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TRANSVERSALLY SYMMETRIC IMMERSIONS

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Abstract. We introduce the notions of (extrinsic) locally transversally symmetric immersions and submanifolds in a Riemannian manifold equipped with a unit Killing vector field as analogues of those of (extrinsic) locally symmetric immersions and submanifolds. We treat their geometric properties, derive several characterizations and give a list of examples.

Keywords: reflections, isometric flows, (locally) symmetric and transversally symmetric immersions and submanifolds, Killing-transversally symmetric spaces, normal flow space forms

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1. Introduction

Extrinsic symmetric immersions and *submanifolds* in Euclidean spaces [5], [15], rank one symmetric spaces [1], [11, and references therein], [16] and compact symmetric spaces [11] have been studied intensively. These submanifolds are intrinsic symmetric spaces and have parallel second fundamental form. We refer to [10] for more information and further references. In that paper a local theory was developed for general Riemannian manifolds by using reflections with respect to submanifolds.

The main purpose of this paper is to study a similar notion for Riemannian manifolds equipped with a unit Killing vector field and for isometric immersions or submanifolds which are tangent to this field. Such a Killing vector field determines an isometric flow and the study of the transversal geometry of this Riemannian foliation is one of the main topics to be considered. Using this and reflections with respect to submanifolds, naturally associated to the tangent isometric immersion or submanifold, we introduce the notions of an *extrinsic locally transversally symmetric*

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immersion and *submanifold*, and the main purpose of this paper is to study those in detail. It will turn out that again there is a strong relation with some aspects of extrinsic differential geometry. We shall see that such submanifolds are always invariant and the covariant derivative of the second fundamental form vanishes for transversal vectors. Moreover, the submanifold is a *locally Killing-transversally symmetric space.* The local and global theory of these spaces was developed in [6], [8]. These spaces are in a natural way analogues of symmetric spaces, and the φ-symmetric spaces from Sasakian geometry are interesting examples which are analogues of Hermitian symmetric spaces.

In Section 2 we collect some basic material about flow geometry and Killing-transversally symmetric spaces and derive two new and useful recursion formulas for the covariant derivatives of the Riemannian curvature tensor. In Section 3 we introduce *local extrinsic reflections.* The locally transversally symmetric immersions or submanifolds are then defined as those which have *isometric* local extrinsic reflections. Further, we derive the basic properties, prove several characterizations and study the relation to extrinsic symmetric immersions related to the transversal geometry. Moreover, we discuss these results when the ambient space is itself a Killing-transversally symmetric space. In the final Section 4 we consider normal flow space forms as ambient spaces. These last ones were introduced and studied in [7] as analogues of real and complex space forms which also generalize Sasakian space forms. Our theorems in Section 3 and Section 4 also provide a lot of examples of transversally symmetric submanifolds or immersions.

2. Preliminaries

Let (M,g) be an n-dimensional, (connected) and smooth Riemannian manifold with $n \geqslant 2$. Let ∇ denote its Levi Civita connection and R the corresponding Riemannian curvature tensor with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for all $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M.

Further, let ξ be a unit Killing vector field on (M,g) and denote by $\mathfrak{F}_\xi$ the *isometric flow* [17] on the manifold generated by ξ. The leaves of this Riemannian foliation $\mathfrak{F}_\xi$ are geodesics and moreover, a geodesic which is orthogonal to the flow field ξ at one of its points is orthogonal to it at all of its points. These geodesics are called *transversal* or *horizontal* geodesics. The foliation is locally a Riemannian submersion. So, let $m \in (M,g)$ and let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the map $\pi\colon \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ is a submersion. Let g' denote the metric on $\mathcal{U}'$ defined by

$$g'(X', Y') \circ \pi = g(X'^{*}, Y'^{*})$$

for $X', Y' \in \mathfrak{X}(\mathcal{U}')$, where X'^*, Y'^* denote the horizontal lifts of X', Y' with respect to the $(n-1)$-dimensional horizontal distribution on $\mathcal{U}$ determined by $\eta = 0$, η being the dual one-form of ξ with respect to g. Then $\pi\colon (\mathcal{U}, g_{|\mathcal{U}}) \to (\mathcal{U}', g')$ is a Riemannian submersion. So, we may use the tensors A and T introduced by O'Neill in [13] (see also [2], [14], [17]) for our treatment. Note that, since the leaves are geodesics, $T = 0$. Further, for the integrability tensor A we have

$$\begin{aligned} A_U\xi &= \nabla_U\xi, & A_\xi U &= 0, \\ A_X Y &= (\nabla_X Y)^{\mathcal{V}} = -A_Y X, & g(A_X Y, \xi) &= -g(A_X\xi, Y) \end{aligned}$$

where $U \in \mathfrak{X}(M)$, X, Y are horizontal vectors fields and $\mathcal{V}$ denotes the vertical component.

Next, put

$$HU = -A_U\xi$$

and define the $(0,2)$-tensor field h by

$$h(U, V) = g(HU, V)$$

for $U, V \in \mathfrak{X}(M)$. Now h is clearly skew-symmetric since ξ is a Killing vector field. Moreover, we have

$$A_X Y = h(X, Y)\xi = \tfrac{1}{2}\eta([X, Y])\xi$$

for all horizontal $X, Y \in \mathfrak{X}(M)$. This yields

$$h = -\,\mathrm{d}\eta. \tag{2.1}$$

Note that $A = 0$, or equivalently $h = 0$, if and only if the horizontal distribution is integrable. In that case, since $T = 0$, (M, g) is locally a Riemannian product of an $(n-1)$-dimensional and a one-dimensional space. Further, the Levi Civita connection ∇' of g' is determined by

$$\nabla_{X'^*}Y'^* = (\nabla'_{X'}Y')^* + h(X'^*, Y'^*)\xi$$

for $X', Y' \in \mathfrak{X}(\mathcal{U}')$.

From all these formulas one obtains, after a straightforward computation and using the notation $R(X, Y, Z, W) = g(R_{XY}Z, W)$,

Lemma 2.1. [6] *We have*

$$\begin{aligned} (\nabla_\xi h)(X, Y) &= g((\nabla_\xi A)_X Y, \xi) = 0, \\ R(X, Y, Z, \xi) &= (\nabla_Z h)(X, Y), \\ R(X, \xi, Y, \xi) &= g(HX, HY) = -g(H^2X, Y) \end{aligned}$$

for horizontal X, Y, Z.

Using this lemma we obtain that the ξ-*sectional curvature* corresponding to X, i.e., the sectional curvature $K(X,\xi)$ of the two-plane spanned by (X,ξ), is non-negative for all horizontal X. Moreover, since $H\xi = 0$, $K(X,\xi) = 0$ for all horizontal X if and only if $h = 0$. Further, $K(X,\xi) > 0$ for each horizontal X if and only if H has maximal rank $n-1$ and hence, in that case n must be an odd number. (2.1) then implies that this is equivalent to the statement that η is a *contact form* on M. This motivates

Definition 2.1. $\mathfrak{F}_\xi$ is called a contact flow on (M,g) if η is a contact form.

In [6] we introduced another special type of an isometric flow which will be important in our further treatment. We recall its definition.

Definition 2.2. The flow $\mathfrak{F}_\xi$ on (M,g) is said to be a normal flow if

$$R(X,Y,X,\xi) = 0$$

for all horizontal vector fields X, Y on M.

From Lemma 2.1 we then get the following useful characterization:

Proposition 2.1. [6] *The flow $\mathfrak{F}_\xi$ is normal if and only if*

$$(\nabla_U H)V = g(HU,HV)\xi + \eta(V)H^2U \tag{2.2}$$

for all $U,V \in \mathfrak{X}(M)$.

Another characterization may be obtained as follows. Let H' be the $(1,1)$-tensor field on $\mathcal{U}'$ defined by

$$H'X' = \pi_* HX'^*$$

and let h' be the $(0,2)$-tensor field given by

$$h'(X',Y') = g'(H'X',Y')$$

for all $X',Y' \in \mathfrak{X}(\mathcal{U}')$. Then $h = \pi^* h'$. Moreover, it follows that $\mathfrak{F}_\xi$ is normal if and only if $\nabla' H' = 0$ for each $(\mathcal{U}',g')$ [6].

Next we shall establish some results about the curvature tensor R and its covariant derivatives $\nabla^k R$ when (M,g) is equipped with a normal flow $\mathfrak{F}_\xi$. First, we have

$$R_{UV}\xi = \eta(V)H^2U - \eta(U)H^2V, \tag{2.3}$$

$$R_{U\xi}V = g(HU,HV)\xi + \eta(V)H^2U. \tag{2.4}$$

Using this we obtain, for any horizontal vector x, the following expressions:

$$\begin{cases} R_{x\xi}x = & \|Hx\|^2\xi, \\ (\nabla_x R)_{x\xi}x = & R_{xHx}x - \|Hx\|^2 Hx, \\ (\nabla^2_{xx} R)_{x\xi}x = & 2(\nabla_x R)_{xHx}x. \end{cases} \tag{2.5}$$

Further, we prove two lemmas which will be used later on. We start with

Lemma 2.2. *Let $\mathfrak{F}_\xi$ be a normal flow on (M,g). Then, for all horizontal vectors x on M and all $p \geqslant 1$, we have*

$$(2.6)\quad (\nabla^{2p-1}_{x\ldots x}R)_{x\xi}x = \sum_{i=1}^{p}\left\{c_i^{2p-1}\|Hx\|^{2(i-1)}(\nabla^{2(p-i)}_{x\ldots x}R)_{xHx}x\right\} + c_{p+1}^{2p-1}\|Hx\|^{2p}Hx,$$

$$(2.7)\quad (\nabla^{2p}_{x\ldots x}R)_{x\xi}x = \sum_{i=1}^{p} c_i^{2p}\|Hx\|^{2(i-1)}(\nabla^{2(p-i)+1}_{x\ldots x}R)_{xHx}x,$$

where c_i^{2p}, $i \in \{1,\ldots,p\}$, and c_j^{2p-1}, $j \in \{1,\ldots,p+1\}$, are constants.

Proof. Denote by T^l, $l \in \mathbb{Z}$, the one-forms on M given by

$$\begin{aligned}
&T^l a = 0, \text{ if } l < 0,\\
&T^0 a = R_{xHxxa} - \|Hx\|^2 g(Hx,a),\\
&T^l a = (\nabla^l_{x\ldots x}R)_{xHxxa}, \text{ if } l \geqslant 1
\end{aligned}$$

for all vectors a on M. By induction we now prove the formula

$$(2.8)\qquad (\nabla^k_{x\ldots x}R)_{x\xi}x = \sum_{i\geqslant 1} c_i^k\|Hx\|^{2(i-1)}T^{k-2i+1}.$$

Then, (2.6) and (2.7) follow at once from (2.8).

For $k=1$, (2.8) follows from (2.5). Further, suppose that (2.8) holds for $1 < k \leqslant m$. Let γ denote the geodesic such that $\dot\gamma(t) = x \in T_qM$, $\gamma(t) = q$. Then we get

$$\begin{aligned}
(\nabla^{m+1}_{x\ldots x}R)_{x\xi}x &= \nabla_x\Big((\nabla^m_{x\ldots x}R)_{x\xi}x\Big) + (\nabla^m_{x\ldots x}R)_{xHx}x\\
&= \nabla_x\left(\sum_i c_i^m\|Hx\|^{2(i-1)}T^{m-2i+1}\right) + T^m.
\end{aligned}$$

From (2.2) and (2.4) we have

$$\nabla_x T^0 = T^1.$$

Hence, by using our induction hypothesis, we get

$$\nabla_x T^l = T^{l+1} + \sum_{j\geqslant 1} c_j^l\|Hx\|^{2j}T^{l-2j+1},$$

where $0 \leqslant l \leqslant m$. So, we obtain

$$\begin{aligned}
(\nabla^{m+1}_{x\ldots x}R)_{x\xi}x = &\sum_i c_i^m\Big\{\|Hx\|^{2(i-1)}T^{m-2(i-1)}\\
&+\sum_j c_j^{m-2i+1}\|Hx\|^{2((i+j)-1)}T^{m-2((i+j)-1)}\Big\} + T^m.
\end{aligned}$$

Hence, the required result follows by rearranging this last formula. $\square$

For normal flows the curvature tensor R' of $(\mathcal{U}', g')$ is related to R by

$$(2.9)\qquad \begin{aligned}(R'_{X'Y'}Z')^* &= R_{X'^*Y'^*}Z'^* - g(HY'^*, Z'^*)HX'^* \\ &\quad + g(HX'^*, Z'^*)HY'^* + 2g(HX'^*, Y'^*)HZ'^*\end{aligned}$$

for all $X', Y', Z' \in \mathfrak{X}(\mathcal{U}')$. Then (2.3) and (2.4) yield

$$(2.10)\qquad ((\nabla'_{V'}R')_{X'Y'}Z')^* = ((\nabla_{V'^*}R)_{X'^*Y'^*}Z'^*)^{\mathcal{H}}$$

where $\mathcal{H}$ denotes the horizontal component. For covariant derivatives of higher order, the corresponding expression is more complicated. Here we prove our second lemma.

Lemma 2.3. *Let $\mathfrak{F}_\xi$ be a normal flow on (M, g). Then we have*

$$\begin{aligned}(2.11)\quad (\nabla^{2p-1}_{x'^*\ldots x'^*}R)_{x'^*y'^*x'^*z'^*} &= \Big\{(\nabla'^{2p-1}_{x'\ldots x'}R')_{x'y'x'z'} \\ &+ \sum_{i=1}^{p-1} a_i^{2p-1}\|H'x'\|^{2(i-1)}\Big(h'(x',y')(\nabla'^{2(p-i)-1}_{x'\ldots x'}R')_{x'H'x'x'z'} \\ &+ h'(x',z')(\nabla'^{2(p-i)-1}_{x'\ldots x'}R')_{x'H'x'x'y'}\Big) \\ &+ h'(x',y')h'(x',z')\Big(\sum_{j=1}^{p-2} b_j^{2p-1}\|H'x'\|^{2(j-1)}(\nabla'^{2(p-j)-3}_{x'\ldots x'}R')_{x'H'x'x'H'x'}\Big)\Big\}\circ\pi,\end{aligned}$$

$$\begin{aligned}(2.12)\quad (\nabla^{2p}_{x'^*\ldots x'^*}R)_{x'^*y'^*x'^*z'^*} &= \Big\{(\nabla'^{2p}_{x'\ldots x'}R')_{x'y'x'z'} \\ &+ \sum_{i=1}^{p} a_i^{2p}\|H'x'\|^{2(i-1)}\Big(h'(x',y')(\nabla'^{2(p-i)}_{x'\ldots x'}R')_{x'H'x'x'z'} \\ &+ h'(x',z')(\nabla'^{2(p-i)}_{x'\ldots x'}R')_{x'H'x'x'y'}\Big) \\ &+ h'(x',y')h'(x',z')\Big(\sum_{j=1}^{p-1} b_j^{2p}\|H'x'\|^{2(j-1)}(\nabla'^{2(p-j-1)}_{x'\ldots x'}R')_{x'H'x'x'H'x'} \\ &+ b_p^{2p}\|H'x'\|^{2p}\Big)\Big\}\circ\pi\end{aligned}$$

for all vectors x', y', z' on $\mathcal{U}'$ and $p \geqslant 1$, where a_i^{2p}, $i \in \{1, \ldots, p\}$, a_j^{2p-1}, b_j^{2p}, $j \in \{1, \ldots, p-1\}$, and b_k^{2p-1}, $k \in \{1, \ldots, p-2\}$, are constants.

Proof. Denote by T'^l, $l \in \mathbb{Z}$, the one-forms on $\mathcal{U}'$ given by

$$\begin{aligned} T'^l a' &= 0, \text{ if } l < 0, \\ T'^0 a' &= R'_{x'H'x'x'a'} - 4\|H'x'\|^2 h'(x', a'), \\ T'^l a' &= (\nabla'^l_{x'\ldots x'}R')_{x'H'x'x'a'}, \text{ if } l \geqslant 1\end{aligned}$$

for all tangent vectors a' on $\mathcal{U}'$. By induction we prove the formula

$$(2.13)\qquad (\nabla^k_{x'^*\dots x'^*}R)_{x'^*y'^*x'^*z'^*} = \Big\{(\nabla'^k_{x'\dots x'}R')_{x'y'x'z'}$$
$$+\sum_{i\geqslant 1} a_i^k\|H'x'\|^{2(i-1)}\Big(h'(x',y')T'^{k-2i}z' + h'(x',z')T'^{k-2i}y'\Big)$$
$$+h'(x',y')h'(x',z')\Big(\sum_{j\geqslant 1} b_j^k\|H'x'\|^{2(j-1)}T'^{(k-2)-2j}H'x'\Big)\Big\}\circ\pi.$$

(2.11) and (2.12) follow from (2.13).

First, note that (2.13) holds for $k=1$ because of (2.10). Therefore, we now suppose that (2.13) holds for all k with $1<k\leqslant m$. Let γ' denote the geodesic on $\mathcal{U}'$ tangent to x' and denote by x', y', z' the parallel vectors x', y', z' along γ'. Then, since

$$\nabla_{x'^*}y'^* = h(x'^*, y'^*)\xi,$$

we get

$$(2.14)\quad (\nabla^{m+1}_{x'^*\dots x'^*}R)_{x'^*y'^*x'^*z'^*} = x'^*\Big((\nabla^m_{x'^*\dots x'^*}R)_{x'^*y'^*x'^*z'^*}\Big)$$
$$-h(x'^*,y'^*)(\nabla^m_{x'^*\dots x'^*}R)_{x'^*\xi x'^*z'^*} - h(x'^*,z'^*)(\nabla^m_{x'^*\dots x'^*}R)_{x'^*y'^*x'^*\xi}.$$

Hence, taking into account that $\nabla'_{x'}T'^l = T'^{l+1}$ for all $l\geqslant 0$ and using the induction hypothesis, we have

$$x'^*\Big((\nabla^m_{x'^*\dots x'^*}R)_{x'^*y'^*x'^*z'^*}\Big) = \Big\{(\nabla'^{m+1}_{x'\dots x'}R')_{x'y'x'z'}$$
$$+\sum_{i\geqslant 1} a_i^m\|H'x'\|^{2(i-1)}\Big(h'(x',y')T'^{(m+1)-2i}z' + h'(x',z')T'^{(m+1)-2i}y'\Big)$$
$$+h'(x',y')h'(x',z')\Big(\sum_{j\geqslant 1} b_j^m\|H'x'\|^{2(j-1)}T'^{((m+1)-2)-2j}H'x'\Big)\Big\}\circ\pi.$$

From Lemma 2.2 we get

$$(\nabla^m_{x'^*\dots x'^*}R)_{x'^*\xi x'^*a'^*} = \sum_{l\geqslant 1} c_l^m\|Hx'^*\|^{2(l-1)}T^{m-2l+1}a'^*$$

and, applying again the induction procedure, we conclude

$$T^{m-2l+1}a'^* = \Big\{T'^{(m+1)-2l}a' + \sum_{r\geqslant 1} a_r^{(m+1)-2l}\|H'x'\|^{2r}T'^{(m+1)-2(l+r)}a'$$
$$+\sum_{r\geqslant 1} a_r^{(m+1)-2l}\|H'x'\|^{2(r-1)}h'(x',a')T'^{((m+1)-2)-2((l+r)-1)}H'x'$$
$$+h'(x',a')\Big(\sum_{s\geqslant 1} b_s^{m-2l+1}\|H'x'\|^{2s}T'^{((m+1)-2)-2(l+s)}H'x'\Big)\Big\}\circ\pi.$$

Now, substituting these last formulas in (2.14) and rearranging, we get that (2.13) holds for $k = m + 1$. This completes the proof. $\square$

Finally, we give some preliminaries about locally Killing-transversally symmetric spaces. Let $m \in (M, g)$ and denote by $\sigma = \sigma_m : [-\delta, \delta] \to M$ the geodesic flow line through $m = \sigma(0)$ where δ is sufficiently small. Let $N\sigma = T^\perp\sigma$ denote the normal bundle of σ and denote by $\exp_\sigma$ the exponential map of $N\sigma$ defined by

$$\exp_\sigma(t, v) = \exp_{\sigma(t)} v$$

for $t \in [-\delta, \delta]$ and $v \in N_{\sigma(t)}\sigma = T^\perp_{\sigma(t)}\sigma$. Next, let $\mathcal{U}_\sigma(r)$ denote the tubular neighborhood of radius r about σ, i.e.,

$$\mathcal{U}_\sigma(r) = \{\exp_\sigma(t, v) \mid v \in T^\perp_{\sigma(t)}\sigma,\ \|v\| < r,\ t \in [-\delta, \delta]\}.$$

Further, let

$$B^\perp_{\sigma(t)}(r) = \{v \in T^\perp_{\sigma(t)}\sigma \mid \ \|v\| < r\}$$

and consider the open solid tube

$$V_\sigma(r) = \bigcup_{t \in [-\delta, \delta]} B^\perp_{\sigma(t)}(r)$$

about the zero section of N_σ. In what follows we suppose that r is sufficiently small so that $\exp_\sigma$ is a diffeomorphism of $V_\sigma(r)$ onto $\mathcal{U}_\sigma(r)$. Then the local diffeomorphism

$$s_\sigma : \mathcal{U}_\sigma(r) \longrightarrow \mathcal{U}_\sigma(r), p = \exp_\sigma(t, v) \longmapsto s_\sigma(p) = \exp_\sigma(t, -v)$$

is called the (local) reflection with respect to σ. (See [19] for further information and references.) In what follows we shall denote s_σ by s_m. A Riemannian manifold (M, g) equipped with a flow $\mathfrak{F}_\xi$ as above and such that the local reflection s_m is an isometry for all $m \in M$ is called a *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*) [6]. These spaces may be characterized as follows.

Proposition 2.2. [6] *The following statements are equivalent:*
(i) $(M, g, \mathfrak{F}_\xi)$ *is a locally* KTS-*space;*
(ii) $\mathfrak{F}_\xi$ *is normal and* $(\nabla_X R)(X, Y, X, Y) = 0$ *for all horizontal* X, Y.

Proposition 2.3. [6] *Let* $\mathfrak{F}_\xi$ *be a normal flow on* (M, g). *Then* $(M, g, \mathfrak{F}_\xi)$ *is a locally* KTS-*space if and only if each base space* $\mathcal{U}'$ *of a local Riemannian submersion* $\pi : \mathcal{U} \to \mathcal{U}' = \mathcal{U}/\xi$ *is a locally symmetric space.*

This means, according to the terminology used in [18], that $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is a *normal transversally symmetric foliation.*

Let f be an immersion of an n-dimensional differentiable manifold M into an $\bar{n}$-dimensional Riemannian manifold $(\overline{M}, \bar{g})$ and denote by g the induced metric from $\bar{g}$ on M. Then f becomes an isometric immersion. M, $(\overline{M}, \bar{g})$ and f will be assumed to be analytic where necessary. Further, denote by $\overline{R}$ and R the curvature tensors of the corresponding Levi Civita connections $\overline{\nabla}$ and ∇, respectively. Moreover, we denote by α the second fundamental form of (M, f), by $\nabla^{\perp}$ the normal connection in the normal bundle $N(M)$ and by $R^{\perp}$ its curvature tensor. In what follows, and if the argument is local, we shall sometimes identify M with its image under f to simplify the notation. We recall the well-known Gauss and Weingarten formulas:

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{3.1}$$

$$\overline{\nabla}_X U = -S_U X + \nabla^{\perp}_X U \tag{3.2}$$

where $X, Y \in \mathfrak{X}(M)$, S_U is the shape operator of (M, f) corresponding to the (local) normal field U and it is related to α by $\bar{g}(\alpha(X, Y), U) = g(S_U X, Y)$.

Also, we recall the Gauss and Codazzi equations:

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + \bar{g}(\alpha(X, W), \alpha(Y, Z)) - \bar{g}(\alpha(X, Z), \alpha(Y, W)), \tag{3.3}$$

$$(\overline{R}_{XY} Z)^{\perp} = -(\tilde{\nabla}_X \alpha)(Y, Z) + (\tilde{\nabla}_Y \alpha)(X, Z) \tag{3.4}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$, $U, V \in \mathfrak{X}(M)^{\perp}$. Here $\tilde{\nabla}$ denotes the covariant derivative defined by

$$(\tilde{\nabla}_X \alpha)(Y, Z) = \nabla^{\perp}_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) \tag{3.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Then f is said to be *totally geodesic* if $\alpha = 0$, *parallel* if $\tilde{\nabla}\alpha = 0$ and *minimal* if tr $\alpha = 0$.

Now, let $\mathfrak{F}_{\bar{\xi}}$ be an isometric flow on $(\overline{M}, \bar{g})$ determined by a unit Killing vector field $\bar{\xi}$. An isometric immersion $f\colon (M, g) \to (\overline{M}, \bar{g})$ is said to be *tangent* if $\bar{\xi}$ is tangent to $f(M)$. In that case we shall denote by ξ the unit Killing vector field on (M, g) induced by $\bar{\xi}$, i.e., $f_* \xi = \bar{\xi} \circ f$, and by η the one-form on M given by $\eta = f^* \bar{\eta}$. In the rest of this paper we will always consider tangent isometric immersions. Such an immersion is said to be *η-parallel* if $(\tilde{\nabla}_X \alpha)(Y, Z) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$ and orthogonal to ξ (see [9]).

Next, let $\sigma = \sigma_m \colon [-\delta, \delta] \to M$ be an integral curve of ξ through $m \in M$, $\sigma(0) = m$. Denote by $\bar{\sigma} = \bar{\sigma}_{f(m)}$ the image $f \circ \sigma$ in $\overline{M}$ and by $\exp_{\bar{\sigma}}$ the exponential map of the normal bundle $N\bar{\sigma}$ of $\bar{\sigma}$. Also, let $\mathcal{U}_{\sigma}$ and $\overline{\mathcal{U}}_{\bar{\sigma}}$ be the tubular neighborhoods about σ and $\bar{\sigma}$ on M and $\overline{M}$, respectively and let us suppose that they have the same sufficiently small radius r. Further, we consider the following two topologically embedded submanifolds $\mathcal{U}_{\bar\sigma}^-$ and $\mathcal{U}_{\bar\sigma}^+$ on $\overline{M}$ contained in $\overline{\mathcal{U}}_{\bar\sigma}$ and of dimension n and $(\bar n - n + 1)$, respectively:

$$\begin{aligned}\mathcal{U}_{\bar\sigma}^- &= \{\exp_{\bar\sigma}(t,x) \mid x \in T_{\bar\sigma}M,\ \eta(x)=0,\ \|x\|<r,\ -\delta<t<\delta\},\\ \mathcal{U}_{\bar\sigma}^+ &= \{\exp_{\bar\sigma}(t,u) \mid u \in N_{\bar\sigma}M,\ \|u\|<r,\ -\delta<t<\delta\}.\end{aligned}$$

It is clear that when f is totally geodesic, then $\mathcal{U}_{\bar\sigma}^-$ coincides with $f(\mathcal{U}_\sigma)$.

Let $\mathcal{T}_{\mathcal{U}_{\bar\sigma}^+}(s)$ be the tubular neighborhood or the (open) solid tube of radius s around $\mathcal{U}_{\bar\sigma}^+$ defined by

$$\mathcal{T}_{\mathcal{U}_{\bar\sigma}^+}(s) = \{\exp_{\mathcal{U}_{\bar\sigma}^+}(p,x) \mid x \in N_p\mathcal{U}_{\bar\sigma}^+,\ \|x\|<s,\ p\in\mathcal{U}_{\bar\sigma}^+\}$$

where s is supposed to be smaller than the distance from $\mathcal{U}_{\bar\sigma}^+$ to its nearest focal point and $\exp_{\mathcal{U}_{\bar\sigma}^+}$ denotes the exponential map of the normal bundle $N(\mathcal{U}_{\bar\sigma}^+)$ of $\mathcal{U}_{\bar\sigma}^+$, i.e.,

$$\exp_{\mathcal{U}_{\bar\sigma}^+}(p,x) = \exp_p^{\overline{M}} x.$$

Now, let φ_m denote the (local) reflection with respect to $\mathcal{U}_{\bar\sigma}^+$ defined on $\mathcal{T}_{\mathcal{U}_{\bar\sigma}^+}(s)$ by

$$\varphi_m\colon q = \exp_{\mathcal{U}_{\bar\sigma}^+}(p,x) \mapsto \varphi_m(q) = \exp_{\mathcal{U}_{\bar\sigma}^+}(p,-x)$$

for all $p\in\mathcal{U}_{\bar\sigma}^+$ and all $x\in N_p\mathcal{U}_{\bar\sigma}^+$ with $\|x\|<s$. Then φ_m is called the *local extrinsic reflection* at m for $(f,\mathfrak{F}_{\bar\xi})$. It is an involutive diffeomorphism and $\mathcal{U}_{\bar\sigma}^+$ belongs to its fixed point set. The restriction of φ_m to $\mathcal{U}_{\bar\sigma}^-$ is a local reflection with respect to $\bar\sigma$ for $\mathcal{U}_{\bar\sigma}^-$ and, if M is totally geodesic, each (intrinsic) local reflection s_m of $(M,g,\mathfrak{F}_\xi)$ satisfies $\varphi_m\circ f = f\circ s_m$ on a set contained in $\mathcal{U}_\sigma$.

Next, let $\Phi_{\bar\sigma}\colon t\in[-\delta,\delta]\mapsto\Phi_{\bar\sigma}(t)\in\operatorname{End}(T_{\bar\sigma(t)}\overline{M})$ be the field of endomorphisms along $\bar\sigma$ given by

$$\Phi_{\bar\sigma}(t)x = -x + 2\bar\eta(x)\bar\xi, \quad \Phi_{\bar\sigma}(t)u = u$$

for all $x\in T_{\bar\sigma}M$ and all $u\in N_{\bar\sigma}M$. Then $\Phi_{\bar\sigma}(t)$ is a rotation field along $\bar\sigma$ and

$$\psi_{\bar\sigma} = \exp_{\bar\sigma}\circ\Phi_{\bar\sigma}\circ\exp_{\bar\sigma}^{-1}$$

is a local diffeomorphism on a sufficiently small tubular neighborhood of $\bar\sigma$. This $\psi_{\bar\sigma}$ is called a (*local*) Φ-*rotation* around $\bar\sigma$ (see [12] for more information). Note that

$$\varphi_{m*|\bar\sigma(t)} = \psi_{\bar\sigma*|\bar\sigma(t)} = \Phi_{\bar\sigma}(t) \tag{3.6}$$

for all $t\in[-\delta,\delta]$. In particular, $\psi_{\bar\sigma}$ fixes all points of $\mathcal{U}_{\bar\sigma}^+$.

We have

Lemma 3.1. *The local extrinsic reflection φ_m on a sufficiently small neighborhood of $f(m)$ is an isometry if and only if $\psi_{\bar\sigma}$ is an isometry, and then $\varphi_m=\psi_{\bar\sigma}$.*

P r o o f. First, let φ_m be an isometry. Using (3.6) we get

$$\varphi_m = \exp_{\bar\sigma} \circ (\varphi_{m*})_{\bar\sigma} \circ \exp_{\bar\sigma}^{-1} = \exp_{\bar\sigma} \circ \Phi_{\bar\sigma} \circ \exp_{\bar\sigma}^{-1} = \psi_{\bar\sigma}$$

and hence, $\psi_{\bar\sigma}$ is an isometry.

To prove the converse, we first show that

$$(\psi_{\bar\sigma *})_p x = -x \tag{3.7}$$

for all $p \in \mathcal{U}_{\bar\sigma}^+$ and all $x \in N_p\mathcal{U}_{\bar\sigma}^+$. To do this, let γ be a curve in $\mathcal{U}_{\bar\sigma}^+$ from $f(m)$ to p. Since $\psi_{\bar\sigma}$ is an isometry, we have that $\mathcal{U}_{\bar\sigma}^+$ is totally geodesic and $(\psi_{\bar\sigma *})_p = P_\gamma \circ \Phi_{\bar\sigma}(0) \circ P_\gamma^{-1}$ where P_γ denotes the parallel translation along γ. Hence, (3.7) follows.

On the other hand, since $\mathcal{U}_{\bar\sigma}^+$ belongs to the fixed point set of the isometry $\psi_{\bar\sigma}$, we have

$$\psi_{\bar\sigma} = \exp_{\mathcal{U}_{\bar\sigma}^+} \circ \psi_{\bar\sigma * | \mathcal{U}_{\bar\sigma}^+} \circ \exp_{\mathcal{U}_{\bar\sigma}^+}^{-1} .$$

So, this and (3.7) yield $\psi_{\bar\sigma} = \varphi_m$, which completes the proof. □

A criterion for an isometric local rotation was derived in [12]. From this and Lemma 3.1 we then obtain the following characterization for isometric local extrinsic reflections.

Proposition 3.1. *Let $f\colon (M,g) \to (\overline{M}, \bar g, \mathfrak{F}_{\bar\xi})$ be a tangent isometric immersion. Then the local reflection φ_m at m for $(f, \mathfrak{F}_{\bar\xi})$ is an isometry if and only if*
(i) *$\Phi_{\bar\sigma}$ is parallel along $\bar\sigma$;*
(ii) $(\overline{\nabla}^k_{x\ldots x}\overline{R})_{xuxv} = (\overline{\nabla}^k_{\Phi_{\bar\sigma}x\ldots\Phi_{\bar\sigma}x}\overline{R})_{\Phi_{\bar\sigma}x\Phi_{\bar\sigma}u\Phi_{\bar\sigma}x\Phi_{\bar\sigma}v}$
for all $x \in N_{\bar\sigma(t)}\bar\sigma$, all $u, v \in T_{\bar\sigma(t)}\overline{M}$, all $t \in [-\delta, \delta]$ and all $k \in \mathbb{N}$.

We now continue by considering a special class of tangent isometric immersions which will play an important role. Such an immersion f is said to be *invariant* (with respect to $\mathfrak{F}_{\bar\xi}$) if $\overline{H}X$ is tangent to M $(= f(M))$ for any tangent vector X of M, i.e., $\overline{H}T_mM \subset T_mM$ for all $m \in M$ [9]. In that case $\overline{H}N_mM \subset N_mM$ and the tensor field H of M related to $\overline{H}$ by $f_* \circ H = \overline{H} \circ f_*$ coincides with the one defined in Section 2 for the flow $\mathfrak{F}_\xi$ on (M,g). Moreover, since f is tangent to $\bar\xi$, we have $\overline{H}X = HX - \alpha(X,\xi)$ for all X tangent to M. Hence, $\alpha(X,\xi) = 0$ for all $X \in \mathfrak{X}(M)$ or equivalently, $S_U\xi = 0$ for all $U \in N(M)$. Moreover, $\alpha(X,\xi) = 0$ for all $X \in \mathfrak{X}(M)$ implies that f is invariant. From this we have

Proposition 3.2. *Let $f\colon (M,g) \to (\overline{M}, \bar g, \mathfrak{F}_{\bar\xi})$ be a tangent isometric immersion. Then f is invariant if and only if the rotation field $\Phi_{\bar\sigma}$ around $\bar\sigma = \bar\sigma_{f(m)} = f \circ \sigma_m$ is parallel for all $m \in M$.*

Proof. From (3.1) and (3.2) we easily obtain

$$\Phi'_{\bar{\sigma}} X = -2\alpha(X, \xi), \quad \Phi'_{\bar{\sigma}} U = -2 S_U \xi$$

for all X, U along $\bar{\sigma}$ where X is tangent to M and orthogonal to ξ and U is normal to M. This yields the required result. □

Next, we turn to the definition of the main notion of this paper.

Definition 3.1. A tangent isometric immersion f of (M, g) into $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ is said to be (extrinsic) locally transversally symmetric if all local extrinsic reflections φ_m, $m \in M$, are isometries. If moreover, M is a submanifold of $\overline{M}$, then M is called an (extrinsic) locally transversally symmetric submanifold.

Then we have

Proposition 3.3. *Locally transversally symmetric immersions are η-parallel and invariant.*

Proof. It follows from Propositions 3.1 and 3.2 that f is invariant. Moreover, since each local extrinsic reflection φ_m is an isometry, we get for all horizontal $x, y, z \in T_m M$:

$$(\tilde{\nabla}_x \alpha)(y, z) = -(\tilde{\nabla}_{\varphi_{m*} x} \alpha)(\varphi_{m*} y, \varphi_{m*} z) = -\varphi_{m*}(\tilde{\nabla}_x \alpha)(y, z).$$

This yields $(\tilde{\nabla}_x \alpha)(y, z) = 0$ and so, f is η-parallel. □

Now, we shall derive a criterion for locally transversally symmetric immersions. Let $\Phi \colon m \mapsto \Phi_m$ be the endomorphism field along M defined by

$$\Phi_m x = -x + 2\bar{\eta}(x)\bar{\xi}, \quad \Phi_m u = u$$

for all $x \in T_m M$ and all $u \in N_m M$. Observe that $\Phi_{\bar{\sigma}(t)} = \Phi_{\bar{\sigma}}(t)$ where $\Phi_{\bar{\sigma}}$ is the rotation field along $\bar{\sigma}$. From Propositions 3.1 and 3.2 we then get

Theorem 3.1. *A tangent isometric immersion $f \colon (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ is locally transversally symmetric if and only if*

(i) *f is invariant;*

(ii) $(\overline{\nabla}^k_{x \ldots x} \overline{R})_{xuxv} = (\overline{\nabla}^k_{\Phi x \ldots \Phi x} \overline{R})_{\Phi x \Phi u \Phi x \Phi v}$

for all tangent vectors x, u, v of $\overline{M}$ along M with x orthogonal to ξ, and all $k \in \mathbb{N}$.

Next, we concentrate on the relation between locally transversally symmetric immersions and the properties of naturally associated mappings on the base spaces of the local Riemannian submersions. We first recall the definition of these mappings. For a tangent isometric immersion we choose sufficiently small open neighborhoods $\mathcal{U}$ and $\overline{\mathcal{U}}$ of $m \in M$ and $f(m)$, respectively, such that ξ is regular on $\mathcal{U}$ and $\bar{\xi}$ on $\overline{\mathcal{U}}$

and moreover, $f(\mathcal{U}) \subset \bar{\mathcal{U}}$. Then the *transverse mapping* $f' : \mathcal{U}' = \mathcal{U}/\xi \to \bar{\mathcal{U}}' = \bar{\mathcal{U}}/\bar{\xi}$ is defined by

$$f' \circ \pi = \bar{\pi} \circ f_{|\mathcal{U}}$$

where $\pi : \mathcal{U} \to \mathcal{U}'$ and $\bar{\pi} : \bar{\mathcal{U}} \to \bar{\mathcal{U}}'$ are the corresponding local Riemannian submersions [9]. This f' is an isometric immersion of $(\mathcal{U}', g')$ into $(\bar{\mathcal{U}}', \bar{g}')$.

Further, let $r, s > 0$ be so small that the open tubular neighborhood $\mathcal{T}_{\mathcal{U}^+_{\bar{\sigma}}}(s)$ around $\mathcal{U}^+_{\bar{\sigma}} = \mathcal{U}^+_{\bar{\sigma}}(r)$ is contained in $\bar{\mathcal{U}}$. Let $m' \in \mathcal{U}'$ and denote by $\mathcal{U}'^+_{m'}$ the submanifold of $\bar{\mathcal{U}}'$ given by

$$\mathcal{U}'^+_{m'} = \{\exp_{f'(m')} u' \mid u' \in N_{m'}\mathcal{U}',\ \|u'\| < r\}.$$

Following [10], the reflection $\varphi'_{m'}$ with respect to $\mathcal{U}'^+_{m'}$, defined in a sufficiently small open tubular neighborhood around $\mathcal{U}'^+_{m'}$, is called the *local extrinsic symmetry* at m' for f'. Moreover, f' is said to be an (*extrinsic*) *locally symmetric immersion* if $\varphi'_{m'}$ is an isometry. Such immersions are always parallel and $(\mathcal{U}', g')$ is an (intrinsic) locally symmetric space.

Using these notions we prove

Theorem 3.2. *Let $\mathfrak{F}_{\bar{\xi}}$ be a normal flow on $(\overline{M}, \bar{g})$ and let $f : (M, g) \to (\overline{M}, \bar{g})$ be a locally transversally symmetric immersion. Then each local transverse mapping of f is a locally symmetric immersion and with the induced structure $(g, \mathfrak{F}_{\xi})$, M is a locally* KTS*-space. Moreover, we have*

$$\varphi_m \circ f = f \circ s_m$$

where s_m denotes the (*intrinsic*) *local reflection at m of $(M, g, \mathfrak{F}_{\xi})$. In particular, φ_m maps $f(\mathcal{U})$ onto itself.*

To prove this theorem we first need some preliminary considerations. Let γ be a transversal geodesic $\gamma(s) = \exp_{\bar{\sigma}(t)}(sx)$, $x \in N_{\bar{\sigma}(t)}\bar{\sigma}$, through $\bar{\sigma}(t)$ for some $t \in [-\delta, \delta]$. We shall also denote $\dot{\gamma}(s)$ by x. From Proposition 2.1 it follows at once that $\overline{H}x$ has a constant length along γ. Moreover, although $\overline{H}x$ is not parallel along γ (we have $\overline{\nabla}_x(\overline{H}x) = \|\overline{H}x\|^2\bar{\xi}$), the plane spanned by $\bar{\xi}$ and $\overline{H}x$ is parallel along γ. Further, if f is an invariant immersion, then $\overline{H}x \in T_{\bar{\sigma}(t)}M$ if x does and $\overline{H}x \in N_{\bar{\sigma}(t)}M$ if x does. Hence, we obtain

Lemma 3.2. *Let $\mathfrak{F}_{\bar{\xi}}$ be a normal flow on $(\overline{M}, \bar{g})$ and $f : (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ an invariant immersion. If $\mathcal{U}^-_{\bar{\sigma}}$ (resp. $\mathcal{U}^+_{\bar{\sigma}}$) is totally geodesic, then $\bar{\xi}$ is tangent to $\mathcal{U}^-_{\bar{\sigma}}$ (resp. $\mathcal{U}^+_{\bar{\sigma}}$).*

Further, we have

Lemma 3.3. *Let $\mathfrak{F}_{\xi}$ be an isometric flow on a Riemannian manifold (M, g) and let P be a connected, relatively compact and topologically embedded submanifold. If ξ is tangent to P, then the reflection φ_P with respect to P preserves ξ, i.e., $\varphi_{P*}\xi = \xi$.*

Proof. Let γ be the geodesic defined in a sufficiently small tubular neighborhood of P by $\gamma(t) = \exp_m(tu)$ where $m \in P$, $u \in N_m P$ and $\|u\| = 1$. Let $\{\alpha_s\}$ be the (local) one-parameter group of isometries generated by ξ. Then, for each parameter s, the curve $\alpha_s \circ \gamma$ is also a geodesic and, since ξ is tangent to P, $\alpha_s(m) \in P$ and its initial velocity $(\alpha_s \circ \gamma)'(0) = \alpha_{s*}u$ is a unit vector orthogonal to P. Hence, we get

$$\varphi_P \circ \alpha_s(\gamma(t)) = \alpha_s(\gamma(-t)) = \alpha_s \circ \varphi_P(\gamma(t))$$

and this implies the result. $\square$

Now we are ready to give the

Proof of Theorem 3.2. Starting from

$$\bar{\pi}(\exp_{\bar{\sigma}}(t, x)) = \exp_{f'(m')} \bar{\pi}_* x$$

for $x \in N\bar{\sigma}$ and $m' = \pi(m)$, we get $\bar{\pi}(\mathcal{U}_{\bar{\sigma}}^{+}) = \mathcal{U}_{m'}^{'+}$. Moreover, since φ_m is an isometry and $\mathcal{U}_{\bar{\sigma}}^{+}$ belongs to its fixed point set, $\mathcal{U}_{\bar{\sigma}}^{+}$ is totally geodesic. So, Lemma 3.2 then yields that $\bar{\xi}$ is tangent to $\mathcal{U}_{\bar{\sigma}}^{+}$. Now, we obtain

$$\bar{\pi}(\exp_{\mathcal{U}_{\bar{\sigma}}^{+}}(p, x)) = \exp_{\mathcal{U}_{m'}^{'+}}(\bar{\pi}(p), \bar{\pi}_* x)$$

for all $x \in N_p \mathcal{U}_{\bar{\sigma}}^{+}$, $\|x\| < s$ and $p \in \mathcal{U}_{\bar{\sigma}}^{+}$. From this we have

$$\bar{\pi}\big(\mathcal{T}_{\mathcal{U}_{\bar{\sigma}}^{+}}(s)\big) = \mathcal{T}'_{\mathcal{U}_{m'}^{'+}}(s).$$

Now, let $\varphi'_{m'}$ denote the local extrinsic symmetry at m' for f' defined on $\mathcal{T}'_{\mathcal{U}_{m'}^{+}}(s)$. This $\varphi'_{m'}$ and the local extrinsic reflection at m for $(f, \mathfrak{F}_{\bar{\xi}})$ are related by

$$\bar{\pi} \circ \varphi_m = \varphi'_{m'} \circ \pi. \tag{3.8}$$

So, using also Lemma 3.3, it follows that $\varphi'_{m'}$ is an isometry and hence, f' is a locally symmetric immersion. Proposition 4.1 in [10] yields that $\mathcal{U}'$ is (intrinsic) locally symmetric and then it follows from Proposition 2.3 that $(M, g, \mathfrak{F}_{\xi})$ is a locally KTS-space. Further, applying [10, Proposition 3.3], we then obtain

$$\varphi'_{m'} \circ f' = f' \circ s'_{m'} \tag{3.9}$$

where $s'_{m'}$ denotes the intrinsic local symmetry at m' of $\mathcal{U}'$. Since s_m and $s'_{m'}$ are related by

$$\pi \circ s_m = s'_{m'} \circ \pi,$$

(3.8) and (3.9) yield

$$\bar{\pi} \circ \varphi_m \circ f = \bar{\pi} \circ f \circ s_m.$$

Finally, using again Lemma 3.3, we see that φ_m maps $f(\mathcal{U})$ onto itself and further, $\varphi_m \circ f$ and $f \circ s_m$ are isometries of $\mathcal{U}$ onto $f(\mathcal{U})$. Since

$$(\varphi_m \circ f)_{*m} = (f \circ s_m)_{*m},$$

it then follows that $\varphi \circ f = f \circ s_m$ and this completes the proof. □

From this we derive the following criterion for locally transversally symmetric immersions.

Theorem 3.3. *Let $\mathfrak{F}_{\bar{\xi}}$ be a normal flow on $(\overline{M}, \bar{g})$ and $f\colon (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ a tangent isometric immersion. Then f is locally transversally symmetric if and only if*

(i) *f is invariant;*

(ii) *each local transverse immersion of f is locally symmetric.*

P r o o f. If f is locally transversally symmetric, (i) and (ii) follow from Proposition 3.3 and Theorem 3.2.

Conversely, using Lemma 2.2 and taking into account that f is invariant, we get that the condition (ii) in Theorem 3.1 is equivalent to

$$(\overline{\nabla}^k_{x\ldots x}\overline{R})_{xuxv} = (\overline{\nabla}^k_{\Phi x\ldots\Phi x}\overline{R})_{\Phi x\Phi u\Phi x\Phi v}, \quad k \in \mathbb{N}, \tag{3.10}$$

for all tangent vectors x, u, v of $\overline{M}$ along M and orthogonal to ξ. Next, let $f'\colon \mathcal{U}' = \mathcal{U}/\xi \to \bar{\mathcal{U}}' = \bar{\mathcal{U}}/\bar{\xi}$ be a local transverse mapping of f. Then, applying Lemma 2.3 and the fact that f is invariant, we have that (3.10) holds if

$$(\overline{\nabla}'^k_{x'\ldots x'}\overline{R}')_{x'y'x'z'} = (\nabla'^k_{\Phi'x'\ldots\Phi'x'}\overline{R}')_{\Phi'x'\Phi'y'\Phi'x'\Phi'z'} \tag{3.11}$$

holds for all $k \in \mathbb{N}$ and all vectors x', y', z' on $\bar{\mathcal{U}}'$ along $\mathcal{U}'$, where Φ' is the $(1,1)$-tensor field on $\mathcal{U}'$ defined by

$$\Phi' x' = \bar{\pi}_* \Phi x'^*.$$

Now, by virtue of [10, Theorem 4.2], (3.11) means that f' is locally symmetric. Thus the result follows from Theorem 3.1. □

For locally KTS-spaces we may obtain some simpler characterizations.

Corollary 3.1. *Let $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a locally* KTS*-space. A tangent isometric immersion $f\colon (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ is locally transversally symmetric if and only if*

(i) *f is invariant and η-parallel;*

(ii) *$\overline{R}_{uv}u$ is normal to $f(M)$ for all $u, v \in N(M)$.*

P r o o f. First, let f be locally transversally symmetric. Then the result follows from Proposition 3.3 and the fact that the local extrinsic reflections are isometries.

Conversely, let $f'\colon \mathcal{U}' = \mathcal{U}/\xi \to \bar{\mathcal{U}}' = \bar{\mathcal{U}}/\bar{\xi}$ be a local transverse mapping of f. From [9, Proposition 3.1] it follows that, since f is invariant and η-parallel, f' is a parallel immersion. Further, from (2.9) we get

$$(\overline{R}'_{a'b'}a')^* = \overline{R}_{a'^*b'^*}a'^* + 3h(a'^*, b'^*)Ha'^* \tag{3.12}$$

for all a', b' tangent to $\bar{\mathcal{U}}'$. Since f is invariant, (3.12) and (ii) imply that $\overline{R}'_{u'v'}u'$ is normal to $\mathcal{U}'$ for all $u', v' \in N(\mathcal{U}')$. Applying now [10, Corollary 5.2] we conclude that f' is locally symmetric and then Theorem 3.3 completes the proof. □

Corollary 3.2. *Let $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a locally* KTS*-space. A tangent isometric immersion $f\colon (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ is locally transversally symmetric if and only if*

(i) *f is invariant;*

(ii) *$\mathcal{U}^+_{\bar{\sigma}}$ is totally geodesic for all $\bar{\sigma} = \bar{\sigma}_{f(m)}$, $m \in M$;*

(iii) *$\overline{R}_{xy}x$ is tangent to $f(M)$ for all horizontal x, y tangent to $f(M)$.*

P r o o f. If f is locally transversally symmetric, then f is invariant (Proposition 3.3). The extrinsic reflection φ_m at m is an isometry which yields that $\mathcal{U}^+_{\bar{\sigma}}$ is totally geodesic, since it belongs to the fixed point set of φ_m, and moreover (iii) holds.

Conversely, Lemma 3.2 shows that $\bar{\xi}$ is tangent to $\mathcal{U}^+_{\bar{\sigma}}$. Then, using the fact that $\mathcal{U}^+_{\bar{\sigma}}$ is totally geodesic and $\bar{\pi}\colon \bar{\mathcal{U}} \to \bar{\mathcal{U}}' = \bar{\mathcal{U}}/\bar{\xi}$ is a Riemannian submersion, we get that $\mathcal{U}'^+_{m'} = \bar{\pi}(\mathcal{U}^+_{\bar{\sigma}})$ is totally geodesic in $\bar{\mathcal{U}}'$. Moreover, from (3.12) and the fact that f is invariant, (iii) implies that $\overline{R}'_{x'y'}x'$ is tangent to $\mathcal{U}'$ for all x', y' tangent to $\mathcal{U}'$. Then it follows from [10, Corollary 5.1] that f' is a locally symmetric immersion and so, the required result follows from Theorem 3.3. □

Finally, we prove

Corollary 3.3. *Let $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ be a locally* KTS*-space. A tangent isometric immersion $f\colon (M, g) \to (\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ is locally transversally symmetric if and only if*

(i) *f is invariant;*

(ii) *$\mathcal{U}^+_{\bar{\sigma}}$ and $\mathcal{U}^-_{\bar{\sigma}}$ are totally geodesic for all $\bar{\sigma} = \bar{\sigma}_{f(m)}$, $m \in M$.*

P r o o f. Let f be locally transversally symmetric. Then f is invariant and $\bar{s}_{f(m)} \circ \varphi_m$, where $\bar{s}_{f(m)}$ is the local reflection with respect to $\bar{\sigma}$, is an isometry. Since $\mathcal{U}^-_{\bar{\sigma}}$ belongs to its fixed point set, it is totally geodesic. The result follows now by (ii) of Corollary 3.2.

Conversely, (ii) and the Codazzi equation (3.4) imply (iii) in Corollary 3.2 and hence the result follows. □

4. Transversally symmetric immersions in normal flow space forms

In this final section we shall treat transversally symmetric immersions in a special class of locally KTS-spaces. We start with some preliminaries. A Riemannian manifold (M, g) equipped with a contact flow $\mathfrak{F}_\xi$ is said to be a *flow space form* if the H-sectional curvature is pointwise constant, i.e., the sectional curvature of the two-plane spanned by X and HX for horizontal $X \in T_m M$ is independent of X for each point $m \in M$. The *normal flow space forms* have been studied in [7] where two cases are considered according to whether the ξ-sectional curvature is constant or not. In what follows we consider these two cases separately.

A. Normal flow space forms with constant ξ-sectional curvature c^2.
In this case, if the H-sectional curvature equals k, then $(M, c^2 g, \varphi = c^{-1} H, c^{-1}\xi, c\eta)$ is a Sasakian manifold of constant φ-sectional curvature kc^{-2} and so, (M, g) is obtained by a homothetic change of metric from Sasakian space forms. See [7] for more information and references. Hence, as is proved in [7] for $\dim M \geqslant 5$, the H-sectional curvature is a global constant k and the curvature tensor is given by

$$
\begin{aligned}
(4.1)\quad R_{UV}W = {} & \frac{k+3c^2}{4}\{g(U,W)V - g(V,W)U\} \\
& + \frac{k-c^2}{4}\{\eta(V)\eta(W)U - \eta(U)\eta(W)V + g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\} \\
& + \frac{k-c^2}{4c^2}\{g(W,HU)HV - g(W,HV)HU - 2g(U,HV)HW\}.
\end{aligned}
$$

For $\dim M = 3$ we shall assume that k is also a global constant. Then (4.1) also holds in this case. It also follows that these normal flow space forms are locally KTS-spaces.

In what follows we shall denote such a space by $M^{2n+1}(c^2, k)$ where $c^2 = 1$ corresponds to the Sasakian space forms. For each (c^2, k) the normal flow space form is locally isomorphic to one of the following model spaces:

$$(S^{2n+1} = SU(n+1)/SU(n))(c^2, k)$$

for $k + 3c^2 > 0$, $H(n,1)(k)$ for $k + 3c^2 = 0$, where $H(n,1)$ is the $(2n+1)$-dimensional Heisenberg group,

$$(U(1,n)/U(n))^{\sim}(c^2, k) = (SU(1,n)^{\sim}/SU(n))(c^2, k)$$

for $k + 3c^2 < 0$, where $^{\sim}$ denotes the universal covering.

We refer to [7] for more details.

Next, we give two lemmas. Let P be a connected, relatively compact and topologically embedded submanifold of $M^{2n+1}(c^2,k)$. Then we have

Lemma 4.1. *If $k \neq c^2$, then the reflection φ_P with respect to P is isometric if and only if P is either a totally geodesic invariant submanifold or a totally geodesic anti-invariant submanifold with ξ normal to P and $\dim P = n$.*

Here P is said to be anti-invariant if $HT_pP \subset N_pP$ for all $p \in P$. The proof of this lemma follows from its analogue for Sasakian space forms given in [3].

In the case $k = c^2$, (4.1) implies that $M^{2n+1}(c^2,k)$ is a manifold of constant curvature k and then the following result is well-known (see for example [4]):

Lemma 4.2. *If (M,g) is a space of constant curvature, then φ_P is isometric if and only if P is a totally geodesic submanifold.*

Now we are ready to prove our results for this case.

Theorem 4.1. *Let f be a tangent isometric immersion of a Riemannian manifold M into $\overline{M}^{2n+1}(c^2,\bar{k})$. Then the following statements are equivalent:*
(i) *f is locally transversally symmetric;*
(ii) *f is invariant and η-parallel;*
(iii) *f is invariant and $\mathcal{U}^+_{\bar{\sigma}}$ is totally geodesic for all $\bar{\sigma} = \bar{\sigma}_{f(m)}$, $m \in M$;*
(iv) *$\mathcal{U}^+_{\bar{\sigma}}$ is totally geodesic and invariant for all $\bar{\sigma} = \bar{\sigma}_{f(m)}$, $m \in M$.*

P r o o f. (i) $\Leftrightarrow$ (ii) follows from Corollary 3.1 and (4.1). Corollary 3.2 yields (i) $\Leftrightarrow$ (iii). Finally, (i) $\Leftrightarrow$ (iv) follows at once from Lemma 3.2 and Lemma 4.1. $\square$

Note that since the isometric flow $\mathfrak{F}_{\bar{\xi}}$ on $\overline{M}^{2n+1}(c^2,\bar{k})$ is a contact flow and f is an invariant immersion, the induced flow $\mathfrak{F}_{\xi}$ on M is also a contact flow and hence, M must be odd-dimensional. Moreover, it follows from the Gauss equation (3.3) and $\alpha(X,\xi)=0$ that the ξ-sectional curvature on M equals c^2. From Theorem 4.1 and using [9, Corollary 4.1 and Proposition 4.4] we get

Corollary 4.1. *Let f be a locally transversally symmetric immersion of a Riemannian manifold M^{2n+1} into $\overline{M}^{2(n+r)+1}(c^2,\bar{k})$. If $\bar{k}+3c^2$ is non-positive, then f is totally geodesic. Moreover, if $M = M^{2n+1}(c^2,k)$, then either $k=\bar{k}$ and the immersion is totally geodesic, or $\bar{k} = 2k+3c^2$.*

R e m a r k. We note that all non-totally geodesic complete invariant submanifolds (M,f) embedded in $S^{2n+1}(c^2,\bar{k})$ with f full and η-parallel, which by Theorem 4.1 (ii) are locally transversally symmetric submanifolds, have been completely classified, up to congruences, in [9, Theorem 4.2].

B. Normal flow space forms with non-constant ξ-sectional curvature and globally constant H-sectional curvature.

This class was also discussed in detail in [7] where it was proved that these space forms are also locally KTS-spaces. We recall some further useful facts. In the complete case, $(\overline{M}, \bar{g}, \mathfrak{F}_{\bar{\xi}})$ admits smooth distributions $\overline{\mathfrak{H}}_1$ and $\overline{\mathfrak{H}}_2$ such that for each $\overline{m} \in \overline{M}$, $\overline{\mathfrak{H}}(\overline{m}) = \overline{\mathfrak{H}}_1(\overline{m}) \oplus \overline{\mathfrak{H}}_2(\overline{m})$ is an $\overline{H}$-invariant decomposition of the horizontal subspace $\overline{\mathfrak{H}}(\overline{m})$ and each sectional curvature $K(\overline{\mathfrak{H}}_i, \bar{\xi})$, $i = 1, 2$, is a positive constant $\bar{c}_i^2$ ($\bar{c}_1^2 > \bar{c}_2^2$). Further, such spaces are precisely the Riemannian manifolds $(\overline{M}^{2N+1}, \bar{g})$ equipped with a normal contact flow $\mathfrak{F}_{\bar{\xi}}$ which is transversally modelled on the Riemannian product $\mathbb{C}P^{N_1}(h_1) \times \mathbb{C}H^{N_2}(h_2)$ where $|h_2| < h_1$, $N_1 + N_2 = N$ and the ξ-sectional curvatures $\bar{c}_i^2$, $i = 1, 2$, are given by

$$\bar{c}_i^2 = (-1)^{i+1} h_i \frac{h_1 - h_2}{3(h_1 + h_2)}.$$

In the rest of this section we shall denote such a space by $\overline{M}(N_1, N_2; h_1, h_2)$. The H-sectional curvature $\bar{k}$ is the strictly negative constant $\bar{k} = 2h_1 h_2 (h_1 + h_2)^{-1}$ and the curvature tensor is given by

$$\begin{aligned}
(4.2)\quad R_{UV}W = \sum_{i=1}^{2} \Bigg\{ & \frac{h_i}{4}\Big(g(X_i, Z_i)Y_i - g(Y_i, Z_i)X_i\Big) \\
& + (-1)^i \frac{3(h_1 + h_2)}{4(h_1 - h_2)}\Big(g(HY_i, Z_i)HX_i - g(HX_i, Z_i)HY_i - 2g(HX_i, Y_i)HZ_i\Big) \\
& + (-1)^i \frac{h_i(h_1 - h_2)}{3(h_1 + h_2)}\Big\{\big(g(Y_i, Z_i)\eta(U) - g(X_i, Z_i)\eta(V)\big)\xi \\
& + \eta(W)\big(\eta(V)X_i - \eta(U)Y_i\big)\Big\}\Bigg\} \\
& + g(HV, W)HU - g(HU, W)HV - 2g(HU, V)HW
\end{aligned}$$

for vector fields $U = \sum_{i=1}^{2} X_i + \eta(U)\xi$, $V = \sum_{i=1}^{2} Y_i + \eta(V)\xi$, $W = \sum_{i=1}^{2} Z_i + \eta(W)\xi$ on $\overline{M}$.

Further, we will need

Lemma 4.3. [9] *Let f be an invariant immersion of a Riemannian manifold (M^{2n+1}, g) into a complete normal flow space form $\overline{M}(N_1, N_2; h_1, h_2)$. We have*

(a) *If the ξ-sectional curvature on M is a constant c^2, then c^2 is either $\bar{c}_1^2$ or $\bar{c}_2^2$, and $f_* T_m M \subset \overline{\mathfrak{H}}_1(f(m))$ or $\subset \overline{\mathfrak{H}}_2(f(m))$, respectively, for all $m \in M$. Also, M is locally transversally immersed into $\mathbb{C}P^{N_1}(h_1)$ or $\mathbb{C}H^{N_2}(h_2)$.*

(b) *If the ξ-sectional curvature on M is non-constant and M is a complete locally* KTS*-space, then there exist smooth distributions $\mathfrak{H}_1$ and $\mathfrak{H}_2$ on M such that for each $m \in M$, $\mathfrak{H}(m) = \mathfrak{H}_1(m) \oplus \mathfrak{H}_2(m)$ is an H-invariant decomposition* *of the horizontal subspace $\mathfrak{H}(m)$, the sectional curvatures $K(\mathfrak{H}_i, \xi) = \bar{c}_i^2$ and $f_*\mathfrak{H}_i \subset \overline{\mathfrak{H}}_i$, for $i = 1, 2$.*

From this lemma and (4.2) we get the following: First, from (a) we may conclude that then we can formulate a similar result as in Theorem 4.1 for locally transversally symmetric submanifolds M of $\overline{M}(N_1, N_2; h_1, h_2)$ with constant ξ-sectional curvature. Further, using (b) and Corollaries 3.1 and 3.2, we also obtain

Theorem 4.2. *Let f be a tangent isometric immersion of a complete locally* KTS-*space M into a complete normal flow space form $\overline{M}(N_1, N_2; h_1, h_2)$. Then the following statements are equivalent:*
(i) *f is locally transversally symmetric;*
(ii) *f is invariant and η-parallel;*
(iii) *f is invariant and $\mathcal{U}_{\bar{\sigma}}^{+}$ is totally geodesic for all $\bar{\sigma} = \bar{\sigma}_{f(m)}$, $m \in M$.*

This and [9, Theorem 4.4] yield

Corollary 4.2. *A complete locally* KTS-*space M of dimension $2n+1$ is a locally transversally symmetric submanifold of a complete normal flow space form $\overline{M}(N_1, N_2; h_1, h_2)$ if and only if*
(i) *M is transversally modelled on a Riemannian product ${M'}_1^{n_1} \times \mathbb{C}H^{n_2}(h_2)$, $n_1 + n_2 = n$, where M_1' is a Hermitian symmetric space, and*
(ii) *each local transverse immersion f' of f is a product immersion $f_1' \times f_2'$ where f_1' is a parallel K hler immersion of $\mathcal{U}_1' \subset M_1'$ into $\mathbb{C}P^{N_1}(h_1)$ and f_2' is a totally geodesic immersion of $\mathcal{U}_2' \subset \mathbb{C}H^{n_2}(h_2)$ into $\mathbb{C}H^{N_2}(h_2)$.*

Remark. In [9] many examples of locally transversally symmetric submanifolds (i.e., invariant and η-parallel submanifolds) of a complete, simply connected normal flow space form $\overline{M}(N_1, N_2; h_1, h_2)$ are given.

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