CUT-VERTICES AND DOMINATION IN GRAPHS

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Summary. The paper studies the domatic numbers and the total domatic numbers of graphs having cut-vertices.

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We shall study the domatic number $d(G)$ and the total domatic number $d_t(G)$ of a graph G. A survey of the related theory is given in [3]. We consider finite, undirected graphs without loops or multiple edges.

A subset D of the vertex set $V(G)$ of a graph G is called dominating (total dominating), if for each $x \in V(G) - D$ (for each $x \in V(G)$, respectively) there exists a vertex $y \in D$ adjacent to x. A partition $\mathcal{D}$ of $V(G)$ is called a domatic (total domatic) partition of G, if each class of $\mathcal{D}$ is a dominating (total dominating, respectively) set.

The maximum number of classes of a domatic (total domatic) partition of $V(G)$ was in [1] ([2]) named the domatic (total domatic, respectively) number of G, and it is denoted by $d(G)$ ($d_t(G)$, respectively). Note that $d(G)$ is well-defined for every finite, undirected graph, while $d_t(G)$ is defined only for graphs without isolated vertices.

Consider in G a vertex v of minimum valency $\delta(G)$. Then a dominating set must contain v or a neighbour of v, thus it is obvious that $d(G) \leqslant \delta(G) + 1$. A total dominating set must contain a neighbour of v, thus $d_t(G) \leqslant \delta(G)$.

We shall consider the case when a graph G is the union of two graphs G_1, G_2 having exactly one common vertex a; this vertex a is a cut-vertex of G. The graphs obtained from G_1 and G_2 by deleting a will be denoted respectively by G_1', G_2'.

Theorem 1. *With the above notation, for every graph G the domatic numbers satisfy*

$$\min\{d(G_1), d(G_2)\} \leqslant d(G) \leqslant 1 + \min\{d(G_1'), d(G_2')\}. \tag{1}$$

The gaps in the inequalities can be arbitrarily large:

(2) For any positive integer q there exists a graph G such that

$$d(G) = \min\{d(G_1), d(G_2)\} + q.$$

(3) For any positive integer q there exists a graph G such that

$$d(G) = \min\{d(G_1'), d(G_2')\} - q.$$

Proof. (1): Let $d_1 = d(G_1)$, $d_2 = d(G_2)$. Let $\{D_1^1, \ldots, D_{d_1}^1\}$ be a domatic partition of G_1 with d_1 classes, let $\{D_1^2, \ldots, D_{d_2}^2\}$ be a domatic partition of G_2 with d_2 classes. Without loss of generality assume $d_1 \leqslant d_2$. For $i = 1, \ldots, d_1 - 1$ define $D_i = D_i^1 \cup D_i^2$ and let $D_{d_1} = D_{d_1}^1 \cup \bigcup_{i=d_1}^{d_2} D_i^2$. The sets $D_1, \ldots, D_{d_1}$ evidently form a domatic partition of G and thus $d(G) \geqslant d_1 = \min\{d(G_1), d(G_2)\}$.

For the right side inequality in (1) let $d = d(G)$ and consider a domatic partition $D_1, \ldots, D_d$ of G; without loss of generality let $a \in D_d$. For $i = 1, \ldots, d$ let $D_i^1 = D_i \cap V(G_1), D_i^2 = D_i \cap V(G_2)$. Consider D_i^1 for $1 \leqslant i \leqslant d-1$. Any vertex $x \in V(G_1') - D_i^1$ must be adjacent to a vertex of D_i; as x cannot be adjacent to any vertex of $V(G_2')$, x necessarily is adjacent to a vertex of D_i^1 and thus D_i^1 is a dominating set in G_1'. Therefore $\{D_1^1, \ldots, D_{d-2}^1, D_{d-1}^1 \cup D_d^1\}$ is a domatic partition of G_1' and $d(G_1') \geqslant d(G) - 1$. Analogously $d(G_2') \geqslant d(G) - 1$. This proves (1).

Next, we shall construct graphs demonstrating (2) and (3).

(2): Let the vertex set of G_1 be $V(G_1) = \{a, u_1^1, \ldots, u_{q+1}^1, v_1^1, \ldots, v_{q+1}^1\}$. The set $V(G_1) - \{a\}$ induces the complete subgraph G_1' with $2q+2$ vertices. The vertex a is adjacent to the vertices $u_1^1, \ldots, u_{q+1}^1$. The graph G_2 is isomorphic to G_1 and has the vertex a in common with it. There exists an isomorphism φ of G_1 onto G_2 such that $\varphi(a) = a$. For $i = 1, \ldots, q+1$ denote $u_i^2 = \varphi(u_i^1)$, $v_i^2 = \varphi(v_i^1)$. The vertex a has degree $q+1$ in G_1, therefore $d(G_1) \leqslant q+2$. Consider the partition of $V(G_1)$ formed by the sets $\{u_i^1\}$ for $i = 1, \ldots, q+1$ and by the set $\{a, v_1^1, \ldots, v_{q+1}^1\}$. This is evidently a domatic partition of G_1 with $q+2$ classes and thus $d(G_1) = q+2$. As $G_2 \cong G_1$, also $d(G_2) = q+2 = \min\{d(G_1), d(G_2)\}$. The vertex v_1^1 has degree $2q+1$ in G and therefore $d(G) \leqslant 2q+2$. Consider the partition of $V(G)$ formed by the set $\{a, u_1^1, v_1^2\}$, the sets $\{u_i^1, v_i^2\}$ for $i = 2, \ldots, q+1$ and the sets $\{u_i^2, v_i^1\}$ for $i = 1, \ldots, q+1$. This is a domatic partition of G with $2q+2$ classes and thus $d(G)=2q+2$. This implies assertion (2).

(3): Let both G'_1, G'_2 be complete graphs with $q+2$ vertices. Let G_1 be obtained from G'_1 by adding the vertex a and joining it by an edge to exactly one vertex of G'_1; analogously let G_2 be constructed. Then $d(G'_1)=d(G'_2)=\min\{d(G'_1),d(G'_2)\}=q+2$. For G we have $d(G)\leqslant 3$, because the vertex a has degree 2. We can easily construct a domatic partition of G with three classes and thus $d(G)=3$. This implies assertion (3) and Theorem 1 is proven. □

We shall now express analogous assertions for the total domatic number.

Theorem 2. *With the above notation, for every graph G without isolated vertices the total domatic numbers satisfy*

(1) $\min\{d_t(G_1),d_t(G_2)\}\leqslant d_t(G)\leqslant 1+\min\{d_t(G'_1),d_t(G'_2)\}$.

(2) *For any positive integer q there exists a graph G such that*

$$d_t(G)=\min\{d_t(G_1),d_t(G_2)\}+q.$$

(3) *For any positive integer q there exists a graph G such that*

$$d_t(G)=\min\{d_t(G'_1),d_t(G'_2)\}-q.$$

Proof. (1): The proof is analogous to the proof of Theorem 1.

(2): The vertex set of G_1 is

$$V(G_1)=\{a,u^1_1,\dots,u^1_{q+2},v^1_1,\dots,v^1_{q+1},w^1_1,\dots,w^1_{q+1},x^1_1,\dots,x^1_{q+1}\}.$$

The set $V(G_1)-\{a\}$ induces a complete bipartite graph G'_1 on the bipartition classes $\{u^1_1,\dots,u^1_{q+2},w^1_1,\dots,w^1_{q+1}\}$, $\{v^1_1,\dots,v^1_{q+1},x^1_1,\dots,x^1_{q+1}\}$. The vertex a is adjacent to the vertices $u^1_1,\dots,u^1_{q+2}$. The graph G_2 is isomorphic to G_1 and has the vertex a in common with it. There exists an isomorphism φ of G_1 onto G_2 such that $\varphi(a)=a$. For $i=1,\dots,q+1$ denote $u^2_i=\varphi(u^1_i), v^2_i=\varphi(v^1_i), w^2_i=\varphi(w^1_i), x^2_i=\varphi(x^1_i)$ and $u^2_{q+2}=\varphi(u^1_{q+2})$. The vertex a has degree $q+2$ in G_1, therefore $d_t(G_1)\leqslant q+2$. Consider the partition of $V(G_1)$ formed by the sets $\{u^1_i,v^1_i\}$ for $i=1,\dots,q+1$ and by the set $\{a,u^1_{q+2},w^1_1,\dots,w^1_{q+1},x^1_1,\dots,x^1_{q+1}\}$. It is evident that this is a total domatic partition of G_1 with $q+2$ classes and thus $d_t(G_1)=q+2$. As $G_2\cong G_1$, also $d_t(G_2)=q+2=\min\{d_t(G_1),d_t(G_2)\}$. The vertex w^1_1 has degree $2q+2$ in G, therefore $d_t(G))\leqslant 2q+2$.

Consider the partition of $V(G)$ formed by the set $\{a,u^1_1,v^1_1,w^2_1,x^2_1,u^1_{q+2},u^2_{q+2}\}$, the sets $\{u^1_i,v^1_i,w^2_1,x^2_1\}$ for $i=2,\dots,q+1$ and the sets $\{u^2_i,v^2_i,w^1_i,x^1_i\}$ for $i=1,\dots,q+1$.

This is a total domatic partition of G with $2q+2$ classes and thus $d_t(G) = 2q+2$. This implies assertion (2).

(3): The proof is analogous to the proof of Theorem 1(3); the graphs G_1', G_2' are complete bipartite graphs in which each bipartition class has $q+2$ vertices. This proves Theorem 2. □

Now we shall consider the case when a graph H is obtained from two disjoint graphs H_1, H_2 by joining a vertex a_1 of H_1 with a vertex a_2 of H_2 by a bridge b. By H_1' we denote the graph obtained from H_1 by deleting a_1, by H_2' the graph obtained from H_2 by deleting a_2 .

Theorem 3. *For the domatic numbers of H, H_1, H_2 the following inequalities hold:*

$$\min\{d(H_1), d(H_2)\} \leqslant d(H) \leqslant 1 + \min\{d(H_1), d(H_2)\}.$$

P r o o f. The proof of the first inequality is analogous to the proof of Theorem 1. We shall prove the second inequality. Let $d(H) = d$ and let $\{D_1, \dots, D_d\}$ be a domatic partition of H with d classes. For $i = 1, \dots, d$ let $D_i^1 = D_i \cap V(H_1), D_i^2 = D_i \cap V(H_2)$. Without loss of generality let $a_1 \in D_1$. Consider the case when $a_2 \in D_1$, too. For $1 \leqslant i \leqslant d$ each vertex x of H_1 not belonging to D_i^1 is adjacent to some vertex y of D_i. If $x \neq a_1$, then x is adjacent to no vertex of H_2 and $y \in D_i^1$. If $x = a_1$ then $i \neq 1$ and x is adjacent to exactly one vertex a_2 of H_2 and $a_2 \in D_1^2$, i.e. $a_2 \notin D_i^2$; the vertex x must be again adjacent to $y \in D_i^1$. The partition $D_1^1, \dots, D_d^1$ is a domatic partition of H_1 and $d(H_1) \geqslant d(H)$. Now let $a_2 \notin D_1$; without loss of generality let $a_2 \in D_d$. Analogously to the preceding case we prove that $D_1^1, \dots, D_{d-1}^1$ are dominating sets in H_1; the set D_d^1 need not be, because a_1 may be adjacent to only one vertex of D_d, namely a_2, and to no vertex of D_d^1. The partition $\{D_1^1, \dots, D_{d-2}^1, D_{d-1}^1 \cup D_d^1\}$ is a domatic partition of H_1 and $d(H_1) \geqslant d(H) - 1$. Analogously $d(H_2) \geqslant d(H) - 1$ and thus the assertion is proved. □

Theorem 4. *For the graphs H, H_1, H_2 in the above notation the equality*

$$d(H) = 1 + \min\{d(H_1), d(H_2)\}$$

holds if and only if the following condition is fulfilled: For each $i \in \{1,2\}$ such that $d(H_i) = \min\{d(H_1), d(H_2)\}$ there exists a partition $\{D_1^i, \dots, D_{d+1}^i\}$ (where $d = d(H_i)$) of the vertex set of H_i such that $D_1^i, \dots, D_d^i$ are dominating sets in H_i and D_{d+1}^i is a dominating set in H_i' but not in H_i.

P r o o f. Suppose that $d(H) = 1 + \min\{d(H_1), d(H_2)\}$. Let i and d have the described meaning. Consider a domatic partition $\{D_1, \dots, D_{d+1}\}$ of H. For each

$j = 1, \dots, d+1$ let $D_j^i = D_j \cap V(H_i)$. Without loss of generality let the end vertex of b not belonging to H_i be in D_{d+1}. Let $1 \leqslant j \leqslant d$. For each vertex $x \in V(H_i) \setminus D_j^i$ there exists a vertex $y \in D_j$ adjacent to it. A vertex of H_i can be adjacent to no vertex outside of H_i except that end vertex of b which belongs to D_{d+1} and thus not to D_j; therefore $y \in D_j^i$ and all the sets $D_1^i, \dots, D_d^i$ are dominating in H_i. For each vertex $x \in V(H_i) \setminus D_{d+1}^i$ there also exists a vertex $y \in D_{d+1}$ adjacent to it. No vertex of H_i' can be adjacent to a vertex outside of H_i and thus $y \in D_{d+1}^i$; the set D_{d+1}^i is dominating in H_i'. It cannot be dominating in H_1, because then the domatic number of H_i would be $d+1$.

Now suppose that the condition is fulfilled. Without loss of generality let $d(H_1) = \min\{d(H_1), d(H_2)\}$. Then in H_1 there exists a partition $\{D_1^1, \dots, D_{d+1}^1\}$ with the described property. Choose the subscripts in such a way that $a_1 \in D_1^1$. If $d(H_2) = d(H_1)$, then such a partition $\{D_1^2, \dots, D_{d+1}^2\}$ by assumption exists also in H_2. If $d(H_2) > d(H_1)$, then there exists a domatic partition $\{D_1^2, \dots, D_{d+1}^2\}$ of H_2. In both cases choose the subscripts in such a way that $a_2 \in D_1^2$. Now define $D_1 = D_1^1 \cup D_{d+1}^2, D_{d+1} = D_{d+1}^1 \cup D_1^2, D_j = D_j^1 \cup D_j^2$ for $j = 2, \dots, d$. Then the partition $\{D_1, \dots, D_{d+1}\}$ is a domatic partition of H and $d(H) = d+1 = 1 + \min\{d(H_1), d(H_2)\}$. □

Theorem 5. *Let for the graphs H, H_1, H_2 in the above notation the equality $d(H) = 1 + d(H_1)$ hold. Then there exists a vertex of H_1 non-adjacent to a_1 with the property that by joining it by an edge to a_1 a graph $\hat{H}_1$ with domatic number $d(\hat{H}_1) = d(H_1) + 1$ is obtained from H_1.*

P r o o f. Consider the partition $\{D_1^1, \dots, D_{d+1}^1\}$ introduced above. Let $u \in D_{d+1}^1$. As D_{d+1}^1 is a dominating set in H_1' but not in H_1, the vertex a_1 is not adjacent to u. If we join a_1 and u by an edge, then a_1 is adjacent to a vertex of D_{d+1}^1 and D_{d+1}^1 is dominating in the resulting graph H_1. Then $\{D_1^1, \dots, D_{d+1}^1\}$ is a domatic partition in $\hat{H}_1$ and $d(\hat{H}_1) = d(H_1) + 1$. (As we have added only one edge, it cannot be greater.) □

Note that the inverse assertion is not true. An example is a circuit C_4 of length 4. Its domatic number is 2, after adding one chord it is 3, but no graph having a circuit C_4 as a terminal block has domatic number greater than 2.

Theorem 6. *For the total domatic numbers of H, H_1, H_2 the following inequalities hold:*

$$\min\{d_t(H_1), d_t(H_2)\} \leqslant d_t(H) \leqslant 1 + \min\{d_t(H_1), d_t(H_2)\}.$$

The p r o o f is analogous to the proof of Theorem 3. □

Before stating the next theorem, we shall express a slight modification of the definition of a total dominating set.

Let G be a graph, and let G_0 be a subgraph of G. We say that a subset D of $V(G)$ is total dominating for G_0, if for each vertex $x \in V(G_0)$ there exists a vertex $y \in D$ adjacent to x.

Note that in this definition we do not suppose that $D \subseteq V(G_0)$ but only $D \subseteq V(G)$.

Theorem 7. *If for the graphs H, H_1, H_2 in the above notation the equality*

$$d_t(H) = 1 + \min\{d_t(H_1), d_t(H_2)\}$$

holds, then for each $i \in \{1, 2\}$ such that $d_t(H_i) = \min\{d_t(H_1), d_t(H_2)\}$ there exists a partition $\{D_1^i, \ldots, D_{d+1}^i\}$ (where $d = d_t(H_i)$) of the vertex set of H_i such that $D_1^i, \ldots, D_d^i$ are total dominating sets in H_i and D_{d+1}^i is a total dominating set for H_i' but not for H_i.

The proof is analogous to the first part of the proof of Theorem 4.

Note that Theorem 7 differs from Theorem 4 by the fact that it is only an implication, not an equivalence. Before investigating the inverse assertion, we introduce some notation.

If a graph H_i with a vertex a_i has the property that $d_t(H_i) = d$ and there exists a partition as described in Theorem 7, we say that the pair (H_i, a_i) is in the class $\kappa(d)$. If $(H_i, a_i) \in \kappa(d)$ and the described partition has the property that $a_i \in D_{d+1}^i$ (or $a_i \notin D_{d+1}^i$), we write $(H_i, a_i) \in \kappa_1(d)$ (or $(H_i, a_i) \in \kappa_0(d)$, respectively). Obviously $\kappa_0(d) \cup \kappa_1(d) = \kappa(d)$, note that $\kappa_0(d) \cap \kappa_1(d) \neq \emptyset$ may occur.

Theorem 8. *Let H, H_1, H_2 be graphs in the above notation. The equality*

$$d_t(H) = 1 + \min\{d_t(H_1), d_t(H_2)\}$$

holds if and only if at least one of the following three cases occurs:

(i) *exactly one of the pairs (H_1, a_1), (H_2, a_2) is in $\kappa(d)$ and the graph from the other pair has total domatic number greater than d;*

(ii) *both the pairs (H_1, a_1), (H_2, a_2) are in $\kappa_0(d)$;*

(iii) *both the pairs (H_1, a_1), (H_2, a_2) are in $\kappa_1(d)$.*

Proof. Suppose that the above mentioned equality holds, say $d_t(H_1) \leqslant d_t(H_2)$ and $d_t(H) = 1 + d_t(H_1)$. Then by Theorem 7 $(H_1, a_1) \in \kappa(d)$. With the same notation as in Theorem 7 we let $\mathcal{D} = \{D_1, \ldots, D_{d+1}\}$ be a total domatic partition of H and let $D_j^1 = D_j \cap V(H_1)$, $D_j^2 = D_j \cap V(H_2)$ for $j = 1, \ldots, d+1$. The notation is chosen such that D_{d+1}^1 is a total dominating set for H_1' but not for H_1. Then a_1 is adjacent to no vertex of D^1_{d+1} and necessarily $a_2 \in D_{d+1}$. Hence if $a_1 \in D^1_{d+1}$, then a_1, a_2 belong to the same class of $\mathcal{D}$; otherwise they belong to different classes.

If also $(H_2, a_2) \in \kappa(d)$, then one of the classes $D^2_1, \dots, D^2_{d+1}$ is total dominating for H'_2 but not for H_2; let this class be D^2_k for some k, $1 \leqslant k \leqslant d+1$. Then a_1 must be in D_k. If $a_1 \in D^1_{d+1}$, then $k = d+1$ and both (H_1, a_1), (H_2, a_2) are in $\kappa_1(d)$. If $a_1 \notin D^1_{d+1}$, then $k \neq d+1$ and both (H_1, a_1), (H_2, a_2) are in $\kappa_0(d)$. If $(H_2, a_2) \notin \kappa(d)$ and hence by Theorem 7 $d_t(H_2) > d$ then (i) is satisfied. We have proved that one of the cases (i), (ii), (iii) occurs.

Conversely, assume that $(H_1, a_1) \in \kappa_0(d)$. Construct the described partition $\{D^1_1, \dots, D^1_{d+1}\}$ such that $a_1 \notin D^1_{d+1}$; choose the notation so that $a_1 \in D^1_1$. If $d_t(H_2) > d$, choose a total domatic partition $\{D^2_1, \dots, D^2_{d+1}\}$ of H_2; choose the notation so that $a_2 \in D^2_{d+1}$. If we define $D_j = D^1_j \cup D^2_j$ for $j = 1, \dots, d+1$, then $\{D_1, \dots, D_{d+1}\}$ is a total domatic partition of H and $d_t(H) = d+1$. If $(H_2, a_2) \in \kappa_0(d)$, then construct the described partition $\{D^2_1, \dots, D^2_{d+1}\}$ for H_2 such that $a_2 \in D^2_1$. If we put $D_1 = D^1_1 \cup D^2_{d+1}$, $D_{d+1} = D^1_{d+1} \cup D^2_1$, $D_j = D^1_j \cup D^2_j$ for $j = 2, \dots, d$, then $\{D_1, \dots, D_{d+1}\}$ is a total domatic partition of H and $d_t(H) = d+1$.

Suppose $(H_1, a_1) \in \kappa_1(d)$. Construct the described partition $\{D^1_1, \dots D^1_{d+1}\}$ such that $a_1 \in D^1_{d+1}$; if $d_t(H_2) > d$, choose a total domatic partition $\{D^2_1, \dots, D^2_{d+1}\}$ of H_2; again choose the notation so that $a_2 \in D^2_{d+1}$. If we define $D_j = D^1_j \cup D^2_j$ for $j = 1, \dots, d+1$, then $\{D_1, \dots, D_{d+1}\}$ is a total domatic partition of H and $d_t(H) = d+1$. If $(H_2, a_2) \in \kappa_1(d)$, then construct the described partition $\{D^2_1, \dots, D^2_{d+1}\}$ for H_2 such that $a_2 \in D^2_{d+1}$. Now we define again $D_j = D^1_j \cup D^2_j$, and $\{D_1, \dots, D_{d+1}\}$ is a total domatic partition of H and $d_t(H) = d+1$. This proves Theorem 8. □

A vertex x of the graph G is called saturated, if it is adjacent to all other vertices of G.

Theorem 9. *Let for the graphs H, H_1, H_2 in the above notation the equality $d_t(H) = 1 + d_t(H_1)$ hold. If a_1 is not saturated in H_1, then there exists a vertex of H_1 non-adjacent to a_1 with the property that by joining it by an edge to a_1 a graph $\hat{H}_1$ with total domatic number $d_t(\hat{H}_1) = d_t(H_1) + 1$ is obtained from H_1.*

The proof is analogous to the proof of Theorem 5. If a_1 is saturated in H_1, then the unique subset of $V(H_1)$ which is total dominating for H'_1 but not for H_1 can be only the set $\{a_1\}$ and thus $D^1_{d+1} = \{a_1\}$ and $D^1_{d+1} \cap V(H'_1) = \emptyset$.

At the end of the paper we shall prove a theorem on circuits. Let C_n be the circuit of length n. Its vertices will be denoted by $u_1, \dots, u_n$ so that the edges of C_n are (u_i, u_{i+1}) for $i = 1, \dots, n-1$ and (u_n, u_1). It is known (cf. [2]) that $d_t(C_n) = 2$ if and only if $n \equiv 0 \pmod 4$; otherwise $d_t(C_n) = 1$. □

In the following theorem the circuit C_n will be considered as a graph H_1 or H_2 in the notation introduced above; in this sense we shall write the pair (C_n, a) and the classes $\kappa(1)$, $\kappa_0(1)$ and $\kappa_1(1)$.

Theorem 10. *Let C_n be a circuit of length $n \not\equiv 0 \pmod 4$, let a be an arbitrary vertex of C_n. Then*
(1) $(C_n, a) \in \kappa_1(1) \setminus \kappa_0(1)$ *for* $n \equiv 3 \pmod 4$;
(2) $(C_n, a) \in \kappa_0(1) \setminus \kappa_1(1)$ *for* $n \equiv 1 \pmod 4$;
(3) $(C_n, a) \notin \kappa(1)$ *for* $n \equiv 2 \pmod 4$.

Proof. Without loss of generality put $a = u_n$. Suppose that $(C_n, a) \in \kappa(1)$. Then there exists a partition $\{D_1, D_2\}$ of $V(C_n)$ such that D_1 is a total dominating set in C_n and D_2 is total dominating for the path obtained from C_n by deletion of u_n, but not for C_n. None of the vertices adjacent to u_n belongs to D_2, therefore $u_1 \in D_1$, $u_{n-1} \in D_1$. Suppose that $(C_n, a) \in \kappa_0(1)$, i.e. $u_n \in D_1$. Each vertex of C_n distinct from u_n must be adjacent to a vertex of D_1 and to a vertex of D_2. As $u_n \in D_1$, $u_1 \in D_1$, we have $u_i \in D_2$ for $i \equiv 2 \pmod 4$ or $i \equiv 3 \pmod 4$ and $u_i \in D_1$ for $i \equiv 0 \pmod 4$ or $i \equiv 1 \pmod 4$; in all cases $i \neq n$. But as was mentioned above, $u_{n-1} \in D_1$. This is possible only if $n - 1 \equiv 0 \pmod 4$ or $n - 1 \equiv 1 \pmod 4$, i.e., if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$. If $n \equiv 2 \pmod 4$, then also $u_{n-2} \in D_1$ and u_{n-1} is adjacent to two vertices u_{n-2} and u_n of D_1; this is a contradiction. Therefore $(C_n, a) \in \kappa_0(1)$ implies $n \equiv 1 \pmod 4$, and conversely for $n \equiv 1 \pmod 4$ the described partition exists so that $(C_n, a) \in \kappa_0(1)$.

Next, assume that $(C_n, a) \in \kappa_1(1)$, i.e. $u_n \in D_2$. Then $u_i \in D_1$ for $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$ and $u_i \in D_2$ for $i \equiv 0 \pmod 4$ or $i \equiv 3 \pmod 4$ again for all $i \neq n$. We have $u_{n-1} \in D_1$ and thus $n - 1 \equiv 1 \pmod 4$ or $n - 1 \equiv 2 \pmod 4$, i.e. $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$. If $n \equiv 2 \pmod 4$, then $u_{n-2} \in D_2$ and u_{n-1} is adjacent to two vertices u_{n-2} and u_n of D_2; this is a contradiction. Therefore $(c_n, a) \in \kappa_1(1)$ implies that $n \equiv 3 \pmod 4$, and conversely for $n \equiv 3 \pmod 4$ the described partition exists and $(C_n, a) \in \kappa_1(1)$. We have proved that $(C_n, a) \in \kappa_0(1)$ if and only if $n \equiv 1 \pmod 4$ and $(C_n, a) \in \kappa_1(1)$ if and only if $n \equiv 3 \pmod 4$. This proves Theorem 10. ☐

We are now able to illustrate Theorem 8 by Figures 1–5 below.

In Fig. 1 we see a graph H with $H_1 \cong H_2 \cong C_5$, in Fig. 2 with $H_1 \cong H_2 \cong C_7$. The set D_1 (or D_2) is the set of all vertices labelled by 1 (or 2, respectively). From Theorems 8 and 10 we see that $d_t(H) = 2$ in both cases. In Fig. 3 there is a graph H and $H_1 \cong C_5$, $H_2 \cong C_7$; its total domatic number is 1. Figures 4 and 5 demonstrate (ii) and (iii) in Theorem 8 for a graph H with $(H_1, a_1) \in \kappa_0(1) \cap \kappa_1(1)$. Here H_1 is a C_4, one vertex of which is joined to a new vertex, a_1.

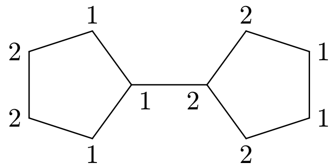

Fig. 1

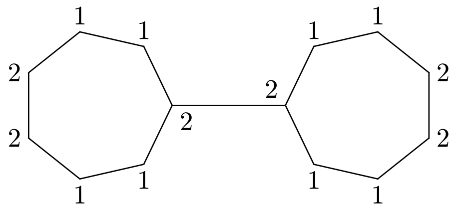

Fig. 2

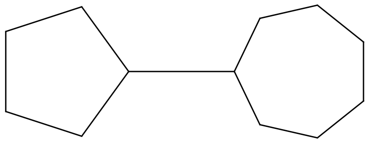

Fig. 3

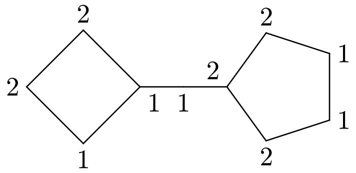

Fig. 4

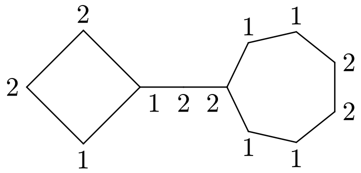

Fig. 5

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