. By Theorem 2.3, S/R_H is a partially ordered group, where R_H denotes the Dubreil equivalence with respect to H and $\preceq$ is the partial order given above. Following the proof of Theorem 24.1 in [4], we obtain that the mapping $\psi\colon S/R_H \to G$, $(xR_H)\psi = x\varphi$ is an isomorphism such that ψ and ψ^{-1} are order preserving. $\square$

Corollary 2.5. *Let $\varphi\colon S \to G$ be an isotone epimorphism where G is a group with compatible partial order $\leqslant$. Then $\leqslant$ is trivial if and only if the anticone $H = \{x \in S;\ x\varphi \leqslant 1_G\}$ is unitary in S.*

P r o o f. By Theorem 2.4, since ψ is an isomorphism, $I_H = 1_G\varphi^{-1}$. If $\leqslant$ is trivial then clearly $H = I_H$, by definition of H. Conversely, if $H = I_H$ and $a\varphi \leqslant b\varphi$ $(a, b \in S)$ then, by Theorem 2.4, $aR_H \preceq bR_H$, i.e., $H : b \subseteq H : a$. Hence $ax \in I_H$ for any $x \in S$ such that $bx \in I_H$. So $(bx)\varphi = 1_G = (ax)\varphi$ giving $b\varphi = a\varphi$. Thus, $\leqslant$ is trivial if and only if $H = I_H$, and this is equivalent to H being unitary, by Proposition 2.1. $\square$

E x a m p l e 2.6. Let B be a band, $(G,\leqslant)$ a partially ordered group and let $S = B\times G$ be their direct product. Then the natural partial order on S is given by

$$(e,a)\leqslant_S (f,b) \Longleftrightarrow e\leqslant_B f \text{ and } \ a=b.$$

Notice that $\leqslant_S$ is not compatible with multiplication, in general. The projection $\varphi\colon\ S\to G$ defined by $(e,a)\varphi = a$ is an isotone epimorphism. By Theorem 2.4, the set $H=\{(e,a)\in S;\ a\leqslant 1_G\}$ is an anticone of S and the mapping $\psi\colon\ S/R_H\to G$ defined by $xR_H\longmapsto x\varphi$ is an isomorphism such that ψ and ψ^{-1} are order preserving. By Corollary 2.5, the anticone H is not unitary if the partial order $\leqslant$ on G is not trivial.

E x a m p l e 2.7. Let S be an inverse semigroup. Then the natural partial order of S has the form

$$a\leqslant_S b \Longleftrightarrow a=eb \text{ for some } e\in E_S \text{ (see [16])}.$$

It was shown in [17] that $H=\{h\in S;\ e\leqslant h \text{ for some } e\in E_S\}$ is the least anticone of S yielding the greatest isotone homomorphic group image of S. The latter is given by the congruence σ on S defined by

$$a\sigma b \Longleftrightarrow ea=eb \text{ for some } e\in E_S;$$

in fact, $R_H=\sigma$ by [3]. We show that H is unitary in S. Let $h, ha\in H$. Then $e\leqslant_S h$, $f\leqslant_S ha$ for some $e,f\in E_S$, whence $e=jh$, $f=iha$ for some $i,j\in E_S$. Since the idempotents of S commute, we get $jf=ijha=iea$, where $ie\in E_S$. Thus $jf\leqslant_S a$ with $jf\in E_S$; hence $a\in H$ and so H is unitary. It follows by Corollary 2.5, that any compatible partial order on the homomorphic group image S/σ of S is trivial.

We next introduce a class of semigroups which contain (unitary) anticones: the class of E-inversive, E-unitary semigroups.

(i) A semigroup S is called *E-inversive* if for every $a\in S$ there exists $x\in S$ such that $ax\in E_S$ (see [5], Ex. 3.2 (8)). In this case there also exists $y\in S$ such that $ay, ya\in E_S$. Examples are provided by periodic (in particular, finite) or regular semigroups (see [14]).

(ii) S is called *E-unitary* if E_S is unitary in S, that is, if $e, ea\in E_S$ implies that $a\in E_S$, and if $e, ae\in E_S$ implies that $a\in E_S$. In fact, these two conditions on S are equivalent (see the beginning of Section 3 in [14]).

Let S be an E-unitary semigroup and let $a,b\in S$ be such that $ab\in E_S$. Then

$$(ba)^3 = bababa = b(ab)^2a = b(ab)a = (ba)^2$$

and

$$(ba)^4 = (ba)^2.$$

Hence $(ba)^2 \in E_S$ and $(ba)(ba)^2 = (ba)^3 = (ba)^2 \in E_S$. It follows that $ba \in E_S$. So E_S is reflexive.

If S is also E-inversive, easy calculations show that E_S is a neat subsemigroup of S and $I_{E_S} = E_S$. Hence E_S is an anticone of S. Also, if H is an anticone of S, then by Theorems 2.3 and 2.4, $H = \{x \in S;\ x\varphi \leqslant 1_G\}$, φ being the natural homomorphism $\varphi\colon S \to S/R_H = G$. Since $e\varphi = 1_G$ for every idempotent $e \in S$, it follows that $E_S \subseteq H$. Thus we have

Proposition 2.8. *Every E-inversive, E-unitary semigroup S has a (least) anticone, namely $H = E_S$.*

Notice that since by Theorem 2.3 every anticone of a semigroup S gives rise to a group G which is an isotone homomorphic image of S the result of Proposition 2.8 is implicitly contained in [1] Theorem 3.1.

3. Generalized F-semigroups

We will now specialize our study to the case of semigroups S containing an anticone H with a greatest element, i.e., an anticone which (by Remark 2.2) is a principal order ideal of $(S, \leqslant_S)$. Such an anticone will be called a *principal anticone*. This additional condition leads to the concept of generalized F-semigroups. We call a semigroup a *generalized F-semigroup* if there exists a group congruence ϱ on S such that the identity ϱ-class $1_G \in G = S/\varrho$ has a greatest element ξ. The element ξ will be called a *pivot* of S.

If a semigroup S has a principal anticone H whose greatest element is ξ, i.e., $H = (\xi] = \{x \in S;\ x \leqslant_S \xi\}$, then R_H is a group congruence. Using the natural homomorphism of S onto the group S/R_H whose identity is I_H, we have

$$\begin{aligned} t, ta \in H \Longrightarrow t, ta \leqslant_S \xi &\Longrightarrow tR_H \cdot aR_H = \xi R_H = tR_H \\ &\Longrightarrow aR_H = 1_{S/R_H} = I_H \\ &\Longrightarrow a \in I_H \subseteq H \qquad \text{[by Proposition 2.1]}. \end{aligned}$$

Hence H is unitary and so, by Proposition 2.1, $H = I_H$. It follows that the identity R_H-class I_H has a greatest element, namely ξ. So S is a generalized F-semigroup.

Conversely, let S be a generalized F-semigroup, ϱ a corresponding group congruence on S and $\varphi\colon S \to G = S/\varrho$ the natural epimorphism. Considering on G the identity relation for $\leqslant$ we have by Theorem 2.4 that $H = \{x \in S;\ x\varphi = 1_G\}$ is an anticone of S. By hypothesis, the identity ϱ-class $1_G \in S/\varrho$, that is, $H = 1_G\varphi^{-1}$, has a greatest element, say, ξ. Therefore H is a principal (hence unitary) anticone and $H = I_H = (\xi]$.

We have proved the following characterization:

Theorem 3.1. *Let S be a semigroup. Then S is a generalized F-semigroup if and only if S has a principal (unitary) anticone H. In this case $H = I_H = (\xi]$, where ξ is a pivot of S.*

Remark 3.2. 1. A unitary anticone is not necessarily principal. Indeed, consider any E-unitary inverse semigroup S. By Proposition 2.8, E_S is a unitary anticone and by [10] Proposition 7.1.3, E_S contains a greatest element if and only if S has an identity.

2. Since for any anticone H of a semigroup S, I_H is unitary (by Proposition 2.1), the natural partial order on I_H is just the restriction of $\leqslant_S$ to I_H.

3. If S is a generalized F-semigroup then any group G appearing in the definition admits only the identity relation as a compatible partial order (by Theorem 3.1 and Corollary 2.5). Hence the negative cone of G consists of the identity alone.

Our next aim is to show that the group in the definition of a generalized F-semigroup is unique. We show even more:

Theorem 3.3. *Let S be a generalized F-semigroup and ϱ a corresponding group congruence. Then ϱ is the least group congruence on S. In particular, both the congruence and the pivot of S are uniquely determined.*

Proof. Let τ be any group congruence on S and let $a, b \in S$ be such that $a\varrho b$. If $c \in (a\varrho)^{-1} = (b\varrho)^{-1}$ then $c\varrho = (a\varrho)^{-1}$ so that $(c\varrho)\cdot(a\varrho) = I_H$, the identity of S/R_H (H being the principal (unitary) anticone of S corresponding to ϱ in Theorem 3.1). Therefore, $ca \in I_H = H = (\xi]$ by Theorem 3.1, that is, $ca \leqslant_S \xi$. Similarly, $cb \leqslant_S \xi$. If ψ denotes the natural homomorphism corresponding to τ, then it follows that $(c\psi)\cdot(a\psi) = \xi\psi = (c\psi)\cdot(b\psi)$ (see the beginning of Section 2). Therefore, $a\psi = b\psi$ (by cancellation), that is, $a\tau b$. $\square$

Due to the definition, the knowledge of semigroups T containing a greatest element is relevant to the study of generalized F-semigroups. A characterization of such semigroups T was given in [18]. Here we provide an independent proof. For this purpose, we first show

Lemma 3.4. *Let S be a semigroup with a greatest element, say, ξ. Then $\xi^3 = \xi^2$ and $\xi^2 \in E_S$.*

P r o o f. By hypothesis $\xi^2 \leqslant_S \xi$. If $\xi^2 = \xi$ then $\xi \in E_S$. If $\xi^2 <_S \xi$ then $\xi^2 = x\xi = \xi y = x\xi^2$ for some $x, y \in S$. Thus $\xi^3 = x\xi^2 = \xi^2$ and so $\xi^2 \in E_S$. $\square$

Theorem 3.5 ([18]). *A semigroup S admits a greatest element if and only if S is one of the following types:*

(i) *S is a band with identity;*

(ii) *$S = T \cup \{\xi\}$, where T is a band with identity e such that $\xi^2 = e$ and $a\xi = \xi a = a$ for every $a \in T$.*

P r o o f. If S is a semigroup of type (i) then the identity $e \in S$ is the greatest element of S. On the other hand, if S is of type (ii) then $a\xi = \xi a = a$ for every $a \in T$ implies that $a \leqslant \xi$ (since $a \in E_S$). Thus ξ is the greatest element of S.

Conversely, let S be a semigroup with greatest element ξ. Then $a \leqslant \xi$ for every $a \in S$. If $\xi \in E_S$, it follows by [15], Lemma 2.1, that $a \in E_S$. Hence S is a band with identity ξ, i.e., S is of type (i). If $\xi \notin E_S$ then we have the following results:

1. $T = S \setminus \{\xi\}$ is a subsemigroup of S:

Let $a, b \in T$; then $a \leqslant_S \xi$ and so $a = x\xi = \xi y = xa$ for some $x, y \in S$. Assume that $ab \notin T$. Then $ab = \xi$ and

$$a = x\xi = x \cdot ab = xa \cdot b = ab = \xi,$$

a contradiction. Thus $ab \in T$.

2. $a\xi = a\xi^2$, $\xi a = \xi^2 a$ for every $a \in S$:

If $a = \xi$ then by Lemma 3.4

$$a\xi = \xi^2 = \xi^3 = \xi \cdot \xi^2 = a\xi^2$$

and similarly $\xi a = \xi^2 a$.

If $a \neq \xi$ then $a <_S \xi$ and so $a = x\xi = \xi y = xa$ for some $x, y \in S$. It follows by Lemma 3.4 that

$$a\xi = x\xi \cdot \xi = x\xi^2 = x\xi^3 = x\xi \cdot \xi^2 = a\xi^2$$

and similarly $\xi a = \xi^2 a$.

3. $\xi^2 \in T$ is the identity of T:

Since $\xi \notin E_S$, we have $\xi^2 \in S \setminus \{\xi\} = T$. Let $a \in T$. Then $a <_S \xi$ and so $a = x\xi = \xi y = xa$ for some $x, y \in S$. Therefore, by 2,

$$a\xi^2 = a\xi = x\xi \cdot \xi = x\xi^2 = x\xi = a.$$

Similarly, $\xi^2 a = a$.

4. $T = S \setminus \{\xi\}$ is a band:

By 2 and Lemma 3.4, $a <_S \xi$ for every $a \in T$ implies that $a \leqslant_S \xi^2$. Since $\xi^2 \in E_S$ by Lemma 3.4, it follows by [15], Lemma 2.1 that $a \in E_S$. Hence by 1, T is a band.

We have shown that $S = T \cup \{\xi\}$, where T is a band with identity ξ^2 such that $a\xi = a\xi^2 = a$ and $\xi a = \xi^2 a = a$ for every $a \in T$. Therefore, S is of type (ii). $\square$

Corollary 3.6. *If S is a generalized F-semigroup with the pivot ξ then either $(\xi] = E_S$ or $(\xi] = E_S \cup \{\xi\}$ with $\xi^2 \in E_S$ and $e\xi = \xi e = e$ for all $e \in E_S$.*

Proof. By Theorem 3.1, $H = (\xi]$ is a principal anticone of S, hence a subsemigroup of S with the greatest element ξ (note that by Remark 2.2, the natural partial order on H is the restriction of $\leqslant_S$ to H). Therefore $\xi^2 \in E_S$ by Lemma 3.4. Since $e\varphi = 1_G$ for any $e \in E_S$, $E_S \subseteq (\xi]$. The assertion now follows from Theorem 3.5. $\square$

This description of the identity class yields the following properties of a generalized F-semigroup.

Proposition 3.7. *Every generalized F-semigroup S with the pivot ξ is E-inversive. Furthermore, E_S is a subsemigroup of S with the greatest element ξ^2.*

Proof. By Corollary 3.6, either $(\xi] = E_S$ or $(\xi] = E_S \cup \{\xi\}$ where ξ^2 is the identity of E_S. By the proof of Theorem 3.5, $T = E_S$ is a subsemigroup of S. It follows that E_S contains a greatest element: ξ^2. We show now that S is E-inversive. Let $a \in S$ and let $\varphi\colon S \to G = S/\varrho$ be the surjective homomorphism satisfying $1_G\varphi^{-1} = (\xi]$. Then we have

$$\begin{aligned} a\varphi \in G &\Longrightarrow (a\varphi)^{-1} = b\varphi \text{ for some } b \in S \\ &\Longrightarrow ab \in 1_G\varphi^{-1} = (\xi] \\ &\Longrightarrow ab \in E_S \text{ or } ab = \xi \\ &\Longrightarrow ab \in E_S \text{ or } a \cdot bab = \xi^2 \in E_S. \end{aligned}$$

Hence S is E-inversive. $\square$

The two properties given in Proposition 3.7 are not sufficient for a semigroup to be a generalized F-semigroup. For example, consider the multiplicative monoid S of natural numbers together with 0; then S is E-inversive and $E_S = \{0, 1\}$ is a subsemigroup with the greatest element 1. If S were a generalized F-semigroup with pivot ξ then Proposition 3.7 would imply $\xi^2 = 1$ and so $\xi = 1$. Hence $(\xi] = \{0, 1\}$, which is not unitary, a contradiction (see Theorem 3.1).

The next theorem establishes a characterization of a generalized F-semigroup in terms of the idempotents of S.

Theorem 3.8. *Let S be a semigroup. Then S is a generalized F-semigroup with the pivot ξ if and only if S is E-inversive, ξ is an upper bound of E_S and $E_S \cup \{\xi\}$ is unitary.*

P r o o f. Necessity follows by Proposition 3.7, Corollary 3.6 and Theorem 3.1.

Conversely, let S be E-inversive, let ξ be an upper bound of E_S and let $E_S \cup \{\xi\}$ be unitary. Suppose first that $\xi \in E_S$. Then S is an E-inversive and E-unitary semigroup. It follows by Proposition 2.8 that $H = E_S$ is a (unitary) anticone with the greatest element ξ. Thus by Theorem 3.1, S is a generalized F-semigroup with the pivot ξ. Suppose now that $\xi \notin E_S$. We show that $H = E_S \cup \{\xi\}$ is a principal anticone of S.

1. H is a subsemigroup of S:

Let $h, k \in H$. Since S is E-inversive, there exists $x \in S$ such that $hkx \in E_S \subseteq H$. Since H is unitary, we have successively $kx \in H$, $x \in H$ and finally $hk \in H$.

2. H is reflexive:

Let $a, b \in S$ be such that $ab \in H$. Consider first the case $ab \in E_S$. Then

$$(ba)^3 = b(ab)^2a = (ba)^2 \Longrightarrow (ba)^2 \in E_S \subseteq H.$$

Since $(ba)(ba)^2 = (ba)^2 \in H$ and since H is unitary, we have that $ba \in H$. Consider next the case $ab = \xi$. By 1, H is a subsemigroup (with the greatest element ξ). Thus by Lemma 3.4, $\xi^3 = \xi^2$,

$$(ba)^4 = b(ab)^3a = b\xi^3a = b\xi^2a = (ba)^3$$

and so $(ba)^3 \in E_S \subseteq H$. Thus $(ba)^3(ba) = (ba)^3 \in H$; since H is unitary, it follows that $ba \in H$.

3. H is neat:

This follows from 2 and the fact that S is E-inversive and $E_S \subseteq H$.

4. $I_H = H$:

Since by 1, H is a subsemigroup of S, we have $H \subseteq H : x$ for any $x \in H$. Also, because H is unitary, $H : x \subseteq H$.Thus $H = H : x$ for any $x \in H$. Thus $H \subseteq I_H$. Conversely, let $a \in I_H$; then $H : a = H$ and $h \in H = H : a \Longrightarrow ah \in H \Longrightarrow a \in H$ (since H is unitary).

We have shown that H is an anticone. Since, by hypothesis, $\xi \in H$ is an upper bound of $E_S \subseteq E_S \cup \{\xi\} = H$, ξ is the greatest element of H. Sufficiency now follows by Theorem 3.1. $\square$

Notice that in Theorem 3.8 the attribute "with the pivot ξ" is essential. In fact, consider the following example.

E x a m p l e 3.9. Let $T = \{0, 1\}$ be the two-element semilattice and let $S = \{0, 1, a\}$ with $a0 = 0a = 0$, $a1 = 1a = 1$, $a^2 = 1$ (see Theorem 3.5). Then $a \in S$ is the greatest element of S and S satisfies the conditions of Theorem 3.8 with $\xi = a$. Hence S is a generalized F-semigroup with the pivot $\xi = a$. Now, 1 is also an upper bound of E_S, but $E_S \cup \{1\} = E_S$ is not unitary in S since $a \cdot 1 = 1 \in E_S$, $a \notin E_S$. This means that S is not a generalized F-semigroup with the pivot $\xi = 1$.

As an immediate consequence of Theorem 3.8, we give a characterization of those elements of a semigroup S which may serve as the pivot of S. Notice that by Theorem 3.3 there is at most one such element.

Corollary 3.10. *Let S be a semigroup. Then S is a generalized F-semigroup with the pivot ξ if and only if* (i) *ξ^2 is the greatest idempotent of S and $\xi^2 \leqslant_S \xi$,* (ii) *for any $a \in S$ there exists $a' \in S$ such that $aa' \leqslant_S \xi^2$,* (iii) *$E_S \cup \{\xi\}$ is unitary in S.*

Note that the conditions of Corollary 3.10 also characterize those order ideals of a semigroup $(S, \cdot, \leqslant_S)$ which are (principal) anticones of S.

As a special case of Theorem 3.8, consider a semigroup S such that E_S has a greatest element. Then we obtain

Corollary 3.11. *Let S be a semigroup containing a greatest idempotent, say e. Then S is a generalized F-semigroup with the pivot e if and only if S is E-inversive and E-unitary.*

The condition imposed on S in Corollary 3.11 is certainly satisfied if S has an identity. In this case it is easy to show that the identity, being a maximal element of $(S, \leqslant_S)$, is the pivot of S. Thus, we obtain a characterization of generalized F-monoids:

Corollary 3.12. *Let S be a monoid. Then S is a generalized F-semigroup if and only if S is E-inversive and E-unitary.*

Next we study generalized F-semigroups which are regular. We begin with the more general situation where only the pivot of S is regular. First we show

Proposition 3.13. *For a generalized F-semigroup with the pivot ξ the following conditions are equivalent:*

(i) *ξ is regular;* (ii) *ξ is (the greatest) idempotent;* (iii) *S is E-unitary.*

P r o o f. By hypothesis, there exists a group G and a surjective homomorphism $\varphi\colon S \to G$ such that $1_G\varphi^{-1} = (\xi]$.

(i) $\Longrightarrow$ (ii). Let $\xi' \in S$ be such that $\xi = \xi\xi'\xi$. Since $\xi\xi' \in E_S$, we have that $(\xi\xi')\varphi = 1_G$ so that $\xi\xi' \in (\xi]$. Hence $\xi\xi' \leqslant_S \xi$ and so

$$\xi\xi' = x\xi = \xi y = x\xi\xi'$$

for some $x, y \in S^1$. Thus $\xi = x\xi = \xi\xi' \in E_S$. (It follows by Theorem 3.8 that ξ is the greatest idempotent.)

(ii) $\Longrightarrow$ (iii). This follows from Corollary 3.10.

(iii) $\Longrightarrow$ (i). Since by Theorem 3.1, $(\xi]$ is a semigroup with the greatest element ξ, we have $\xi^3 = \xi^2 \in E_S$ by Lemma 3.4. Thus, by hypothesis, $\xi^2\xi \in E_S$ implies that $\xi \in E_S$. Hence ξ is regular. □

As a consequence of Proposition 3.13, the conditions of Corollary 3.11 characterize the generalized F-semigroups with a regular pivot. Also they yield a characterization of regular generalized F-semigroups:

Theorem 3.14. *Let S be a regular semigroup. Then S is a generalized F-semigroup if and only if S is an E-unitary monoid.*

Proof. Let S be a regular semigroup. Then S is E-inversive. If S is an E-unitary monoid it follows from Corollary 3.12 that S is a generalized F-semigroup.

Conversely, if S is a regular generalized F-semigroup with the pivot ξ then by Proposition 3.13, ξ is the greatest idempotent of S and S is E-unitary. Following the proof of Proposition 7.1.3 in [10], we show that ξ is the identity of S. Let $a \in S$ and $a' \in S$ be such that $a = aa'a$. Since $aa', a'a \in E_S$ we have by Corollary 3.6 that $aa', a'a \leqslant_S \xi$ and so $a'a\xi = a'a$ and $\xi aa' = aa'$. Hence, $a\xi = \xi a = a$ and so ξ is the identity of S. □

Example 3.15. Let B be a band with an identity 1_B, let G be a group with the identity 1_G and let $S = B \times G$ be their direct product. Then S is a regular monoid with identity $(1_B, 1_G)$ and $E_S = \{(e, 1_G) \in S;\ e \in B\}$. Simple calculations show that S is E-unitary. Thus S is a generalized F-semigroup. The corresponding group is the given group G and $(1_B, 1_G)$ is the greatest element of its identity class since $\varphi\colon\ S \to G$, $(e, a)\,\varphi = a$, is a surjective homomorphism.

A construction of all *regular* generalized F-semigroups is given in [8].

4. Examples

In this section we characterize in several classes of semigroups those members which are generalized F-semigroups. Moreover, two types of constructions are investigated with the aim to produce generalized F-semigroups: inflations of semigroups and strong semilattices of monoids. The proofs concerning the last two cases are not given because they consist of extensive calculations.

1. Every group G is a (generalized) F-semigroup (the identity relation on G is the desired group congruence).

2. Every semigroup S with a greatest element is a generalized F-semigroup (the universal relation on S is the corresponding group congruence).

3. A band B is a generalized F-semigroup if and only if B has an identity (this is a consequence of 2 and of Theorem 3.5).

In the class of all monoids the generalized F-semigroups were characterized by Corollary 3.12. For a much bigger class of semigroups, we have

4. Let S be a semigroup containing a maximal element m, which is idempotent. Then S is a generalized F-semigroup if and only if S is E-inversive, E-unitary and has a greatest idempotent (this follows from Theorem 3.8 and Corollary 3.11).

5. Let S be a trivially ordered semigroup (i.e., the natural partial order of S is the identity relation). Then S is a generalized F-semigroup if and only if S is a group. (Necessity: Since by Theorem 3.8, S is E-inversive and $E_S = \{\xi\}$, S is regular by [14], Proposition 3; hence S is a group by [16], Lemma II.2.10.)

Examples of trivially ordered semigroups S (without zero) are provided by weakly cancellative semigroups, right-(left-) simple semigroups, right-(left-) stratified semigroups, in particular, completely simple semigroups (see [7]).

6. Let S be a semigroup with zero. Then S is a generalized F-semigroup if and only if S has a greatest element (that is, S is of type (i) or (ii) in Theorem 3.5).

In the class of all regular semigroups, the generalized F-semigroups were characterized by Theorem 3.14 as the E-unitary monoids. The inverse case deserves to be mentioned separately. Note that every E-unitary inverse semigroup is isomorphic to a McAlister P-semigroup P, and that P has an identity if and only if Y has a greatest element (see [10] Theorem 7.1.1). Thus we obtain

7. Let S be an inverse semigroup. Then S is a generalized F-semigroup if and only if S is isomorphic to a P-semigroup $P(Y, G; X)$ such that Y has a greatest element with respect to $\leqslant_X$.

R e m a r k 4.1. This result provides a method for the construction of all generalized F-inverse semigroups. Take a lower directed partially ordered set X (see [16], Lemma VII.1.3), a principal order ideal Y of X, which is also a subsemilattice, and a group G acting on the left by order-automorphisms on X such that $G \cdot Y = X$; then $S = P(Y, G; X)$ is a generalized F-inverse semigroup. Conversely, every such semigroup can be constructed in this way. It is worthwhile to note the difference of this construction from that of all F-inverse semigroups: by [11], Theorem 2.8, a semigroup S is F-inverse if and only if S is isomorphic to $P(Y, G; X)$ constructed as above with X a semilattice instead of a lower directed partially ordered set (see also [16], Proposition VII.5.11).

In the following, for two constructions necessary and sufficient conditions on the ingredients are given, which allow to produce further examples of generalized F-semigroups.

8. Inflations of semigroups.

Let T be a semigroup; for every $\alpha \in T$ let T_α be a set such that $T_\alpha \cap T_\beta = \emptyset$ for all $\alpha \neq \beta$ in T and $T_\alpha \cap T = \{\alpha\}$ for any $\alpha \in T$. On $S = \bigcup_{\alpha \in T} T_\alpha$ there is a multiplication defined by

$$a \cdot b = \alpha\beta \text{ if } a \in T_\alpha, b \in T_\beta.$$

Then S is a semigroup called an inflation of T. If T satisfies the condition that for every $\alpha \in T$ there exist $\beta, \gamma \in T$ such that $\alpha = \beta\alpha = \alpha\gamma$ (for example, if T has an identity or if T is regular), the natural partial order on S was characterized in [7]:

$$a \leqslant_S b \ (a \in T_\alpha, b \in T_\beta) \text{ if and only if } a = b \text{ or } a = \alpha \leqslant_T \beta.$$

In particular, if $a, b \in T_\alpha$ then $a \leqslant_S b$ if and only if $a = \alpha$.

As can be expected, the structure of S depends heavily on that of T, in particular, the property to be a generalized F-semigroup.

Theorem 4.2. *Let $S = \bigcup_{\alpha \in T} T_\alpha$ be an inflation of the semigroup T such that for every $\alpha \in T$ there exist $\beta, \gamma \in T$ with $\alpha = \beta\alpha = \alpha\gamma$. Then S is a generalized F-semigroup if and only if*

(i) *T is a generalized F-semigroup with the pivot ξ,*

(ii) *$|T_\alpha| = 1$ for every $\alpha \in T$ with $\alpha <_T \xi$;*

(iii) *$|T_\xi| \leqslant 2$.*

A particular case of inflations should be mentioned.

Corollary 4.3. *Let G be a group and let $S = \bigcup_{g\in G} T_g$ be an inflation of G. Then S is a generalized F-semigroup if and only if $|T_{1_G}| \leqslant 2$.*

9. Strong semilattices of monoids.

Let Y be a semilattice and for every $\alpha \in Y$ let S_α be a monoid (whose identity is 1_α) such that $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha \neq \beta$ in Y. For any $\alpha, \beta \in Y$ with $\beta \leqslant_Y \alpha$, let $\varphi_{\alpha,\beta}\colon S_\alpha \to S_\beta$ be a homomorphism such that $\varphi_{\alpha,\alpha} = \mathrm{id}_{S_\alpha}$ for every $\alpha \in Y$ and $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for $\gamma \leqslant_Y \beta \leqslant_Y \alpha$ in Y. On $S = \bigcup_{\alpha\in Y} S_\alpha$ there is a multiplication defined by

$$a \cdot b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) \text{ if } a \in S_\alpha, b \in S_\beta,$$

where $\alpha\beta = \inf\{\alpha, \beta\}$ in Y. The semigroup S is called a strong semilattice of monoids and is denoted by $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$. By [15], the natural partial order on S is characterized by

$$a \leqslant_S b \ (a \in S_\alpha, b \in S_\beta) \quad \text{if and only if } \alpha \leqslant_Y \beta, a \leqslant_\alpha b\varphi_{\beta,\alpha},$$

where $\leqslant_\alpha$ denotes the natural partial order on S_α ($\alpha \in Y$).

Proposition 4.4. *Let S be a strong semilattice of monoids. Then S is a generalized F-semigroup if and only if S is an E-inversive, E-unitary monoid.*

Theorem 4.5. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of monoids. Then S is a generalized F-semigroup if and only if*

(i) *Y has a greatest element ω and for every $\alpha \in Y$, $\varphi_{\omega,\alpha}$ is a monoid-homomorphism;*

(ii) *S_α is E-unitary for any $\alpha \in Y$ and $\varphi_{\alpha,\beta}$ is idempotent pure for all $\beta \leqslant_Y \alpha$ in Y; i. e., if $a\varphi_{\alpha,\beta} \in E_{S_\alpha}$ then $A \in E_{S_\alpha}$;*

(iii) *For every $\alpha \in Y$ and $a \in S_\alpha$ there exist $\beta \leqslant_Y \alpha$ in Y and $x \in S_\beta$ such that $(a\varphi_{\alpha,\beta})x \in E_{S_\beta}$.*

Remark 4.6. Concerning condition (iii) notice that it is possible that no component S_α of S is E-inversive but that S is so. For example, let Y be a chain, unbounded from below, let $S_\alpha = (\mathbb{N}, \cdot)$ ($0 \notin \mathbb{N}$), let $\varphi_{\alpha,\alpha} = \mathrm{id}_{S_\alpha}$ for every $\alpha \in Y$, and for all $\beta <_Y \alpha$, $a \in S_\alpha$, let $a\varphi_{\alpha,\beta} = 1_\beta$ (the identity of S_β). Then for any $a \in S$, say $a \in S_\alpha$, we have $a1_\beta = 1_\beta \in E_S$ whenever $\beta <_Y \alpha$.

Two particular cases of this construction should be mentioned.

Corollary 4.7. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of unipotent monoids (i.e., $E_{S_\alpha} = \{1_\alpha\}$ for every $\alpha \in Y$). Then S is a generalized F-semigroup if and only if*

(i) *Y has a greatest element;*

(ii) *$\varphi_{\alpha,\beta}$ is idempotent pure for all $\beta \leqslant_Y \alpha$ in Y;*

(iii) *for every $\alpha \in Y$ and $a \in S_\alpha$ there exists $\beta \leqslant_Y \alpha$ in Y and $x \in S_\beta$ such that $(a\varphi_{\alpha,\beta})x \in E_{S_\beta}$.*

The other particular case is a specialization of Corollary 4.7, supposing that every S_α $(\alpha \in Y)$ is a group, that is, S is a Clifford semigroup.

Corollary 4.8. *Let $S = [Y; G_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of groups. Then S is a generalized F-semigroup if and only if Y has a greatest element and $\varphi_{\alpha,\beta}$ is injective for all $\beta \leqslant_Y \alpha$ in Y.*

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