ALL GRAPHS IN WHICH EACH PAIR OF DISTINCT VERTICES HAS EXACTLY TWO COMMON NEIGHBORS

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Abstract. We find all connected graphs in which any two distinct vertices have exactly two common neighbors, thus solving a problem by B. Zelinka.

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We consider finite undirected graphs without loops and multiple edges. The symbol $V(G)$ denotes the vertex set of a graph G, while the symbol $A(G)$ denotes the adjacency matrix of G. If $u \in V(G)$, then by $N(u)$ we denote the set of vertices of G adjacent to u and $\overline{N}(u) = V(G) \setminus \bigl(N(u) \cup \{u\}\bigr)$. If $M \subseteq V(G)$, then by $\langle M \rangle$ we denote the subgraph of G induced by the set M. For other undefined notions, see, for example, [2].

According to [3], at the Czechoslovak conference on graph theory at Zemplínska Šírava in June 1991, P. Hliněný proposed the problem to describe all connected graphs G with the property that for any two distinct vertices of G there exist exactly two vertices which are adjacent to both of them in G. He conjectured that there are only two such graphs, whose adjacency lists are shown in Fig. 1a and Fig. 1b. In [3], B. Zelinka disproved this conjecture by giving another graph with this property, whose adjacency list is shown in Fig. 1c. Here we shall show that these three graphs are the only connected graphs with this property.

The following results are proved in [3].

Proposition 1. *Let G be a graph in which any two distinct vertices have exactly two common neighbours. Then for each $u \in V(G)$ the graph $\langle N(u) \rangle$ is regular of degree 2.*

u:	a b c
a:	u b c
b:	u a c
c:	u a b

Fig. 1a

u:	a b c d e f
a:	u b c g h i
b:	u a c j k l
c:	u a b m n o
d:	u e f g j m
e:	u d f h k n
f:	u d e i l o
g:	a d h i j m
h:	a e g i k n
i:	a f g i l o
j:	b d k l g m
k:	b e j l h n
l:	b f j k i o
m:	c d n o g j
n:	c e m o h k
o:	c f m n i l

Fig. 1b

u:	a b c d e f
a:	u b f g h i
b:	u a c g j k
c:	u b d j l m
d:	u c e h l n
e:	u d f k n o
f:	u a e i m o
g:	a b h k l o
h:	a d g i l n
i:	a f h m j n
j:	b c k m i n
k:	b e g j n o
l:	c d m h g o
m:	c f j l i o
n:	d e h k i j
o:	e f k m g l

Fig. 1c

Figure 1. All connected graphs in which each pair of vertices has exactly two common neighbors.

Proposition 2. *Let G be a graph in which any two distinct vertices have exactly two common neighbours. Then no graph $\langle N(u)\rangle$ for $u \in V(G)$ contains a circuit C_4.*

Theorem 1. *Let G be a connected graph in which any two distinct vertices have exactly two common neighbours. Let G contain a vertex u of degree $r \geqslant 5$. Then G is regular of degree r and its number of vertices is $n = \frac{1}{2}(r^2 - r + 2)$.*

Let G be a connected graph in which any two distinct vertices have exactly two common neighbors. First, suppose that the largest vertex degree of G is less than 5. From Proposition 1 it follows that every vertex of G has degree at least 3, while from Proposition 2 it follows that no vertex of G has degree 4. Therefore, G is a cubic graph. If u is an arbitrary vertex of G, the graph $\langle N(u)\rangle \cong C_3$, and each vertex of $N(u)$ is adjacent to u and other two vertices of $N(u)$. Thus, the component of G containing u is isomorphic to K_4, and since G is connected, we conclude that G is isomorphic to K_4, whose adjacency list is shown in Fig. 1a.

Next, suppose that G is a connected regular graph of degree $r \geqslant 5$ with $n = \frac{1}{2}(r^2 - r + 2)$ vertices, by Theorem 1. It is well-known that the (i,j)-element of the matrix $A(G)^2$ represents the number of walks of length 2 between vertices i and j

of G. Then

$$(A(G)^2)_{i,j} = \begin{cases} r, & i = j, \\ 2, & i \neq j. \end{cases}$$

If I denotes the identity matrix and J denotes the all-one matrix of corresponding dimensions, we can write

$$A(G)^2 = (r-2)I + 2J.$$

The matrix $(r-2)I + 2J$ has a simple eigenvalue $r - 2 + 2n = r^2$ and a multiple eigenvalue $r-2$ of multiplicity $n-1$. Thus, the adjacency matrix $A(G)$ has a simple eigenvalue r, since G is regular of degree r, an eigenvalue $\sqrt{r-2}$ of multiplicity k and an eigenvalue $-\sqrt{r-2}$ of multiplicity l, $k + l = n - 1$. The sum of eigenvalues of $A(G)$ is equal to zero (see, e.g., [1]), and from $r + (k-l)\sqrt{r-2} = 0$ we get that

$$l - k = \frac{r}{\sqrt{r-2}} \in \mathbb{N}.$$

It follows that $\sqrt{r-2}$ must be a rational number, and thus an integer, so that $r = a^2 + 2$ for some $a \in \mathbb{N}$. Now

$$\frac{r}{\sqrt{r-2}} = a + \frac{2}{a} \in \mathbb{N}$$

and thus $a \in \{1, 2\}$, or equivalently, $r \in \{3, 6\}$. Since we supposed that $r \geqslant 5$, it follows that G is a regular graph of degree $r = 6$ with $n = 16$ vertices.

Thus, if u is an arbitrary vertex of G, the graph $\langle N(u) \rangle$ is either isomorphic to $2C_3$ or to C_6. Suppose that for some vertex u of G it holds that $\langle N(u) \rangle \cong 2C_3$. Let $\{a, b, c\}$ be one of the two circuits in $N(u)$. Then $\{u, b, c\} \subseteq N(a)$ and since u, b and c form C_3, we conclude that it must hold that $\langle N(a) \rangle \cong 2C_3$. Continuing in this manner, from the connectivity of G it follows that $\langle N(v) \rangle \cong 2C_3$ for every vertex v of G. Let $\{d, e, f\}$ form the other circuit in $N(u)$. In the set $\{u, a, b, c, d, e, f\}$ vertices from the same circuit of $\langle N(u) \rangle$ have two common neighbours—the vertex u and the third vertex of the circuit, while vertices from different circuits of $\langle N(u) \rangle$ have just one common neighbour—the vertex u. Therefore, for each pair $\{s, t\}$ of vertices from different circuits there exists exactly one vertex in $\overline{N}(u)$ adjacent to both s and t. Denoting by $cn(s, t)$ the common neighbour of s and t in $\overline{N}(u)$, we may suppose that

$$\begin{gathered} g = cn(a, d),\ h = cn(a, e),\ i = cn(a, f), \\ j = cn(b, d),\ k = cn(b, e),\ l = cn(b, f), \\ m = cn(c, d),\ n = cn(c, e),\ o = cn(c, f). \end{gathered}$$

Thus, $N(a) = \{u, b, c, g, h, i\}$ and since u, b and c form C_3, it follows that g, h and i must form another C_3. Similarly, considering $N(b)$, $N(c)$, $N(d)$, $N(e)$ and $N(f)$ we get that circuits of length 3 are formed by the following sets of vertices:

$$\{j, k, l\}, \{m, n, o\}, \{g, j, m\}, \{h, k, n\}, \{i, l, o\}.$$

It is easy to check that in the graph constructed in this way, whose adjacency list is shown in Fig. 1b, each pair of vertices has exactly two common neighbours. Since it is regular of degree 6, it must be isomorphic to G.

Next, suppose that $\langle N(u)\rangle \cong C_6$ for every vertex u of G. As before, let $N(u) = \{a, b, c, d, e, f\}$ and let a, b, c, d, e, f in that order form C_6. In the set $\{u, a, b, c, d, e, f\}$ vertices from C_6 at distance two have two common neighbours—the vertex u and a vertex from C_6 between them, while other pairs of vertices from C_6 have one common neighbour—the vertex u. Therefore, for each pair $\{s, t\}$ of vertices from C_6, that are not at distance two, there exists a vertex in $\overline{N}(u)$ adjacent to both s and t. We may suppose that

$$\begin{aligned}
&g = cn(a, b),\ j = cn(b, c),\ l = cn(c, d),\\
&n = cn(d, e),\ o = cn(e, f),\ i = cn(f, a),\\
&h = cn(a, d),\ k = cn(b, e),\ m = cn(c, f).
\end{aligned}$$

Thus, $N(a) = \{u, b, f, g, h, i\}$ and $\langle N(a)\rangle$ contains the edges $\{g, b\}$, $\{b, u\}$, $\{u, f\}$ and $\{f, i\}$. In order that $\langle N(a)\rangle \cong C_6$, $\langle N(a)\rangle$ must also contain the edges $\{g, h\}$ and $\{h, i\}$. Similarly, considering $N(b)$, $N(c)$, $N(d)$, $N(e)$ and $N(f)$ we get that the following pairs of vertices must be adjacent:

$$\{g, k\}, \{k, j\}, \{j, m\}, \{m, l\}, \{l, h\}, \{h, n\}, \{n, k\}, \{k, o\}, \{i, m\}, \{m, o\}.$$

In a graph constructed this far, the vertices h, k and m have degree 6, while the remaining vertices of $\overline{N}(u)$ have degree 4. Now, we have that $\{a, b, k, h\} \subset N(g)$ and $\langle N(g)\rangle$ contains edges $\{h, a\}$, $\{a, b\}$ and $\{b, k\}$. The vertex i cannot belong to $N(g)$, as there is an edge $\{a, i\}$ and then $\langle N(g)\rangle$ will not be regular of degree 2. For the same reason, the existence of the edge $\{b, j\}$ implies that the vertex j cannot belong to $N(g)$. Also, the vertex n cannot belong to $N(g)$, as there are edges $\{h, n\}$ and $\{k, n\}$ and then $\langle N(g)\rangle$ will contain C_5, which is impossible. Therefore, $N(g)$ contains the vertices l and o, which are adjacent. Similarly, we can get that $N(i)$ contains the vertices j and n, which are adjacent. It is easy to check that in the graph constructed in this way, whose adjacency list is shown in Fig. 1c, each pair of vertices has exactly two common neighbours. Since it is regular of degree 6, it must be isomorphic to G.

Thus, we have proved the following

Theorem 2. *There exist exactly three connected graphs, whose adjacency lists are shown in Fig.* 1, *in which each pair of distinct vertices has exactly two common neighbours.*

References

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