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A REFINEMENT OF THE RADIAL POHOZAEV IDENTITY

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Abstract. In this article we produce a refined version of the classical Pohozaev identity in the radial setting. The refined identity is then compared to the original, and possible applications are discussed.

Keywords: Green's function, positive solutions, supercritical nonlinearity

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1. Introduction

The Pohozaev identity has been used, among other things, to prove nonexistence of positive solutions for supercritical elliptic equations in star-shaped domains. More precisely, let $N \geqslant 3$ and let $\Omega \subset \mathbb{R}^N$ be a star-shaped domain. It has been proved in 1965 by S. I. Pohozaev [22] that problems

$$-\Delta u = u^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.1}$$

have no positive solutions for $p \geqslant (N+2)/(N-2)$. For $1 < p < (N+2)/(N-2)$ the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact and existence of positive solutions can be proved by variational methods. For $p = (N+2)/(N-2)$ (i.e. $p+1 = 2^* = 2N/(N-2)$, the critical Sobolev exponent) the embedding is not compact any more, and for $p > (N+2)/(N-2)$ the space $H_0^1(\Omega)$ is not a subspace of $L^{p+1}(\Omega)$ and the variational methods used to prove existence of solutions break down.

An interesting fact was observed by Brezis and Nirenberg in [3] relative to the problem

$$-\Delta u = u^{2^*-1} + \lambda u, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.2}$$

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which differs from (1.1) by the linear term λu. The authors proved that (1.2) admits a solution for positive values λ contained in an interval whose endpoints depend on the dimension N in a somewhat unexpected way. This paper gave rise to numerous studies on problems with critical nonlinearities. It has been noticed that for some dimensions N the branch of solutions which bifurcates from the trivial solution exists for all λ between λ_1 (the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions) and zero, while for other "critical" dimensions this branch is bounded away from zero by some $\lambda_* > 0$ (see [2], [4], [5], [6], [9], [10], [13], [14], [15], [16], [20], [23], and the references therein).

For $N = 3$, in the case when $\Omega = B_1$ is the unit ball in $\mathbb{R}^3$, Brezis and Nirenberg were able to find the exact value of λ_*. For this, the authors devised a refinement of the Pohozaev identity. Later, it was noticed that even the refined identity was not sufficient to produce exact values for λ_* in some problems very closely related to (1.2). In [8], the author and R. Lavine were able to prove a resolution of the refined Pohozaev identity in [3], which gave the exact value for λ_* in the case of radial solutions for closely related problems. As it turns out, it is a difficult matter to describe the value λ_* in domains other than the ball (see for example [6] and [12] and the references therein).

The purpose of this article is to discuss the resolution of the refined Pohozaev identity in the radial case, and to discuss some of the conclusions that can be drawn from it. Consider problems

$$-\Delta u + a(|x|)u = f(u), \quad u > 0 \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1 \tag{1.3}$$

where B_1 is the unit ball in $\mathbb{R}^N$ with $N \geqslant 3$ and a is assumed to be a continuous function so that the operator $-\Delta + a(|x|)$ is coercive, i.e. there is $\lambda_1 > 0$ such that

$$\int_{B_1} |\nabla u|^2 + a(|x|)u^2 \,\mathrm{d}x \geqslant \lambda_1 \int_{B_1} u^2 \,\mathrm{d}x$$

for all smooth functions u with compact support in B_1. About the nonlinearity f we assume that it is continuous, nonnegative for all $u \geqslant 0$, satisfies $f(0) = 0$ and $f'(0) = 0$ and has the antiderivative $F(u) = \int_0^u f(t)\,\mathrm{d}t$. A model nonlinearity is $f(u) = u^p$ with $p > 1$. We are interested in the radial solutions of (1.3), that is, solutions of the ODE

$$-u_{rr} - \frac{N-1}{r}\, u_r + a(r)u = f(u), \quad u > 0 \text{ in } (0,1), \quad u_r(0) = u(1) = 0 \tag{1.4}$$

where $r = |x|$. We assume that a is such that the homogeneous linear ODE

$$-\xi_{rr} - \frac{N-1}{r}\, \xi_r + a(r)\xi = 0 \tag{1.5}$$

admits a pair of linearly independent solutions ξ and ζ on the interval $(0,1)$.

For comparison, we illustrate first the original identity of Pohozaev in the radial case and $a \equiv 0$.

Theorem 2.1. *Any solution of* (1.4) *with* $a \equiv 0$ *satisfies*

$$\int_0^1 r^{N-1}\Big(F(u) - \frac{N-2}{2N} f(u)u\Big)\,\mathrm{d}r > 0.$$

P r o o f. The equation (1.4) assumes the form

$$-u_{rr} - \frac{N-1}{r}\,u_r = f(u) \text{ in } (0,1), \quad u_r(0) = u(1) = 0. \tag{2.1}$$

Multiplying (2.1) by $r^N u_r$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(-r^N\frac{u_r^2}{2}\Big) + \frac{N}{2}r^{N-1}u_r^2 - (N-1)r^{N-1}u_r^2 = r^N f(u)u_r$$

or

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(-r^N\frac{u_r^2}{2}\Big) - \frac{N-2}{2}r^{N-1}u_r^2 = \frac{\mathrm{d}}{\mathrm{d}r}(r^N F(u)) - Nr^{N-1}F(u).$$

After dividing by N and rearranging the terms we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{r^N}{2N}u_r^2 + \frac{r^N}{N}F(u)\Big) = r^{N-1}F(u) - \frac{N-2}{2N}r^{N-1}u_r^2. \tag{2.2}$$

On the other hand, multiplying (2.1) by $r^{N-1}u$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}(-r^{N-1}u_r u) + u_r\frac{\mathrm{d}}{\mathrm{d}r}(r^{N-1}u) - (N-1)r^{N-2}u_r u = r^{N-1}f(u)u,$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{N-2}{2N}r^{N-1}u_r u\Big) = \frac{N-2}{2N}r^{N-1}u_r^2 - \frac{N-2}{2N}r^{N-1}f(u)u. \tag{2.3}$$

Adding (2.2) and (2.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{r^N}{2N}u_r^2 + \frac{N-2}{2N}r^{N-1}u_r u + \frac{r^N}{N}F(u)\Big) = r^{N-1}\Big(F(u) - \frac{N-2}{2N}f(u)u\Big).$$

Since $F(0) = 0$, integration on the interval $(0,1)$ yields

$$\frac{1}{2N}u_r^2(1) = \int_0^1 r^{N-1}\Big(F(u) - \frac{N-2}{2N}f(u)u\Big)\,\mathrm{d}r.$$

Since the left hand side is strictly positive by the Uniqueness Theorem for ODE's, the theorem follows. $\square$

The argument for nonexistence of solutions in the supercritical case is based on the inequality

$$F(u) - \frac{N-2}{2N} f(u)u \leqslant 0. \tag{2.4}$$

Indeed, for $f(u) = u^p$ we have $F(u) = (p+1)^{-1}u^{p+1}$, hence

$$F(u) - \frac{N-2}{2N} f(u)u = \Big(\frac{1}{p+1} - \frac{N-2}{2N}\Big)u^{p+1}.$$

While for $1 < p < (N+2)/(N-2)$ the paranthesis above is positive, for $p \geqslant (N+2)/(N-2)$ we have that

$$\int_0^1 r^{N-1}\Big(F(u) - \frac{N-2}{2N} f(u)u\Big)\, \mathrm{d}r$$

in Theorem 2.1 is non-positive. This provides the contradiction to the existence of solutions of (2.1) in the supercritical case. The identity (2.1) becomes a little more complicated in the presence of the linear term (i.e. when $a(r)$ is not identically zero) and the refinement in the Brezis-Nirenberg approach consists in introducing two auxiliary functions.

Note that after multiplication by r^{N-1} the problem (1.4) can be written as

$$-\left(r^{N-1}u_r\right)_r + r^{N-1}a(r)u = r^{N-1}f(u) \text{ in } (0,1), \quad u_r(0) = u(1) = 0. \tag{2.5}$$

Let ξ and ζ be linearly independent solutions of the homogeneous equation

$$-\left(r^{N-1}\xi_r\right)_r + r^{N-1}a(r)\xi = 0 \quad \text{such that} \quad \xi_r(0) = \zeta(1) = 0. \tag{2.6}$$

Observe that for $a \equiv 0$, a pair of such solution is

$$\xi_0(r) \equiv 1, \quad \zeta_0(r) = \frac{r^{-(N-2)} - 1}{N-2}. \tag{2.7}$$

Define the Wronskian of two functions to be

$$W[u,\xi](r) = r^{N-1}\left(u(r)\xi_r(r) - u_r(r)\xi(r)\right),$$

and note that

$$W[\xi_0, \zeta_0](r) \equiv -1.$$

We will assume that $a(r)$ is such that the problem (2.6) has a pair ξ, ζ of linearly independent positive solutions on the interval $(0,1)$ such that

$$\xi_r(0) = \zeta(1) = 0 \text{ and } W[\xi,\zeta](r) \equiv -1 \text{ and the limits} \\ \lim_{r\to 0^+} r^{N-2}\zeta(r) \text{ and } \lim_{r\to 0^+} r^{N-1}\zeta_r(r) \text{ exist and are finite.} \tag{2.8}$$

For functions a that will guarantee (2.8), see Lemma 1 in [7].

The next theorem is the refinement of the Pohozaev identity.

Theorem 2.2. *With ξ and ζ as in* (2.8), *any solution u of* (2.5) *satisfies the identity*

$$\int_0^1 r^{2(N-1)}\xi\zeta(f(u)u + 2F(u))\mathcal{T}(r)\,\mathrm{d}r = 0 \tag{2.9}$$

where

$$\mathcal{T}(r) = \frac{\xi_r}{\xi} + \frac{\zeta_r}{\zeta} + \frac{4(N-1)}{r}\frac{F(u)}{f(u)u+2F(u)}. \tag{2.10}$$

P r o o f. By multiplying the equation (2.6) by $-u$ and the equation (2.5) by ξ and adding, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}W[u,\xi](r) = r^{N-1}f(u)\xi.$$

Similarly we have

$$\frac{\mathrm{d}}{\mathrm{d}r}W[u,\zeta](r) = r^{N-1}f(u)\zeta.$$

By combining the two equalities above it follows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(W[u,\xi]W[u,\zeta]\right) = r^{N-1}f(u)\left(\xi W[u,\zeta] + \zeta W[u,\xi]\right).$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(W[u,\xi]W[u,\zeta]\right) &= r^{2(N-1)}f(u)\left(u(\xi\zeta)_r - 2u_r\xi\zeta\right)\\ &= r^{2(N-1)}f(u)u(\xi\zeta)_r - 2r^{2(N-1)}f(u)u_r\xi\zeta\\ &= r^{2(N-1)}f(u)u(\xi\zeta)_r - 2r^{2(N-1)}\frac{\mathrm{d}}{\mathrm{d}r}(F(u))\xi\zeta\\ &= \frac{\mathrm{d}}{\mathrm{d}r}(-2r^{2(N-1)}F(u)\xi\zeta) + r^{2(N-1)}(f(u)u+2F(u))(\xi\zeta)_r\\ &\quad + 4(N-1)r^{2N-3}F(u)\xi\zeta.\end{aligned}$$

We now obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dr}}(W[u,\xi]W[u,\zeta] + 2r^{2(N-1)}F(u)\xi\zeta) \\
&\qquad = r^{2(N-1)}(f(u)u + 2F(u))(\xi\zeta)_r + 4(N-1)r^{2N-3}F(u)\xi\zeta
\end{aligned}
$$

or

$$
\frac{\mathrm{d}}{\mathrm{dr}}(W[u,\xi]W[u,\zeta] + 2r^{2(N-1)}F(u)\xi\zeta) = r^{2(N-1)}\xi\zeta(f(u)u + 2F(u))\mathcal{T}(r). \tag{2.11}
$$

From the boundary conditions and from (2.8) we have that

$$
\lim_{r\to 0^+} W[u,\xi](r)W[u,\zeta](r) = W[u,\xi](1)W[u,\zeta](1) = 0
$$

and

$$
\lim_{r\to 0^+} r^{2(N-1)}F(u)\xi\zeta = r^{2(N-1)}F(u)\xi\zeta\Big|_{r=1} = 0.
$$

Integrating (2.11) on the interval $(0,1)$ we get

$$
\int_0^1 r^{2(N-1)}\xi\zeta(f(u)u + 2F(u))\mathcal{T}(r)\,\mathrm{d}r = 0.
$$

□

Our technique to prove nonexistence of solutions is to provide ξ and ζ as in (2.8) for which $\mathcal{T}(r) < 0$ for all $r \in (0,1)$, and this in turn will provide a contradiction to the identity in Theorem 2.2.

In the case $a \equiv 0$ we substitute ξ_0 and ζ_0 in (2.10) to obtain

$$
\mathcal{T}_0(r) = -\frac{N-2}{r - r^{N-1}} + \frac{4(N-1)}{r}\frac{F(u)}{f(u)u + 2F(u)}.
$$

Direct algebraic calculations show that $\mathcal{T}_0(r) < 0$ for $r \in (0,1)$ is equivalent to

$$
F(u) - \frac{N-2}{2N}f(u)u - \frac{2(N-1)}{N}r^{N-2}F(u) < 0. \tag{2.12}
$$

Condition (2.12) relaxes somewhat the inequality (2.4) but apparently it is not easy to extract significantly new information (to produce functions $f(u)$ that do not satisfy (2.4) but satisfy (2.12)) from this relaxation.

There are two other directions in which Theorem 2.9 may provide new information. One is for power type nonlinearities and a not identically zero (see [8]), and the other may be a combination of $f(u)$ not necessarily a power function and a not identically zero. A promising advantage of using the refined identity is that it involves the functions ξ and ζ which are basic ingredients in the construction of Green's function. Indeed, it is well known that Green's function of the operator

$$-u_{rr} - \frac{N-1}{r}u_r + a(r)u \quad \text{with boundary conditions} \quad u_r(0) = u(1) = 0$$

is given by

$$G(r,s) = s^{N-1}\xi\left(\min(r,s)\right)\zeta\left(\max(r,s)\right) \qquad \text{for } r,s \in (0,1),$$

where ξ and ζ are functions satisfying (2.8). It has been noticed in previous works (see [18], [19], [12], [6], [24], [25]) that the regular part of Green's function plays an important role for the existence of solutions. In our setting, the regular part of Green's function is

$$H(r,s) = G(r,s) - \Gamma(r,s), \quad \text{where} \quad \Gamma(r,s) = \begin{cases} 0 & \text{if } r \leqslant s, \\ \dfrac{s^{N-1}}{N-2}(r^{2-N} - s^{2-N}) & \text{if } r > s, \end{cases}$$

and on the diagonal $r = s$ it satisfies

$$H(r,r) = G(r,r) = r^{N-1}\xi(r)\zeta(r).$$

We can therefore hope that the nonexistence approach using the refined identity will narrow and eventually eliminate the gap between existence and nonexistence of solutions for problem (1.4).

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