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# PROBABILISTIC ANALYSIS OF SINGULARITIES FOR THE 3D NAVIER-STOKES EQUATIONS

Franco Flandoli, Pisa, Marco Romito, Florence

*Abstract.* The classical result on singularities for the 3D Navier-Stokes equations says that the 1-dimensional Hausdorff measure of the set of singular points is zero. For a stochastic version of the equation, new results are proved. For statistically stationary solutions, at any given time $t$, with probability one the set of singular points is empty. The same result is true for a.e. initial condition with respect to a measure related to the stationary solution, and if the noise is sufficiently non degenerate the support of such measure is the full energy space.

*Keywords*: singularities, Navier-Stokes equations, Brownian motion, stationary solutions

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## 1. Introduction and preliminary remarks

The present note is a review of the paper [12] and some elements from the related works [11], [17]; see also [18] for further results. For a few general references on stochastic Navier-Stokes equations see [2], [20], [21], [9], among many others, while for general references on Navier-Stokes equations on the one hand and infinite dimensional stochastic analysis on the other, see [19] and [5].

**1.1. Could probability tell us something new about classical problems in fluid dynamics?** This difficult challenging problem has a few positive answers and works in progress. A first example is the ergodicity for the 2D stochastic Navier-Stokes equations, proved first in [10] under some assumptions on the noise, and later on by many authors under various sets of assumptions and with different techniques, see for instance [6], [1], [14], [22]. Some ergodic properties are often tacitly assumed in statistical fluid mechanics, but a proof for the deterministic Navier-Stokes equations is still out of reach, in spite of the efforts spent on outstanding theories like the Ruelle-Bowen-Sinai one.

A second example is the probabilistic analysis of singularities for the 3D deterministic and stochastic Navier-Stokes equations developed in [11], [12]. This is the subject of the present note.

Finally, we mention a number of other directions like the probabilistic representations of solutions to Navier-Stokes equations, the vortex method, probabilistic model of turbulence, statistical solutions of Foias-like equations, diffusion of passive scalars, stochastic vortex filaments. Without the aim to list contributions in all these fields, we mention only [4], [13], [15], [16], [7].

Two typical tools, beyond others, are employed: (1) irreducibility, (2) stochastic stationarity. Tool (1) is usually introduced by means of a noise forcing term in the Navier-Stokes equations. It is somewhat an idealization of the real behaviour of a fluid, but it captures in a sort of idealized limit the extreme variability observed in turbulent fluids. Tool (2) has some of the technical advantages of time-invariance, even if the individual realizations (trajectories) may have a very complex time evolution.

**1.2. 3D Navier-Stokes equations and singularities.** Consider the Navier-Stokes equation in a bounded regular domain $D \subset \mathbb{R}^3$

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u + \nabla P = \nu \triangle u + f + \sigma \frac{\partial B}{\partial t}, \\ \operatorname{div} u = 0, \quad u|_{\partial D} = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{1}$$

Speaking in terms of Physics, $u$ is the velocity field, $P$ the pressure, $f$ a slowly varying forcing term, $\frac{\partial B}{\partial t}$ a fast fluctuating forcing term. The kinematic viscosity $\nu$ is assumed to be strictly positive, while the noise intensity $\sigma \geqslant 0$ may be equal to zero (deterministic case), depending on the theorem.

Before giving a rigorous definition of a suitable weak solution, let us mention the concept of singular points. A point $(t,x) \in (0,\infty) \times D$ will be called *regular* if $u$ is locally (essentially) bounded around it. Otherwise, the point $(t,x)$ is called *singular*. The set of singular points of $u$ will be denoted by $S(u)$. We have $S\,(u) \subset (0,\infty) \times D \subset \mathbb{R}^4$. The fundamental result of Caffarelli, Kohn and Nirenberg [3] tells us that the 1-dimensional Hausdorff measure of $S(u)$ is zero provided $u$ is a suitable weak solution:

$$\mathcal{H}^1(S(u)) = 0.$$

This result is a refinement of previous results of Scheffer. Whether $S(u)$ is empty or not is the main open problem. It is empty for time-invariant solutions. In a sense, we shall prove that it is empty also for stochastically stationary solutions.

A singularity corresponds to a local concentration of energy. The global kinetic energy cannot blow up: for $\sigma = 0$, $\frac{1}{2}\int_D |u\,(t,x)|^2 \,\mathrm{d}x$ (plus dissipation energy) is bounded by $\frac{1}{2}\int_D |u_0\,(x)|^2 \,\mathrm{d}x$ plus the work done by the body forces, and the same result is true (with a more involved inequality) also for $\sigma > 0$ under reasonable assumptions on $B$. However, energy may concentrate, it may be transferred to smaller scales, and the (energy)/(unit volume) may blow up at some point: $r^{-3}\int_{B_r(x_0)} |u(t,x)|^2 \, \mathrm{d}x \to \infty$ as $r \to 0$ (this is an open problem). This problem is similar to the concentration of energy in finite regions that can be seen for Hamiltonian systems of $\infty$-many particles. Here and below we denote the ball of center $x_0$ and radius $r$ by $B_r(x_0)$ or simply by $B_r$.

Roughly speaking, the idea of the blow-up control is the following one. We have a local energy balance of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_r}\frac{|u|^2}{2} + \nu\int_{B_r}|\nabla u|^2 \leqslant \int_{\partial B_r}\frac{|u|^2}{2}u\cdot n + \text{work done by forces}$$

which says that the local variation (possible concentration) of kinetic energy $\frac{\mathrm{d}}{\mathrm{d}t}\int_{B_r}\frac{|u|^2}{2}$, plus the local dissipation $\nu\int_{B_r}|\nabla u|^2$, are controlled by the energy flux $\int_{\partial B_r}\frac{|u|^2}{2}u\cdot n$ plus the work terms. On the other hand, we have the Sobolev inequality

$$\int_{B_r}|u|^3 \leqslant C\Big(\int_{B_r}|\nabla u|^2\Big)^{\frac{3}{4}}\Big(\int_{B_r}|u|^2\Big)^{\frac{3}{4}} + \frac{C}{r^{\frac{3}{2}}}\Big(\int_{B_r}|u|^2\Big)^{\frac{3}{2}}$$

which allows us to control terms of the order of the energy flux by the local kinetic and dissipation energy. These two tools together give rise to iterative *nonlinear* relations for the above quantities, on a sequence of nested balls $B_{r_n}$. The resulting inequalities may be closed if some quantity is small. The criterion discovered by Caffarelli, Kohn and Nirenberg is that

$$\limsup_{r\to 0}\frac{1}{r}\int_{t-r^2}^{t+r^2}\int_{B_r(x)}|\nabla u|^2 = 0 \tag{2}$$

(or just smaller than a certain universal constant) implies $(t,x)$ regular. Having established this fact, it is not difficult to prove that $\mathcal{H}^1(S(u)) = 0$.

How may probability enter this problem?

1) As for $\infty$-many particle Hamiltonian systems, one could try to prove a good result in a stationary regime and for many initial conditions with respect to a probability measure. This is the content of this note.

2) Perhaps the emergence of singularities requires a great degree of organization (only special fluid configurations may produce singularities). Perhaps this coherence is broken by the noise. We cannot solve this problem with a true understanding of the geometry of emerging singularities. We can only prove that in the presence of noise that activates all modes, our results hold true for most initial conditions.

**1.3. Suitable weak solutions.** Let $H$ be the Hilbert space

$$H = \{u\colon\, D \to \mathbb{R}^3;\; u \in \left(L^2\left(D\right)\right)^3, \operatorname{div} u = 0,\, \left(u \cdot n\right)|_{\partial D} = 0\}$$

where $n$ is the outer normal to $\partial D$ (see for example Temam [19]), and let $V$ be the space of all $u \in (H^1(D))^3 \cap H$ such that $u|_{\partial D} = 0$. Define the Stokes operator $A\colon D(A) \subset H \to H$ as $Au = \mathcal{P}\triangle u$, where $\mathcal{P}$ is the orthogonal projection from $(L^2(D))^3$ onto $H$ and $D(A) = \left(H^2\left(D\right)\right)^3 \cap V$. Assume for sake of simplicity that $f \in L^2_{\mathrm{loc}}\left(0, \infty; H\right)$. Given any stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P}, (B_t)_{t\geqslant 0})$, where $B$ is a Brownian motion adapted to the filtration $\left(\mathcal{F}_t\right)_{t\geqslant 0}$ with values in $D(A^{\beta})$ for some $\beta > 0$ (see [5] for the definition and basic results of stochastic integration), consider the auxiliary Stokes system, written formally as

$$\begin{cases} \frac{\partial z}{\partial t} + \nabla Q = \nu \triangle z + f + \sigma \frac{\partial B}{\partial t} \\ \operatorname{div} z = 0, \quad z|_{\partial D} = 0, \quad z|_{t=0} = 0. \end{cases} \tag{3}$$

and interpreted rigorously in the mild sense

$$z(t) = \int_0^t \mathrm{e}^{-(t-s)A} f\left(s\right)\, \mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} \sigma \,\mathrm{d}B\left(s\right). \tag{4}$$

Since $B$ is, in particular, a Brownian motion in $H$, the last stochastic integral is well defined and gives us a continuous process in $H$. The auxiliary pressure $Q$ does not appear in (4) since such equation lives in $H$. However (see [12] for the details), from a solution of (4) one may reconstruct a unique pair $(z, Q)$ ($Q$ is unique up to a constant) with $z\left(\omega\right) \in C\left(\left[0, T\right]; L^2\left(D\right)\right)$ (and more, see [12]) and $Q\left(\omega\right) \in L^{\frac{5}{3}}_{\mathrm{loc}}\left(\left(0, T\right) \times D\right)$ for $\mathbf{P}$-a.e. $\omega \in \Omega$, which satisfies equation (3) in the sense of distributions.

Formally, if $(u, P)$ and $(z, Q)$ are solutions to (1) and (3) respectively, then $v = u - z$, $\pi = P - Q$ satisfy the equation

$$\begin{cases} \frac{\partial v}{\partial t} + \left(\left(v + z\right) \cdot \nabla\right)\left(v + z\right) + \nabla \pi = \nu \triangle v, \\ \operatorname{div} v = 0, \quad v|_{\partial D} = 0, \quad v|_{t=0} = u_0, \end{cases} \tag{5}$$

which is a Navier-Stokes type equation with random coefficients. This equation will be interpreted in the sense of distributions.

**Definition 1.** A martingale suitable weak solution of (1) is a process $(u, P)$ defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbf{P}, (B_t)_{t\geqslant 0})$, where $B$ is a Brownian motion adapted to the filtration $\left(\mathcal{F}_t\right)_{t\geqslant 0}$ with values in $D(A^{\beta})$ for a $\beta > 0$, such that

$$\omega \in \Omega \mapsto \left(u\left(\omega\right), P\left(\omega\right)\right) \in L^2\left(0, T; H\right) \times L^{\frac{5}{3}}_{\mathrm{loc}}\left(\left(0, T\right) \times D\right)$$

is a measurable mapping and there exists a set $\Omega_0 \subset \Omega$ of full probability such that for each $\omega \in \Omega_0$ we have

$$u(\omega) \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H^1(D)), \quad P(\omega) \in L^{\frac{5}{3}}_{\mathrm{loc}}((0,T)\times D),$$

and the new variables $v(\omega) = u(\omega) - z(\omega)$ and $\pi(\omega) = P(\omega) - Q(\omega)$ satisfy the modified Navier-Stokes equations (5) in the sense of distributions over $(0,T)\times D$, where $(z,Q)$ is the unique solution given above of (3). Moreover, the following local energy inequality has to hold for all $\omega \in \Omega_0$:

$$\begin{aligned}\int_D |v(t,\omega)|^2\varphi + 2\int_0^t\int_D \varphi|\nabla v(\omega)|^2 \leqslant \int_0^t\int_D |v(\omega)|^2\left(\frac{\partial\varphi}{\partial t} + \triangle\varphi\right)\\ + \int_0^t\int_D \left(|v(\omega)|^2 + 2v(\omega)\cdot z(\omega)\right)\left((v(\omega)+z(\omega))\cdot\nabla\varphi\right)\\ + 2\int_0^t\int_D \varphi z(\omega)\cdot\left((v(\omega)+z(\omega))\cdot\nabla\right)v(\omega) + \int_0^t\int_D 2\pi v(\omega)\cdot\nabla\varphi\end{aligned}$$

for every smooth function $\varphi\colon\ \mathbb{R}^3\times D \to \mathbb{R}$, $\varphi \geqslant 0$, with compact support in $(0,T]\times D$.

It is worth noticing that such solutions exist. A proof of this claim is given in [17]. Also, the concept of martingale solution is equivalent to the one of statistical solution, as given by Foias, Temam, and others. In the previous definition we did not insist on the regularity properties of the auxiliary variables $(z,Q)$; see [12] for the details.

## 2. Main results

### 2.1. Extension of C-K-N theorem to stochastic Navier-Stokes equations.

**Theorem 2.** *Assume that $f \in L^p((0,T\times D))$ for some $p > \frac{5}{2}$ and $B$ is a Brownian motion taking values in $D(A^{\frac{1}{4}+\beta})$ for some $\beta > 0$. Let $u$ be a martingale suitable weak solution of* (1)*. Then, for* $\mathbf{P}$-a.e. *path $u(\omega)$, we have $\mathcal{H}^1(S(u(\omega))) = 0$.*

The interpretation of this statement may be: fast (distributional in time) fluctuations of the forces do not deteriorate the (upper) estimate on singularities.

About the proof, which is quite long, we only notice that one has first to prove some regularity results for the auxiliary Stokes system (3), then one has to adapt the proof of [3] for equation (5). See [12] for details.

**2.2. Improvement for stationary solutions.** We want to study stationary solutions for the Navier-Stokes equations, stationary in the sense of probability or the ergodic theory. Let $u$ be a martingale suitable weak solution of (1) on the time interval $[0,\infty)$. The joint law $\mu$ of $(u,B)$ is a probability measure on $L^2_{\mathrm{loc}}([0,\infty);H)\times C\left([0,\infty);D(A^\beta)\right)$ (for a $\beta>0$), concentrated on a smaller set due to the regularity of $u$. In this setting the pressure $P$ is treated as an auxiliary scalar field. We say that such a solution $u$ is stationary if the joint law in $H^n$ of $(u(t_1+s),...,u(t_n+s))$ is independent of $s\geqslant 0$ for every choice of $n$ and $0\leqslant t_1<...<t_n$. Such laws are well defined since $u$ is weakly continuous in $H$.

We say that $u$ has finite mean dissipation rate if

$$\mathbf{E}\left[\int_0^T\int_D |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\right]<\infty$$

for all $T>0$, where $\mathbf{E}$ is the expectation on $(\Omega,\mathcal{F},\mathbf{P})$. A proof of the existence of a stationary solution with finite mean dissipation rate is given in [17].

**Theorem 3.** *Let $u$ be a stationary martingale suitable weak solution of* (1), *with $B$ a Brownian motion taking values in $D(A^{\frac{1}{4}+\beta})$ for some $\beta>0$. Assume that $u$ has finite mean dissipation rate. Then for every time $t\geqslant 0$ the set of singular points at time $t$ is empty for* $\mathbf{P}$*-almost every trajectory of $u$.*

In other words, if $S_t(u)$ denotes the set of all $x\in D$ such that $(t,x)$ is a singular point for the function $u$, then for all given $t\geqslant 0$, the set $S_t(u(\omega))$ is empty for $\mathbf{P}$-almost every $\omega\in\Omega$.

About the proof (see [12]), by stationarity and finite mean dissipation rate we have that

$$\mathbf{E}\left[\frac{1}{r}\int_{t-r^2}^{t+r^2}\int_D |\nabla u|^2\right]=Cr$$

for some constant $C>0$, so $r^{-1}\int_{t-r^2}^{t+r^2}\int_D|\nabla u|^2$ converges to zero as $r\to 0$ with probability one, by an argument based on the Borel-Cantelli lemma and the monotonicity in $r$ of the previous integral. This leads to (2). Notice that the result is true for all $\sigma\geqslant 0$, hence it is uniquely due to the stationarity and not to the presence of noise.

**2.3. Final results for a.e. initial conditions.** From this theorem, a regularity result for almost any initial condition can be deduced. Let $u$ be a stationary martingale suitable weak solution of (1) and let $\mu$ be the joint law of $(u,B)$ as above. First we define a measure $\mu_0$ on the space $H$ of initial conditions given by a projection of $\mu$. Since weak solutions are continuous from $[0,\infty)$ to $H$ with the weak topology, the law in $H$ of $u(0)$ under $\mu$ is well defined and it will be denoted by $\mu_0$. In a heuristic sense, $\mu_0$ is an invariant measure in $H$ for the Navier-Stokes equations, but we cannot state this in the usual sense since the Navier-Stokes equations do not define a dynamical system or a Markov semigroup (one may use the concept of infinitesimal invariance).

One can prove that $\mu$ disintegrates with respect to $\mu_0$ (see the details in [12]):

$$\mu(\cdot) = \int_H \mu(\cdot|u(0) = u_0)\, \mu_0(\mathrm{d}u_0).$$

For $\mu_0$-a.e. $u_0 \in H$, it turns out that the measure $\mu(\cdot|u(0) = u_0)$ is the law of a martingale suitable weak solution with the initial condition $u_0$ (precisely, it is the joint law of solution-Brownian motion).

As a consequence of the previous theorem one can prove (see [12]):

**Corollary 4.** *For every $t \geqslant 0$, for $\mu_0$-a.e. $u_0 \in H$,*

$$S_t(u) = \emptyset \qquad \mu(\cdot|u(0) = u_0)\text{-a.s.}$$

The interpretation is that we do not see singularities at any given time $t$, not only in the stationary regime (the theorem of the previous section) but also for $\mu_0$-a.e. initial condition. Hence only special (with respect to $\mu_0$) initial conditions may produce a certain kind of singular behaviour.

The weak point of the previous theorem could be that $\mu_0$ is concentrated on a very poor set, like a point or a periodic orbit. In the case of a single point it means that $\mu$ was the delta Dirac mass over a time-invariant solution, and therefore the absence of singularities is a well know fact (easy consequence of the result of [3]). It is therefore interesting to know that under suitable assumptions of non-degeneracy of the noise the support of $\mu_0$ is $H$. This is our first theorem where $\sigma > 0$ and $B$ cannot be just a deterministic function (we stated our previous theorems for a Brownian motion $B$ but they hold true for a suitable class of deterministic functions too, see [12]). We assume that $B$ directly acts on all Fourier components, namely that its covariance is injective. Presumably this condition can be weakened. It implies a form of irreducibility of the dynamic, proved in [8], which implies that trajectories visit all open sets of $H$ with positive probability.

**Theorem 5.** *Assume that $\sigma > 0$ and the Brownian motion $B$ has injective covariance. Then the support of $\mu_0$ is the full space $H$:*

$$\mathrm{supp}(\mu_0) = H.$$

Therefore the set of initial conditions having the property from the corollary is rich. This result is due to the noise, while all the previous ones hold true also for $\sigma = 0$.

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*Authors' addresses*: *Franco Flandoli*, Dipartimento di Matematica Applicata, Università di Pisa, Via Bonanno 25/b, 56126 Pisa, Italy, e-mail: `flandoli@dma.unipi.it`; *Marco Romito*, Dipartimento di Matematica, Università di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy, e-mail: `romito@math.unifi.it`.