THE INTEGRABILITY OF A FIELD OF ENDOMORPHISMS

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Abstract. A Theorem is proved that gives intrinsic necessary and sufficient conditions for the integrability of a zero-deformable field of endomorphisms. The Theorem is proved by reducing to a special case in which the endomorphism field is nilpotent. Many arguments used in the derivation of similar results are simplified, principally by means of using quotient rather than subspace constructions.

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1. Introduction

A field of endomorphisms J on a manifold M is said to be *integrable* if at each point of M there exists a local coordinate system relative to which the matrix representing J is constant. Of course J may be thought of equivalently as a type (1,1) tensor field or a vector-valued one-form. In order for J to be integrable it is necessary that J should be zero-deformable, that is to say, at each point of M there exists a frame relative to which the Jordan normal form of J is constant.

The problem has been considered before. Kobayashi [1] proved that if J is zero-deformable and either non-derogatory nilpotent or semi-simple then J is integrable if and only if the Nijenhuis torsion N_J vanishes. Lehmann-Lejeune proved that for a general zero-deformable J integrability is equivalent to the vanishing of the first structure function of the associated G-structure [3]. Turiel obtained the same result by a clever application of his theory of a normal form for a pair of symplectic forms [4], [5].

The purpose of the present paper is to characterize the integrability of J in terms of a minimal number of necessary and sufficient conditions and in the process to streamline some of the very elaborate arguments that can be found in [1], [3]. We formulate our result as follows.

Theorem 1. *Let J be a zero-deformable field of endomorphisms. Then the following conditions are equivalent.*

(i) *J is integrable*

(ii) *The G-structure defined by J is flat.*

(iii) *There exists a symmetric linear connection relative to which J is parallel.*

(iv) *N_J vanishes and each of the kernels $\ker(N^r), r = 1, \dots, k-1$ is integrable, where N denotes the nilpotent part of J and $N^k = 0$ but $N^{k-1} \neq 0$.*

In fact, the content of our Theorem is that (iv) $\Rightarrow$ (i). The equivalence of (i) and (ii) is more or less a definition and the fact that (ii) $\Rightarrow$ (iii) follows from general considerations about G-structures. The fact that (iii) $\Rightarrow$ (i) is the main conclusion of [3].

One of the benefits of the approach adopted here, which involves working with quotient spaces, is that we only have to use coordinates at the very end in the proof of Theorem 2. At that stage the summation convention applies for repeated indices and otherwise the notation is reasonably standard. The endomorphisms and manifolds concerned are assumed to be of class C^∞.

2. The eigendistributions of an integrable endomorphism

Throughout the remainder of the paper J will denote a zero-deformable endomorphism field. The semi-simple and nilpotent parts of J will be denoted by S and N, respectively. Also the Nijenhuis torsion of J will be denoted by N_J and is defined by

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] + J^2[X,Y].$$

We quote the following result from Lehmann-Lejeune.

Proposition 2.1. *Suppose that N_J vanishes and that P and Q are polynomials with constant real coefficients and that X and Y are arbitrary vector fields. Then*

$$[P(J)X, Q(J)Y] = P(J)[X, Q(J)Y] + Q(J)[P(J)X, Y] - P(J)Q(J)[X,Y].$$

Proof. One shows first of all that

$$[JX, J^{n+1}Y] = J[X, J^{n+1}Y] + J^{n+1}[JX,Y] - J^{n+2}[X,Y].$$

Then one shows that $[J^m X, J^n Y] = J^m[X, J^n Y] + J^n[J^m X, Y] - J^{m+n}[X,Y]$ for all $m, n \geqslant 0$ again by induction. The latter condition implies the general case: again, first of all, for $P = J^m$ and then for arbitrary P. $\square$

Corollary 2.2. *The distributions* $\operatorname{im}(J-\lambda I)^r$, $\operatorname{im}(J-\mu I)^s$ *where* $\lambda, \mu \in \mathbb{R}$ *and* r *and* s *are non-negative integers are mutually integrable if* N_J *vanishes.*

Proof. Put $P=(J-\lambda I)^r$ and $Q=(J-\mu I)^s$ in the preceding Proposition. □

Proposition 2.3. *The generalized eigendistributions of a zero-deformable* J *for which* N_J *vanishes are mutually integrable.*

Proof. Assume that the eigenvalues of J are real and are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The generalized eigendistributions of J are $\ker(J-\lambda_i I)^{r_i}$ where r_i is the smallest integer such that $\ker(J-\lambda_i I)^{r_i+1} = \ker(J-\lambda_i I)^{r_i}$. Now

$$\ker(J-\lambda_i I)^{r_i} = \bigcap_{i\neq j} \operatorname{im}(J-\lambda_j I)^{r_j}$$

and hence the generalized eigendistributions are mutually integrable. □

We shall show next that the vanishing of N_J implies that J is projectable to the generalized eigendistributions of J. We begin with the following general result for the proof of which we refer to [2].

Proposition 2.4. *Let* $\pi\colon P \longrightarrow Q$ *be a smooth surjective submersion of manifolds. Let* K *be a field of endomorphisms on* P. *Then*

(i) *K is projectable to Q provided KV is a vertical vector field whenever V is vertical and KY is a projectable vector field whenever Y is a projectable vector field.*

(ii) *If K is projectable to $\overline{K}$ on Q then the Nijenhuis tensor N_K of K projects to $N_{\overline{K}}$ in the sense that*

$$\pi_*(N_K(X,Y)) = N_{\overline{K}}(\overline{X}, \overline{Y})$$

where X and Y are vector fields on P that project to $\overline{X}$ and $\overline{Y}$, respectively, on Q.

Proposition 2.5. *Let* J *be a zero-deformable field of endomorphisms on* M *such that* N_J *vanishes. Then* J *is projectable to the generalized eigendistributions of* J *and on such a distribution the induced endomorphism field has vanishing Nijenhuis torsion.*

Proof. Suppose that λ is an eigenvalue of J and put $\ker(J-\lambda I)^r = D$ where r is the smallest integer such that $\ker(J-\lambda I)^{r+1}$ is zero but $\ker(J-\lambda I)^r$ is not zero. A vertical vector field V is one that satisfies

$$(J-\lambda I)^r V = 0.$$

Clearly then

$$(J - \lambda I)^r JV = J(J - \lambda I)^r V = 0$$

and so JV is again vertical.

Also a projectable vector field X is one that satisfies

$$(J - \lambda I)^r [X, V] = 0$$

whenever V is vertical. We must show that JX is projectable also. We apply Proposition 2.1 with $P = J$ and $Q = (J - \lambda I)^r$ to deduce

$$[JX, (J\lambda I)^r V] = J[X, (J - \lambda I)^r V] + (J - \lambda I)^r [JX, V] - J(J - \lambda I)^r [X, V]$$

and hence

$$(J - \lambda I)^r [JX, V] = 0$$

as required. □

Remarks 2.6. 1. Since TM splits locally as a direct sum of the generalized eigendistributions of J the preceding Proposition 2.5 shows that J projects to each of these distributions.

2. We have so far been tacitly assuming that the eigenvalues of J are real. If an eigenvalue, say λ, happens to be complex, one obtains an integrable almost complex structure on the sum of the generalized eigendistributions corresponding to λ and $\bar{\lambda}$. Thus by the Newlander-Nirenberg theorem one can introduce complex coordinates and the theory goes over in the obvious way with complex variables formally replacing real variables. These points have been discussed at great length in 2.3 and do not need to be repeated here.

3. The nilpotent case

In view of the previous section the general problem of the integrability of a zero-deformable endomorphism field with vanishing Nijenhuis torsion is reduced to the case where such an endomorphism field is nilpotent. Indeed on the distribution $\ker(J - \lambda I)^r$—refer to Proposition 2.3 for the notation—J projects to $\lambda I + \overline{N}$ where I is the identity on $\ker(J - \lambda I)^r$ and $\overline{N}$ is the projection of the nilpotent part N of J. Note that N is a polynomial in J and hence by Proposition 2.1 the Nijenhuis torsion N_N must vanish and hence by Proposition $N_{\overline{N}}$ must vanish also. The reader may observe also that the entire theory so far is predicated only on the assumptions that J is zero-deformable and that N_J is zero.

Let us now consider condition (iv) of Theorem 1. Since TM has been decomposed as a direct sum of the generalized eigendistributions of J it follows easily that the distributions $\ker(\overline{N}^r), r = 1, \ldots, k-1$, are integrable. The proof of Theorem 1 will then be a consequence of the following Theorem.

Theorem 2. *Let N be a nilpotent zero-deformable endomorphism for which N^k, but not N^{k-1}, is zero. Then necessary and sufficient conditions for the integrability of N are: $\ker N$, $\ker N^2, \ldots, \ker N^{k-1}$ are integrable and the Nijenhuis torsion N_N is zero.*

Proof. Define $Q = M/\ker(N)$ and $\pi\colon\ M \to Q$ to be the quotient map. Define also $\overline{N}$ on Q by

$$\overline{N}_{\overline{X}} = \pi_*(N_X)$$

where $\overline{X}$ is an arbitrary vector field on Q and X is any vector field on M that projects onto $\overline{X}$. We apply Proposition 2.4 to show that $\overline{N}$ is well-defined. Now V is vertical if $NV = 0$ which implies that $N(NV) = 0$ and so NV is vertical. Secondly, Y is projectable if $[V, Y]$ is vertical whenever V is vertical, that is,

$$NY = 0 \Rightarrow N[V, Y] = 0.$$

Now since N_N is zero we find that

$$N[V, NY] = -N[NV, Y] + N^2[V, Y] + [NV, NY] = 0,$$

since $NV = 0$ and $N[V, Y] = 0$ and hence NY is projectable. Again from Proposition 2.5 $N_{\overline{N}}$ vanishes.

Note also that

$$\overline{N}^{k-1}\overline{X} = \pi_*(N^{k-1}X) = 0$$

and hence $\overline{N}$ is nilpotent of order $k-1$.

We show next that $\ker(\overline{N})$ is integrable. Suppose that

$$\overline{N}\overline{X} = \overline{N}\overline{Y} = 0.$$

Then

$$N^2X = N^2Y = 0.$$

Since $\ker N^2$ is integrable we have that

$$N^2[X, Y] = 0$$

and hence

$$0 = \pi_* N[X,Y] = \overline{N}[\overline{X},\overline{Y}].$$

Likewise $\ker(\overline{N}^2),\ldots,\ker(\overline{N}^{k-2})$ are all integrable. To summarize we have shown that $\overline{N}$ satisfies all the hypotheses of Theorem 2 with k replaced by $k-1$.

We now proceed by induction on k noting that when $k=1$ the endomorphism would have to be zero. By induction there exist coordinates (x^i) on Q such that

$$\overline{N} = A^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$$

where A^i_j is a constant nilpotent matrix and $1 \leqslant i,\ j \leqslant m$. By construction there exist coordinates (x^i, y^α) on M such that N is of the form

$$N = A^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j + B^\alpha_j \frac{\partial}{\partial y^\alpha} \otimes \mathrm{d}x^j + C^\alpha_\beta \frac{\partial}{\partial y^\alpha} \otimes \mathrm{d}y^\beta,$$

where B^α_j and C^α_β are functions of (x^i) and (y^α). However, since $\frac{\partial}{\partial y^\alpha}$ span the kernel of π_*, it follows that C^α_β is zero. We can now write

$$N = \Big(A^i_j \frac{\partial}{\partial x^i} + B^\alpha_j \frac{\partial}{\partial y^\alpha}\Big) \otimes \mathrm{d}x^j.$$

If we apply the Nijenhuis condition to $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ we find that $A^i_j \frac{\partial}{\partial x^i} + B^\alpha_j \frac{\partial}{\partial y^\alpha}$ are a collection of coordinate vector fields. We may assume that A^i_j is in Jordan normal form, for example, and change $A^i_j \frac{\partial}{\partial x^i} + B^\alpha_j \frac{\partial}{\partial y^\alpha}$ into vector fields of the form $\frac{\partial}{\partial x^a} + B^\alpha_a \frac{\partial}{\partial y^\alpha}$ $(1 \leqslant a \leqslant r)$ and $B^\alpha_s \frac{\partial}{\partial y^\alpha}$ $(r+1 \leqslant s \leqslant m)$. Now the latter set of $m-r$ vector fields must form a basis for the integrable distribution $\mathrm{im}(N) \cap \ker(N)$ and so can be transformed to a set of coordinate vector fields $\frac{\partial}{\partial \overline{y}^s}$ $(r+1) \leqslant s \leqslant m$ by a transformation of the form

$$\overline{x}^i = x^i,\ \overline{y}^\alpha = \overline{y}^\alpha(x^j, y^\beta).$$

The fact that $\frac{\partial}{\partial x^a} + B^\alpha_a \frac{\partial}{\partial y^\alpha}$ and $B^\alpha_s \frac{\partial}{\partial y^\alpha}$ commute gives

$$\frac{\partial B^\alpha_a}{\partial \overline{y}^s} = 0$$

and the fact that $\frac{\partial}{\partial x^a} + B^\alpha_a \frac{\partial}{\partial y^\alpha}$ and $\frac{\partial}{\partial x^b} + B^\beta_b \frac{\partial}{\partial y^\beta}$ commute shows that by a transformation of the previous type we can arrange that $\frac{\partial}{\partial x^a} + B^\alpha_a \frac{\partial}{\partial y^\alpha}$ is changed into $\frac{\partial}{\partial \overline{x}^a}$. Thus N has been written with constant coefficients as required. $\square$

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