MINIMAL ACYCLIC DOMINATING SETS AND CUT-VERTICES

Vladmir Samodivkin, Sofia

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Abstract. The paper studies minimal acyclic dominating sets, acyclic domination number and upper acyclic domination number in graphs having cut-vertices.

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For the graph theory terminology not presented here, we follow Haynes et al. [3]. All our graphs are finite and undirected with no loops or multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For any vertex v of G its *open neighborhood* $N(v,G)$ is $\{x \in V(G);\ vx \in E(G)\}$ and its *closed neighborhood* $N[v,G]$ is $N(v,G) \cup \{v\}$. For a set $S \subseteq V(G)$ its *open neighborhood* $N(S,G)$ is $\bigcup\limits_{v\in S} N(v,G)$, its *closed neighborhood* $N[S,G]$ is $N(S,G) \cup S$. A subset of vertices A in a graph G is said to be *acyclic* if $\langle A, G\rangle$ contains no cycles. Note that the property of being acyclic is a hereditary property, that is, any subset of an acyclic set is itself acyclic. A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either in D or is adjacent to an element of D. A dominating set D is a *minimal dominating set* if no proper subset $D' \subset D$ is a dominating set. The set of all minimal dominating sets of a graph G is denoted by $\mathrm{MDS}(G)$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [4], [5].

A given graph invariant can often be combined with another graph theoretical property P. Harary and Haynes [3] defined the *conditional domination number* $\gamma(G : P)$ as the smallest cardinality of a dominating set $S \subseteq V(G)$ such that the subgraph $\langle S, G\rangle$ induced by S has property P. One of the many possible properties imposed on S is:

P_{ad}: $\langle S, G\rangle$ has no cycles.

The conditional domination number $\gamma(G : P_{\mathrm{ad}})$ is called the *acyclic domination number* and is denoted by $\gamma_{\mathrm{a}}(G)$. The concept of acyclic domination in graphs was introduced by Hedetniemi et al. [6]. An acyclic dominating set D is a *minimal acyclic dominating set* if no proper subset $D' \subset D$ is an acyclic dominating set. The *upper acyclic domination number* $\Gamma_{\mathrm{a}}(G)$ is the maximum cardinality of a minimal acyclic dominating set of G. The set of all minimal acyclic dominating sets of a graph G is denoted by $\mathrm{MD_aS}(G)$. For every vertex x of a graph G let $\mathrm{MD_aS}(x, G) = \{D \in \mathrm{MD_aS}(G);\ x \in D\}$.

Let us introduce the following assumption

$(*)$ a graph H is the union of two connected graphs H_1 and H_2 having exactly one common vertex x and $|V(H_i)| \geqslant 2$ for $i = 1, 2$.

In this paper we deal with minimal acyclic dominating sets, acyclic domination number and upper acyclic domination number in graphs having cut-vertices. Observe that domination and some of its variations in graphs having cut-vertices has been the topic of several studies—see for example [1, 7, 5 Chapter 16].

1. Minimal acyclic dominating sets

In this section we begin an investigation of minimal acyclic dominating sets in graphs having cut-vertices.

The following lemma will be used in the sequel, without specific reference.

Lemma A [5, Lemma 2.1]. *For any graph G, $\mathrm{MD_aS}(G) \subseteq \mathrm{MDS}(G)$.*

Theorem 1.1. *Let H_1, H_2 and H be graphs satisfying $(*)$. Let $M \in \mathrm{MD_aS}(x, H)$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. Then one of the following holds:*
(i) *$M_j \in \mathrm{MD_aS}(x, H_j)$ for $j = 1, 2$;*
(ii) *there are l and m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MD_aS}(x, H_l)$, and $M_m - \{x\}$ is the unique subset of M_m which belongs to $\mathrm{MD_aS}(H_m)$.*

Proof. Since $x \in M$ then M_j is an acyclic dominating set of H_j, $j = 1, 2$. Let there be $i \in \{1, 2\}$ such that $M_i \notin \mathrm{MD_aS}(x, H)$. Suppose $M_j \notin \mathrm{MD_aS}(x, H_j)$ for $j = 1, 2$. Then there is a vertex $u_1 \in M_1$ and a vertex $u_2 \in M_2$ such that $M_j - \{u_j\}$ is an acyclic dominating set of H_j, $j = 1, 2$. Hence $(M_1 - \{u_1\}) \cup (M_2 - \{u_2\}) = M - (\{u_1\} \cup \{u_2\})$ is an acyclic dominating set of H—a contradiction. So, without loss of generality let $M_1 \notin \mathrm{MD_aS}(x, H_1)$ and $M_2 \in \mathrm{MD_aS}(x, H_2)$. Hence there is a vertex $u \in M_1$ such that $M_1 - \{u\}$ is an acyclic dominating set of H_1. If $u \neq x$ then $M - \{u\}$ is an acyclic dominating set of H, which is a contradiction. Hence $u = x$ and $M_1 - \{x\}$ is an acyclic dominating set of H_1. Suppose $M_1 - \{x\} \notin \mathrm{MD_aS}(H_1)$. Then there is a vertex $w \in M_1 - \{x\}$ such that $M_1 - \{x, w\}$ is an acyclic dominating set of H_1. But then $M - \{w\}$ is an acyclic dominating set of H—a contradiction. Therefore $M_1 - \{x\} \in \mathrm{MD_aS}(H_1)$. Let $v \in M_1 - \{x\}$. Suppose $M_1 - \{v\}$ is an acyclic dominating set of H_1. Then $M - \{v\}$ is an acyclic dominating set of H—a contradiction. □

Theorem 1.2. *Let H_1, H_2 and H be graphs satisfying $(*)$. Let $M \in \mathrm{MD_aS}(H)$, $x \notin M$ and $M_j = M \cap V(H_j)$, $j = 1, 2$. Then one of the following holds:*
(i) *$M_j \in \mathrm{MD_aS}(H_j)$ for $j = 1, 2$;*
(ii) *there are l and m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MD_aS}(H_l)$, $M_m \in \mathrm{MD_aS}(H_m - x)$ and M_m is no dominating set in H_m.*

P r o o f. Clearly, there is $i \in \{1, 2\}$ such that M_i is an acyclic dominating set of H_i. Without loss of generality let $i = 1$. Suppose $M_1 \notin \mathrm{MD_aS}(H_1)$. Then there is $u \in M_1$ such that $M_1 - \{u\}$ is an acyclic dominating set of H_1 and then $M - \{u\}$ is an acyclic dominating set of G—a contradiction. So $M_1 \in \mathrm{MD_aS}(H_1)$. Analogously, if M_2 is an acyclic dominating set of H_2, then $M_2 \in \mathrm{MD_aS}(G_2)$. Now, let M_2 be not an acyclic dominating set of H_2. Then M_2 is an acyclic dominating set of $H_2 - x$. Suppose $M_2 \notin \mathrm{MD_aS}(H_2 - x)$. Then there is $v \in M_2$ such that $M_2 - \{v\}$ is an acyclic dominating set of $H_2 - x$ and hence $M - \{v\}$ is an acyclic dominating set of H—a contradiction. □

Theorem 1.3. *Let H_1, H_2 and H be graphs satisfying $(*)$. Let $M_j \in \mathrm{MD_aS}(H_j)$ for $j = 1, 2$ and $x \notin M_1 \cup M_2$. Then one of the following holds:*
(i) *$M_1 \cup M_2 \in \mathrm{MD_aS}(H)$;*
(ii) *there are $l \in \{1, 2\}$ and $u \in V(H_l)$ such that $\{u\} = N(x, H_l) \cap M_l$, $M_l - \{u\} \in \mathrm{MD_aS}(H_l - x)$ and $(M_1 \cup M_2) - \{u\} \in \mathrm{MD_aS}(H)$.*

P r o o f. Let $M = M_1 \cup M_2$. Then M is an acyclic dominating set of H. Suppose $M \notin \mathrm{MD_aS}(H)$. Hence, there is a vertex $u \in M$ such that $M - \{u\}$ is an acyclic dominating set of H. Without loss of generality let $u \in V(H_1)$. Then $M_1 - \{u\}$ is no acyclic dominating set of H_1 and hence $M_1 - \{u\}$ is an acyclic dominating set of $H_1 - x$. Therefore $\{u\} = N(x, H_1) \cap M_1$. Suppose $M_1 - \{u\} \notin \mathrm{MD_aS}(H_1 - x)$. Then there is a vertex $v \in M_1 - \{u\}$ such that $M_1 - \{u, v\}$ is an acyclic dominating set of $H_1 - x$. Hence $M_1 - \{v\}$ is an acyclic dominating set of H_1—a contradiction. So $M_1 - \{u\} \in \mathrm{MD_aS}(H_1 - x)$. Suppose $M - \{u\} \notin \mathrm{MD_aS}(H)$. Hence there is a vertex w, $w \in M - \{u\}$ that $M - \{u, w\}$ is an acyclic dominating set of H. If $w \in V(H_1)$, then $M_1 - \{u, w\}$ is an acyclic dominating set of $H_1 - x$—a contradiction. Therefore $w \in V(H_2)$ and then $M_2 - \{w\}$ is an acyclic dominating set of H_2—a contradiction. So $M - \{u\} \in \mathrm{MD_aS}(H)$. □

Theorem 1.4. *Let H_1, H_2 and H be graphs satisfying $(*)$. Let $M_j \in \mathrm{MD_aS}(x, H_j)$ for $j = 1, 2$. Then $M_1 \cup M_2 \in \mathrm{MD_aS}(x, H)$.*

Proof. Let $M = M_1 \cup M_2$. Obviously M is an acyclic dominating set of H. Suppose $M \notin \mathrm{MD_aS}(H)$. Then there is a vertex $u \in M$ such that $M - \{u\}$ is an acyclic dominating set of H. First, let $u \neq x$ and without loss of generality let $u \in V(H_1) - \{x\}$. Then $M_1 - \{u\}$ is an acyclic dominating set of H_1—a contradiction. Secondly, let $u = x$. Now, there is $i \in \{1, 2\}$ such that $M_i - \{x\}$ is an acyclic dominating set of H_i, which is a contradiction. So $M \in \mathrm{MD_aS}(H)$ and since $x \in M$ we have $M \in \mathrm{MD_aS}(x, H)$. □

Theorem 1.5. *Let H_1, H_2 and H be graphs satisfying $(*)$. Let $M_1 \in \mathrm{MD_aS}(x, H_1)$, $M_2 \in \mathrm{MD_aS}(H_2)$, $x \notin M_2$ and $M = M_1 \cup M_2$. Then one of the following holds:*

(i) *$M \in \mathrm{MD_aS}(H)$;*
(ii) *$M_1 - \{x\} \in \mathrm{MD_aS}(H_1 - x)$ and $M - \{x\} \in \mathrm{MD_aS}(H)$;*
(iii) *there is $U \subseteq M_2$ such that $(M_2 - U) \cup \{x\} \in \mathrm{MD_aS}(H_2)$ and $M - U \in \mathrm{MD_aS}(H)$;*
(iv) *no subset of M is an acyclic dominating set of H.*

Proof. Let $M \notin \mathrm{MD_aS}(H)$ and let there exist $M_3 \subset M$ such that $M_3 \in \mathrm{MD_aS}(H)$. First, let $x \notin M_3$. Then $M_1 - \{x\}$ is an acyclic dominating set of $H_1 - x$. Suppose $M_1 - \{x\} \notin \mathrm{MD_aS}(H_1 - x)$. Now, there is a vertex $v \in M_1 - \{x\}$ that $M_1 - \{x, v\}$ is an acyclic dominating set of $H_1 - x$. Hence $M_1 - \{v\}$ is an acyclic dominating set of H_1—a contradiction. So, $M_1 - \{x\} \in \mathrm{MD_aS}(H_1 - x)$ and $M - \{x\}$ is an acyclic dominating set of H. Now, suppose $M - \{x\} \notin \mathrm{MD_aS}(H)$. Then there is a vertex $w \in M - \{x\}$ such that $M - \{x, w\}$ is an acyclic dominating set of H. If $w \in V(H_1)$ then $M_1 - \{x, w\}$ is an acyclic dominating set of $H_1 - x$—a contradiction. If $w \in V(H_2)$, then $M_2 - \{w\}$ is an acyclic dominating set of H_2—a contradiction. So $M - \{x\} \in \mathrm{MD_aS}(H)$. Secondly, let $x \in M_3$. Let $U = M - M_3$. If there is $u \in U \cap M_1$, then $M_1 - \{u\}$ is an acyclic dominating set of H_1—a contradiction. Hence, $U \subseteq M_2$. Then $(M_2 - U) \cup \{x\} = M_3 \cap V(H_2)$ is an acyclic dominating set of H_2. Since M is no minimal acyclic dominating set of H we have $U \neq \emptyset$ and hence $M_2 - U$ is no dominating set of H_2. If there is $v \in M_2 - U$ such that $(M_2 - (U \cup \{v\}) \cup \{x\}$ is an acyclic dominating set of H_2 then $M_3 - \{v\}$ is an acyclic dominating set of H—a contradiction. Hence $(M_2 - U) \cup \{x\}$ is a minimal acyclic dominating set of H_2. □

2. Γ_a-sets and γ_a-sets

In this section we present some results concerning the acyclic domination number and the upper acyclic domination number of graphs having cut-vertices.

Let $\mu(G)$ be a numerical invariant of a graph G defined in such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ with a given property P. A set with the property P and with $\mu(G)$ vertices in G is called a μ-set of G. Fricke et al. [2] define a vertex v of a graph G to be

(i) *μ-good*, if v belongs to some μ-set of G and

(ii) *μ-bad*, if v belongs to no μ-set of G.

Theorem 2.1. *Let H_1, H_2 and H be graphs satisfying $(*)$.*

1. *Let x be a Γ_a-good vertex of a graph H. Then $\Gamma_a(H) \leqslant \Gamma_a(H_1) + \Gamma_a(H_2)$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2)$, M is a Γ_a-set of H and $x \in M$, then there are l and m such that $\{l, m\} = \{1, 2\}$, $M \cap V(H_l)$ is a Γ_a-set of H_l and $M \cap V(H_m) - \{x\}$ is a Γ_a-set of H_m.*

2. *Let x be a Γ_a-good vertex of graphs H_1 and H_2. Then $\Gamma_a(H_1) + \Gamma_a(H_2) - 1 \leqslant \Gamma_a(H)$. If $\Gamma_a(H_1) + \Gamma_a(H_2) - 1 = \Gamma_a(H)$, M_j is a Γ_a-set of H_j, $j = 1, 2$ and $\{x\} = M_1 \cap M_2$ then $M_1 \cup M_2$ is a Γ_a-set of H.*

3. *Let x be a Γ_a-bad vertex of a H_1 and H_2. Then $\Gamma_a(H) \geqslant \Gamma_a(H_1) + \Gamma_a(H_2) - 1$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$ and M_j is a Γ_a-set of H_j, $j = 1, 2$ then there are $l \in \{1, 2\}$ and $u \in V(H_l)$ such that $\{u\} = N(x, H_l) \cap M_l$ and $M_1 \cup M_2 - \{u\}$ is a Γ_a-set of H.*

4. *Let x be a Γ_a-bad vertex of H. Then $\Gamma_a(H) \leqslant \max\{\Gamma_a(H_1) + \Gamma_a(H_2), \Gamma_a(H_1 - x) + \Gamma_a(H_2), \Gamma_a(H_1) + \Gamma_a(H_2 - x)\}$.*

Proof. 1. Let M be a Γ_a-set of H, $x \in M$ and $M \cap V(H_j) = M_j$, $j = 1, 2$.

Case $M_j \in \mathrm{MD_aS}(x, H_j), j = 1, 2$: Then $\Gamma_a(H) = |M| = |M_1| + |M_2| - 1 \leqslant \Gamma_a(H_1) + \Gamma_a(H_2) - 1$.

Case there are l, m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MD_aS}(x, H_l)$ and $M_m - \{x\} \in \mathrm{MD_aS}(H_m)$: We have $\Gamma_a(H) = |M| = |M_l| + |M_m - \{x\}| \leqslant \Gamma_a(H_l) + \Gamma_a(H_m)$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2)$, then $|M_l| = \Gamma_a(H_l)$ and $|M_m - \{x\}| = \Gamma_a(H_m)$. Hence M_l is a Γ_a-set of H_l and $M_m - \{x\}$ is a Γ_a-set of H_m.

There are no other possibilities because of Theorem 1.1.

2. Let M_j be a Γ_a-set of H_j, $j = 1, 2$ and $\{x\} = M_1 \cap M_2$. It follows from Theorem 1.4 that $M_1 \cup M_2 \in \mathrm{MD_aS}(x, H)$. Hence $\Gamma_a(H) \geqslant |M_1 \cup M_2| = |M_1| + |M_2| - 1 = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$ then $|M_1 \cup M_2| = \Gamma_a(H)$. Hence $M_1 \cup M_2$ is a Γ_a-set of H.

3. Let M_j be a Γ_a-set of H_j, $j = 1, 2$ and $M = M_1 \cup M_2$. If $M \in \mathrm{MD_aS}(H)$ then $\Gamma_a(H) \geqslant |M| = |M_1| + |M_2| = \Gamma_a(H_1) + \Gamma_a(H_2)$. Otherwise it follows from

Theorem 1.3 that there are $l \in \{1,2\}$ and $u \in V(H_l)$ such that $\{u\} = N(x, H_l) \cap M_l$ and $M - \{u\} \in \mathrm{MD_aS}(H)$. Hence $\Gamma_a(H) \geqslant |M - \{u\}| = |M_1| + |M_2| - 1 = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$. If $\Gamma_a(H) = \Gamma_a(H_1) + \Gamma_a(H_2) - 1$ then $|M - \{u\}| = \Gamma_a(H)$. Hence $M - \{u\}$ is a Γ_a-set of H.

4. Let M be a Γ_a-set of H and $M_j = M \cap V(H_j)$, $j = 1,2$. If $M_j \in \mathrm{MD_aS}(H_j)$, $j = 1,2$ then $\Gamma_a(H) = |M| = |M_1| + |M_2| \leqslant \Gamma_a(H_1) + \Gamma_a(H_2)$. Otherwise it follows from Theorem 1.2 that $M_l \in \mathrm{MD_aS}(H_l)$ and $M_m \in \mathrm{MD_aS}(H_m - x)$ for some l, m such that $\{l,m\} = \{1,2\}$. Hence $\Gamma_a(H) = |M| = |M_l| + |M_m| \leqslant \Gamma_a(H_l) + \Gamma_a(H_m - x)$. □

Theorem 2.2. *Let G be a graph of order at least two. Then for each vertex $v \in V(G)$ we have $\gamma_a(G) - 1 \leqslant \gamma_a(G - v) \leqslant |V(G)| - 1$. If $v \in V(G)$ and $\gamma_a(G) - 1 = \gamma_a(G - v)$ then*

(i) *v is a γ_a-good vertex of the graph G;*

(ii) *if v is not isolated and $u \in N(v, G)$ then u is a γ_a-bad vertex of the graph $G - v$.*

P r o o f. Clearly $\gamma_a(G - v) \leqslant |V(G - v)| = |V(G)| - 1$. Assume $\gamma_a(G - v) < \gamma_a(G)$. Then for an arbitrary γ_a-set M of the graph $G - v$ we have $N[M, G] = V(G) - \{v\}$ and then $N(v, G) \cap M = \emptyset$. Hence $M \cup \{v\}$ is an acyclic dominating set of G and then $\gamma_a(G) \leqslant |M \cup \{v\}| = |M| + 1 = \gamma_a(G - v) + 1 \leqslant \gamma_a(G)$. Therefore $\gamma_a(G) - 1 = \gamma_a(G - v)$ and $M \cup \{v\}$ is a γ_a-set of G. Hence v is a γ_a-good vertex of G. Since $N(v, G) \cap M = \emptyset$ we conclude that each vertex belonging to $N(v, G)$ is a γ_a-bad vertex of $G - v$. □

Theorem 2.3. *Let H_1, H_2 and H be graphs satisfying $(*)$. Then*

1. *$\gamma_a(H) \geqslant \gamma_a(H_1) + \gamma_a(H_2) - 1$.*

2. *Let x be a γ_a-bad vertex of the graph H, $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1$ and let M be a γ_a-set of H. Then there are l, m such that $\{l,m\} = \{1,2\}$, $M \cap V(H_l)$ is a γ_a-set of H_l, $M \cap V(H_m)$ is a γ_a-set of $H_m - x$ and $\gamma_a(H_m - x) = \gamma_a(H_m) - 1$.*

3. *Let x be a γ_a-good vertex of H, $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1$, let M be a γ_a-set of H and $x \in M$. Then $M \cap V(H_j)$ is a γ_a-set of H_j, $j = 1,2$.*

4. *Let x be a γ_a-good vertex of graphs H_1 and H_2. Then $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2) - 1$. If M_j is a γ_a-set of H_j, $j = 1,2$ and $\{x\} = M_1 \cap M_2$ then $M_1 \cup M_2$ is a γ_a-set of the graph H.*

5. *Let x be a γ_a-bad vertex of graphs H_1 and H_2. Then $\gamma_a(H) = \gamma_a(H_1) + \gamma_a(H_2)$. If M_j is a γ_a-set of H_j, $j = 1,2$ then $M_1 \cup M_2$ is a γ_a-set of H.*

P r o o f. 1: Let M be a γ_a-set of H and $M_i = M \cap V(H_i)$, $i = 1,2$.

C a s e $x \notin M$: If $M_j \in \mathrm{MD_aS}(H_j)$ for $j = 1,2$ then $\gamma_a(H) = |M| = |M_1| + |M_2| \geqslant \gamma_a(H_1) + \gamma_a(H_2)$. Otherwise it follows by Theorem 1.2 that there are l, m such that $\{l,m\} = \{1,2\}$, $M_l \in \mathrm{MD_aS}(H_l)$ and $M_m \in \mathrm{MD_aS}(H_m - x)$. Hence

$\gamma_{\rm a}(H) = |M| = |M_l| + |M_m| \geqslant \gamma_{\rm a}(H_l) + \gamma_{\rm a}(H_m - x)$. Now, Theorem 2.2 yields $\gamma_{\rm a}(H) \geqslant \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) - 1$.

Case $x \in M$ and $M_j \in \mathrm{MD_aS}(H_j)$, $j = 1, 2$: It follows that $\gamma_{\rm a}(H) = |M| = |M_1| + |M_2| - 1 \geqslant \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) - 1$.

Case $x \in M$ and there are l, m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MD_aS}(H_l)$ and $M_m - \{x\} \in \mathrm{MD_aS}(H_m)$: We have $\gamma_{\rm a}(H) = |M| = |M_l| + |M_m - \{x\}| \geqslant \gamma_{\rm a}(H_l) + \gamma_{\rm a}(H_m)$.

There are no other possibilities because of Theorem 1.1.

2: Let $M \cap V(H_i) = M_i$, $i = 1, 2$. From the proof of 1 we have that there are l, m such that $\{l, m\} = \{1, 2\}$, $M_l \in \mathrm{MD_aS}(H_l)$, $M_m \in \mathrm{MD_aS}(H_m - x)$, $|M_l| = \gamma_{\rm a}(H_l)$ and $|M_m| = \gamma_{\rm a}(H_m - x) = \gamma_{\rm a}(H_m) - 1$. Hence the result follows.

3: It follows from the proof of 1 that $M \cap V(H_i) \in \mathrm{MD_aS}(H_i)$ and $|M \cap V(H_i)| = \gamma_{\rm a}(H_i)$ for $i = 1, 2$. Hence $M \cap V(H_i)$ is a $\gamma_{\rm a}$-set of H_i, $i = 1, 2$.

4: Let M_j be a $\gamma_{\rm a}$-set of H_j, $j = 1, 2$ and $\{x\} = M_1 \cap M_2$. It follows from Theorem 1.4 that $M_1 \cup M_2 \in \mathrm{MD_aS}(H)$. Hence $\gamma_{\rm a}(H) \leqslant |M_1 \cup M_2| = |M_1| + |M_2| - 1 = \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) - 1$. Now from 1 we have that $\gamma_{\rm a}(H) = \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) - 1$. Then $|M_1 \cup M_2| = \gamma_{\rm a}(H)$. Therefore $M_1 \cup M_2$ is a $\gamma_{\rm a}$-set of H.

5: Suppose $\gamma_{\rm a}(H) = \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) - 1$. If x is a $\gamma_{\rm a}$-bad vertex of H then by 2 there exists $m \in \{1, 2\}$ such that $\gamma_{\rm a}(H_m - x) = \gamma_{\rm a}(H_m) - 1$. Hence by Theorem 2.2 x is a $\gamma_{\rm a}$-good vertex of H_m—a contradiction. If x is a $\gamma_{\rm a}$-good vertex of H, M is a $\gamma_{\rm a}$-set of H and $x \in M$ then by 3 we have $M \cap V(H_s)$ is a $\gamma_{\rm a}$-set of H_s, $s = 1, 2$. But then x is a $\gamma_{\rm a}$-good vertex of H_s, $s = 1, 2$, which is a contradiction.

Hence $\gamma_{\rm a}(H) \geqslant \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2)$.

Let M_j be a $\gamma_{\rm a}$-set of H_j, $j = 1, 2$ and $M = M_1 \cup M_2$.

Case there are $l \in \{1, 2\}$ and $u \in V(H_l)$ such that $\{u\} = N(x, H_l) \cap M_l$, $M_l - \{u\} \in \mathrm{MD_aS}(H_l - x)$ and $M - \{u\} \in \mathrm{MD_aS}(H)$: Let $\{m\} = \{1, 2\} - \{l\}$. Hence $\gamma_{\rm a}(H) \leqslant |M - \{u\}| = |M_l - \{u\}| + |M_m| = |M_l| - 1 + |M_m| = \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) - 1$, which is a contradiction.

Case $M \in \mathrm{MD_aS}(H)$: Then $\gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2) \leqslant \gamma_{\rm a}(H) \leqslant |M| = |M_1| + |M_2| = \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2)$. Hence $\gamma_{\rm a}(H) = \gamma_{\rm a}(H_1) + \gamma_{\rm a}(H_2)$ and then $|M| = \gamma_{\rm a}(H)$. Therefore M is a $\gamma_{\rm a}$-set of H.

The result now follows because of Theorem 1.3. □

Remark 2.4. In [1] Brigham, Chinn and Dutton obtained that, in the above notation, $\gamma(H_1) + \gamma(H_2) \geqslant \gamma(H) \geqslant \gamma(H_1) + \gamma(H_2) - 1$.

Observe that if m is a positive integer then there exists a graph H (in the above notation) such that $m = \gamma_{\rm a}(H) - \gamma_{\rm a}(H_1) - \gamma_{\rm a}(H_2)$. Indeed, let n and p be integers, $m + 1 \leqslant n \leqslant p$, $V(H) = \{x, y, z; a_1, \ldots, a_{m+1}; b_1, \ldots, b_n; c_1, \ldots, c_p\}$, $E(H) = \{xy, xz, yz; xa_1, \ldots, xa_{m+1}; yb_1, \ldots, yb_n; zc_1, \ldots, zc_p\}$, $H_1 = \langle \{x; a_1, \ldots, a_{m+1}\}, H \rangle$

and $H_2 = \langle\{x, y, z; b_1, \ldots, b_n; c_1, \ldots, c_p\}, H\rangle$. Then $\gamma_{\rm a}(H) = 3 + m$, $\gamma_{\rm a}(H_1) = 1$ and $\gamma_{\rm a}(H_2) = 2$. Hence $m = \gamma_{\rm a}(H) - \gamma_{\rm a}(H_1) - \gamma_{\rm a}(H_2)$.

Theorem 2.5. *Let G be a connected graph with blocks $G_1, G_2, \ldots, G_n$. Then $\gamma_{\rm a}(G) \geqslant \sum\limits_{i=1}^{n} \gamma_{\rm a}(G_i) - n + 1$.*

P r o o f. We proceed by induction on the number of blocks n. The statement is immediate if $n = 1$. Let the blocks of G be $G_1, G_2, \ldots, G_n, G_{n+1}$ and without loss of generality let G_{n+1} contain only one cut-vertex of G. Hence Theorem 2.3 implies that $\gamma_{\rm a}(G) \geqslant \gamma_{\rm a}(G_{n+1}) + \gamma_{\rm a}(Q) - 1$ where $Q = \Big\langle \bigcup\limits_{i=1}^{n} V(G_i), G \Big\rangle$. The result now follows from the inductive hypothesis. □

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Author's address: *Vladimir Samodivkin*, Department of Mathematics, University of Architecture, Civil Engineering and Geodesy, Hristo Smirnenski 1 Blv., 1046 Sofia, Bulgaria, e-mail: `vlsam_fte@uacg.bg`.