

LOCALLY SPECTRALLY BOUNDED LINEAR MAPS

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Abstract. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} . We characterize locally spectrally bounded linear maps from $\mathcal{L}(\mathcal{H})$ onto itself. As a consequence, we describe linear maps from $\mathcal{L}(\mathcal{H})$ onto itself that compress the local spectrum.

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1. INTRODUCTION

Throughout this paper, X will denote a complex Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators on X with identity operator I . The local resolvent set of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$, $\varrho_T(x)$, is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U_λ of λ in \mathbb{C} and an analytic function $f: U_\lambda \rightarrow X$ such that the equation $(\mu - T)f(\mu) = x$ holds for all $\mu \in U_\lambda$. The local spectrum of T at x , denoted by $\sigma_T(x)$, is given by

$$\sigma_T(x) := \mathbb{C} \setminus \varrho_T(x),$$

and is a compact subset of $\sigma(T)$. The local spectral radius of T at x is defined by

$$r_T(x) := \limsup_{n \rightarrow +\infty} \|T^n x\|^{1/n},$$

and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that T is said to have the single-valued extension property if for every open set U of \mathbb{C} , the equation

$$(T - \lambda)\varphi(\lambda) = 0 \quad (\lambda \in U),$$

has no nontrivial analytic solution on U . Evidently, every operator $T \in \mathcal{L}(X)$ with empty interior point spectrum enjoys this property. Our references on local spectral theory are the remarkable books of P. Aiena [1] and of K. Laursen and M. Neumann [11].

We will say that a linear map $\varphi: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ is locally spectrally bounded at a fixed nonzero vector $e \in X$ if there is a positive constant M such that $r_{\varphi(T)}(e) \leq Mr_T(e)$ for all $T \in \mathcal{L}(X)$. When X is an infinite dimensional Hilbert space, we prove the following.

Theorem 1.1. *Let \mathcal{H} be an infinite dimensional Hilbert space and let e be a fixed nonzero vector in \mathcal{H} . A continuous surjective linear map $\varphi: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is locally spectrally bounded at e if and only if there are a nonzero scalar c and an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $Ae = e$, and $\varphi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$.*

This theorem is an extension of the result due to Bračić and Müller [7, Theorems 3.3 and 3.4], where they characterized continuous surjective linear maps from $\mathcal{L}(X)$ into itself that preserve the local spectrum (local spectral radius) at a fixed vector in X .

The following results show, unlike in the infinite dimensional case, that the additional assumption of continuity on φ can be omitted, and extend the main results from [4], [6], [9] to this more general scope.

Theorem 1.2. *Let $n \geq 3$ be a positive integer and let $e \in \mathbb{C}^n$ be a fixed nonzero vector. Let $\varphi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a surjective linear map. If φ is locally spectrally bounded at e , then there exist a nonzero scalar c and matrices $A, S \in M_n(\mathbb{C})$ with A invertible and $Ae = Se = e$ such that either $\varphi(T) = cATA^{-1} + c(S - I)\text{tr}(T/n)$ or $\varphi(T) = cAT^{\text{tr}}A^{-1} + c(S - I)\text{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$. Here $\text{tr}(\cdot)$ denotes the usual trace function on $M_n(\mathbb{C})$ and T^{tr} is the transpose of the matrix T .*

Corollary 1.3. *Let $n \geq 3$ be a positive integer and let $e \in \mathbb{C}^n$ be a fixed nonzero vector. Let $\varphi: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be a surjective linear map such that $\varphi(I)$ and I are linearly dependent. Then the map φ is locally spectrally bounded at e if, and only if, there are a nonzero scalar c and an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ae = e$ and $\varphi(T) = cATA^{-1}$ for all $T \in M_n(\mathbb{C})$.*

2. PROOF OF THE MAIN RESULTS

We first fix some notation. The duality between the Banach spaces X and its dual X^* will be denoted by $\langle \cdot, \cdot \rangle$. For $x \in X$ and $f \in X^*$, as usual we denote by $x \otimes f$ the rank at most one operator on X given by $z \mapsto \langle z, f \rangle x$. For $T \in \mathcal{L}(X)$ we will denote by $\ker(T)$, T^* , $\sigma(T)$, $\sigma_{\text{su}}(T)$, and $r(T)$, the null space, the adjoint, the spectrum, the surjectivity spectrum, and the spectral radius of T ; respectively.

The proof of our results uses several auxiliary lemmas. The first is quoted in [3, lemma 2.1]. It concerns spectrally bounded linear maps from a purely infinite C^* -algebra with real rank zero onto a semi-simple Banach algebra. For our purposes, the only relevant example of an algebra having these properties is the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on an infinite-dimensional Hilbert space \mathcal{H} .

Recall that a linear map φ between unital Banach algebras \mathcal{A} and \mathcal{B} is called spectrally bounded if there is a positive constant M such that

$$r(\varphi(a)) \leq Mr(a) \quad (a \in \mathcal{A}),$$

where $r(\cdot)$ denotes the spectral radius function.

Lemma 2.1. *Let \mathcal{H} be an infinite dimensional Hilbert space and let $\varphi: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ be a surjective spectrally bounded linear map. Then there exist a nonzero scalar λ and an epimorphism or an anti-epimorphism $J: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ such that $\varphi(T) = \lambda J(T)$ for all $T \in \mathcal{L}(\mathcal{H})$.*

A few comments must be added to this statement. In [3, lemma 2.1], λ can be any central unitary element; however, since the centre of $\mathcal{L}(\mathcal{H})$ is trivial, λ must be a complex number in our setting. Further, the conclusion of [3, lemma 2.1] is that J is a Jordan epimorphism; since the algebra $\mathcal{C}(\mathcal{H})$ is prime, a well known theorem of Herstein [10] tells us that J must be an epimorphism or an anti-epimorphism in our setting.

The following lemma characterizes spectrally bounded linear maps from $M_n(\mathbb{C})$ onto itself.

Lemma 2.2. *A surjective linear map φ from $M_n(\mathbb{C})$ into itself is spectrally bounded if, and only if, there exist a nonzero scalar $c \in \mathbb{C}$ and an automorphism or an anti-automorphism φ on $M_n(\mathbb{C})$ such that $\varphi(T) = c\varphi(T) + (\varphi(I) - cI) \text{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$.*

Proof. This is a consequence of [13, Remark 4]. □

Lemma 2.3. *Let e be a fixed nonzero vector in X , and let $T \in \mathcal{L}(X)$. If $\lambda \in \sigma_{\text{su}}(T)$, then for every $\varepsilon > 0$, there exists $T' \in \mathcal{L}(X)$ such that $\|T - T'\| < \varepsilon$ and $\lambda \in \sigma_{T'}(e)$.*

Proof. See [7, Lemma 2.2]. □

The following result is inspired by [7].

Lemma 2.4. *Let e be a fixed nonzero vector in X . If $\varphi: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ is a continuous surjective locally spectrally bounded linear map at e , then φ is spectrally bounded.*

Proof. Suppose that φ is locally spectrally bounded at e . Without loss of generality, we can assume that $r_{\varphi(T)}(e) \leq r_T(e)$ for every $T \in \mathcal{L}(X)$, and let us show that $r(\varphi(T)) \leq r(T)$ for all $T \in \mathcal{L}(X)$. To this end, let $T \in \mathcal{L}(X)$ and let $\lambda \in \sigma(\varphi(T))$ satisfy $|\lambda| = r(\varphi(T))$, which means $\lambda \in \sigma_{\text{su}}(\varphi(T))$. By Lemma 2.3, for each integer $n \geq 1$ there exists an operator T'_n in $\mathcal{L}(X)$ such that $\|T'_n - \varphi(T)\| < n^{-1}$ and $\lambda \in \sigma_{T'_n}(e)$. Since φ is continuous and surjective, by the Banach open mapping theorem there exists $\eta > 0$ such that $\eta B(0, 1) \subseteq \varphi(B(0, 1))$, where $B(0, 1)$ denotes the open unit ball of $\mathcal{L}(X)$. Therefore, for each n there exists $T_n \in \mathcal{L}(X)$ such that $\varphi(T_n) = T'_n$ and $\|T_n - T\| \leq \eta^{-1} \|T'_n - \varphi(T)\| \leq \eta^{-1} n^{-1}$. Thus $T_n \rightarrow T$ and $\lambda \in \sigma_{\varphi(T_n)}(e)$ for all $n \geq 1$. So, by the upper semi-continuity of the spectral radius function, we have

$$\begin{aligned} r(T) &\geq \limsup_{n \rightarrow \infty} r(T_n) \\ &\geq \limsup_{n \rightarrow \infty} r_{T_n}(e) \\ &\geq \limsup_{n \rightarrow \infty} r_{\varphi(T_n)}(e) \\ &\geq |\lambda| = r(\varphi(T)). \end{aligned}$$

Hence φ is spectrally bounded from $\mathcal{L}(X)$ onto itself. □

The next lemma is simple, and its proof is straightforward. We include it for the sake of completeness.

Lemma 2.5. *Assume that X is a complex Banach space of dimension at least two, and let $e \in X$ be a nonzero vector of X . If $A \in \mathcal{L}(X^*, X)$ is a bijective operator, then the anti-automorphism $\varphi: T \mapsto AT^*A^{-1}$ is not locally spectrally bounded at e .*

Proof. Assume, on the contrary, that φ is locally spectrally bounded at e , and let M be a positive constant such that $r_{AT^*A^{-1}}(e) \leq Mr_T(e)$ for all $T \in \mathcal{L}(X)$. Note

that for every $T \in \mathcal{L}(X)$,

$$\begin{aligned} r_{T^*}(A^{-1}e) &= \limsup_{n \rightarrow \infty} \|T^{*n}A^{-1}e\|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \|A^{-1}(AT^*A^{-1})^n e\|^{1/n} \\ &\leq \limsup_{n \rightarrow \infty} \|(AT^*A^{-1})^n e\|^{1/n} = r_{AT^*A^{-1}}(e). \end{aligned}$$

Similarly, we have $r_{AT^*A^{-1}}(e) \leq r_{T^*}(A^{-1}e)$, and so

$$(2.1) \quad r_{T^*}(A^{-1}e) = r_{AT^*A^{-1}}(e) \leq Mr_T(e)$$

for all $T \in \mathcal{L}(X)$.

Now, let $x \in X$ be such that x and e are linearly independent and $\langle x, A^{-1}e \rangle = 1$, and let $f \in X^*$ be such that $\langle e, f \rangle = 0$ and $\langle x, f \rangle = 1$. The operator $T := x \otimes f$ satisfies $Te = 0$ and $T^{*n}A^{-1}e = f$ for all $n \geq 1$. Hence, $r_T(e) = 0$ and $r_{T^*}(A^{-1}e) = 1$. This contradicts the inequality (2.1) and completes the proof. \square

Remark 2.6. Just as in the proof of the above lemma one can see that when $X = \mathcal{H}$ is a Hilbert complex space and $A \in \mathcal{H}$ is a bijective operator, the anti-automorphism $\varphi: T \rightarrow AT^{\text{tr}}A^{-1}$ is not locally spectrally bounded at a nonzero fixed vector $e \in \mathcal{H}$. Here T^{tr} denotes the transpose of the operator T relative to a fixed but arbitrary orthonormal basis.

Let us recall the following useful facts that will be often used in the sequel. It is well-known that $T \in \mathcal{L}(X)$ has the single-valued extension property if, and only if, for every $\lambda \in \mathbb{C}$ and every nonzero vector x in $\ker(\lambda - T)$ we have $\sigma_T(x) = \{\lambda\}$; see [1]. Furthermore, if $X = \mathbb{C}^n$ is a finite dimensional space, then for every $x \in \mathbb{C}^n$

$$(2.2) \quad \sigma_T(x) = \bigcup_{1 \leq k \leq p} \{\lambda_k : 1 \leq k \leq p \text{ with } P_k(x) \neq 0\}.$$

Here $\lambda_1, \lambda_2, \dots, \lambda_p$ are the distinct eigenvalues of T , E_k the corresponding root spaces, and $P_k: \mathbb{C}^n \rightarrow E_k$ ($1 \leq k \leq p$) the canonical projections; see [5].

We now have collected all the necessary ingredients and are therefore in a position to prove the main results of this paper.

Proof of Theorem 1.2. Suppose that φ is locally spectrally bounded at e , and let M be a positive constant such that $r_{\varphi(T)}(e) \leq Mr_T(e)$ for all $T \in \mathcal{L}(X)$. By Lemma 2.4, the map φ is spectrally bounded; and so, by Lemma 2.2, there exist a nonzero scalar $c \in \mathbb{C}$ and an invertible matrix $A \in M_n(\mathbb{C})$ such that either $\varphi(T) = cATA^{-1} + (\varphi(I) - cI) \text{tr}(T/n)$ or $\varphi(T) = cAT^{\text{tr}}A^{-1} + (\varphi(I) - cI) \text{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$.

Assume that $\varphi(T) = cATA^{-1} + (\varphi(I) - cI) \operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$, and note that e and Ae are linearly dependent. Indeed, suppose by the way of contradiction that e and $A^{-1}e$ are linearly independent. As $n \geq 3$, we can find a matrix $T_1 \in M_n(\mathbb{C})$ such that $T_1e = 0$, $T_1A^{-1}e = A^{-1}e$, and $\operatorname{tr}(T_1) = 0$. We have $\sigma_{T_1}(e) = \{0\}$ and $\sigma_{\varphi(T_1)}(e) = \{c\}$; and so $r_{T_1}(e) = 0$ and $r_{\varphi(T_1)}(e) = |c|$. This entails that $c = 0$ and contradicts the surjectivity of φ . Now, let us show by the way of contradiction that e and $\varphi(I)e$ are linearly dependent. So, assume on the contrary that e and $\varphi(I)e$ are linearly independent, and note that in this case e and $A^{-1}f$ are linearly independent too, where $f := \varphi(I)e - ce$. It is easy to see that we can find a matrix $T_2 \in M_n(\mathbb{C})$ such that $T_2e = 0$, $T_2A^{-1}f = c^{-1}A^{-1}((1+c)f - \varphi(I)f)$, and $\operatorname{tr}(T_2/n) = 1$. Thus, we have $\sigma_{T_2}(e) = \{0\}$, $r_{T_2}(e) = 0$, $\varphi(T_2)e = f$ and $\varphi(T_2)f = f$. From this together with the equality (2.2), we infer that $\sigma_{\varphi(T_2)}(e) = \{1\}$ and $r_{\varphi(T_2)}(e) = 1$, which contradicts the fact that φ is locally spectrally bounded at e . So, write $\varphi(I)e = ke$ for some nonzero constant $k \in \mathbb{C}$, and let us show that $k = c$. For every $\nu \in \mathbb{C}$ we can find a matrix T_ν such that $T_\nu e = e$ and $\operatorname{tr}(T_\nu/n) = \nu$. As e and $A^{-1}e$ are linearly dependent, it is easy to see that $\varphi(T_\nu)e = (c + (k - c)\nu)e$, and so $r_{\varphi(T_\nu)}(e) = |(c + (k - c)\nu)|$. Therefore, as $r_{T_\nu}(e) = 1$, we have $|(c + (k - c)\nu)| \leq M$ for all $\nu \in \mathbb{C}$, which implies that $k = c$ and $\varphi(I)e = ce$. Hence $\varphi(T) = cATA^{-1} + c(S - I) \operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$, with $Ae = \alpha e$ for some nonzero $\alpha \in \mathbb{C}$ and $Se = e$, where $S := c^{-1}\varphi(I)$. Dividing A by α if necessary, we may assume that $Ae = e$.

The case when φ takes the second form is dealt with similarly; and the proof is complete. \square

P r o o f of Corollary 1.3. Checking the ‘if’ part is straightforward, so we will only deal with the ‘only if’ part. So assume that φ is locally spectrally bounded at e , and write $\varphi(I) = kI$ for some scalar k . By the proof of the above theorem there exist a nonzero scalar c and an invertible matrix A with $Ae = e$ and $\varphi(I)e = ce$ such that either $\varphi(T) = cATA^{-1} + (\varphi(I) - cI) \operatorname{tr}(T/n)$ or $\varphi(T) = cAT^{\operatorname{tr}}A^{-1} + (\varphi(I) - cI) \operatorname{tr}(T/n)$ for all $T \in M_n(\mathbb{C})$. In particular, $k = c$ and so either $\varphi(T) = cATA^{-1}$ or $\varphi(T) = cAT^{\operatorname{tr}}A^{-1}$ for all $T \in M_n(\mathbb{C})$. Lemma 2.5 yields that φ takes only the first form; and the proof is therefore complete. \square

P r o o f of Theorem 1.1. Note that, since the sufficiency condition is obvious, we only need to prove the necessity. So, assume that φ is locally spectrally bounded at e . By Lemma 2.4 the map φ is spectrally bounded, and therefore by Lemma 2.1 there exist a nonzero complex number c and an epimorphism or an anti-epimorphism J on $\mathcal{L}(\mathcal{H})$ such that $\varphi(T) = cJ(T)$ for all $T \in \mathcal{L}(\mathcal{H})$.

Next, let us show by way of contradiction that φ is injective. So, assume that φ is not injective and note that, in this case, $\ker(\varphi)$ is an ideal of $\mathcal{L}(\mathcal{H})$ containing $\mathcal{K}(\mathcal{H})$, the ideal of all compact operators on \mathcal{H} . So, pick an arbitrary $\varepsilon > 0$ and let

f_ε be a linear functional on \mathcal{H} such that $f_\varepsilon(e) = \varepsilon - 1$. We have $(e \otimes f_\varepsilon + I)e = \varepsilon e$, $\sigma_{e \otimes f_\varepsilon + I}(e) = \{\varepsilon\}$, and $\sigma_{\varphi(e \otimes f_\varepsilon + I)}(e) = \sigma_{\varphi(I)}(e) = \{c\}$ since $e \otimes f_\varepsilon \in \mathcal{K}(\mathcal{H})$. In particular, $r_{e \otimes f_\varepsilon + I}(e) = \varepsilon$ and $r_{\varphi(e \otimes f_\varepsilon + I)}(e) = |c|$, and so there is a positive constant M such that $|c| \leq M\varepsilon$ for all $\varepsilon > 0$ since φ is locally spectrally bounded at e . This implies that $c = 0$, which is a contradiction. Thus J is an automorphism or an anti-automorphism. Now by the fundamental isomorphism theorem [12, Theorem 2.5.19] (see also [8]) there exists an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $J(T) = ATA^{-1}$ or $J(T) = AT^{\text{tr}}A^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$. Lemma 2.5 ensures that the second form is excluded, and consequently φ takes only the first. Moreover, in this case, $Ae = \lambda e$ for some nonzero scalar λ ; otherwise, we can find $T \in \mathcal{L}(X)$ such that $Te = 0$ and $TA^{-1}e = A^{-1}e$. This shows that $r_T(e) = 0$ and $r_{\varphi(T)}(e) = 1$, and gives a contradiction. Dividing A by λ if necessary, we may assume that $Ae = e$; and the proof is complete. \square

3. LINEAR LOCAL SPECTRUM COMPRESSORS

This section is devoted to deriving some consequences of the main results of this paper. These consequences describe linear maps from $\mathcal{L}(\mathcal{H})$ onto itself compressing the local spectrum. A linear map φ from $\mathcal{L}(\mathcal{H})$ into itself is said to compress the local spectrum at a fixed nonzero vector $e \in \mathcal{H}$ if

$$\sigma_{\varphi(T)}(e) \subseteq \sigma_T(e)$$

for all $T \in \mathcal{L}(\mathcal{H})$.

The first consequence extends [7, Theorem 3.3] by replacing “preserves the local spectrum” by the weaker hypothesis “compresses the local spectrum” in the Hilbert space setting.

Theorem 3.1. *Let \mathcal{H} be an infinite dimensional Hilbert space and let $e \in \mathcal{H}$ be a fixed nonzero vector. A continuous surjective linear map $\varphi: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ compresses the local spectrum at e if and only if there is an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that $Ae = e$, and $\varphi(T) = ATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$.*

Proof. Assume that φ compresses the local spectrum at e . By using the fact that $r_T(x) \geq \max\{|\lambda|: \lambda \in \sigma_T(x)\}$ for all $T \in \mathcal{L}(\mathcal{H})$ (see for instance [1] or [11]) together with the same argument as in the proof of Lemma 2.4, one can see that φ is spectrally bounded. Now, in the same way as in the end of the proof of Theorem 1.1, one can see that φ is injective and there exist a nonzero scalar c and an invertible operator $A \in \mathcal{L}(\mathcal{H})$ such that either $\varphi(T) = cATA^{-1}$ or $\varphi(T) = cAT^{\text{tr}}A^{-1}$ for all

$T \in \mathcal{L}(\mathcal{H})$. By [9, Lemma 5], the second form is not possible, and $\varphi(T) = cATA^{-1}$ for all $T \in \mathcal{L}(\mathcal{H})$, from which one can see that A can be supposed to satisfy $Ae = e$. The desired conclusion follows from the fact that $\{e\} = \sigma_{\varphi(T)}(e) \subseteq \sigma_I(e) = 1$, which means that $c = 1$.

As the sufficiency condition is obvious, the proof is complete. □

The second consequence describes linear maps on $M_n(\mathbb{C})$ which compress the local spectrum at a fixed nonzero vector, and extends the main result of [9].

Theorem 3.2. *Let n be a positive integer, and let $e \in \mathbb{C}^n$ be a fixed nonzero vector. A linear map φ from $M_n(\mathbb{C})$ into itself compresses the local spectrum at e if and only if there is an invertible matrix $A \in M_n(\mathbb{C})$ such that*

$$(3.3) \quad Ae = e \text{ and } \varphi(T) = ATA^{-1} \quad \text{for all } T \in M_n(\mathbb{C}).$$

Proof. Evidently, the formula (3.3) defines a compressing local spectrum bijective linear map at e .

Conversely, if φ compresses the local spectrum, then in particular, it preserves at least one eigenvalue of each matrix, i.e., $\sigma(\varphi(T)) \cap \sigma(T) \neq \emptyset$ for all $T \in M_n(\mathbb{C})$. Therefore, by [2, Theorem 2], there is an invertible matrix $A \in M_n(\mathbb{C})$ such that $\varphi(T) = ATA^{-1}$ or $\varphi(T) = AT^{\text{tr}}A^{-1}$ for all $T \in M_n(\mathbb{C})$. Moreover, by [9, Lemma 5] the second form is excluded, and consequently φ takes only the first, from which we may assume that $Ae = e$. This completes the proof. □

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References

- [1] *P. Aiena*: Fredholm and Local Spectral Theory, with Applications to Multipliers. Kluwer Academic Publishers, 2004. zbl
- [2] *S. Akbari, M. Aryapoor*: On linear transformations preserving at least one eigenvalue. Proc. Amer. Math. Soc. *132* (2004), 1621–1625. zbl
- [3] *M. Bendaoud, A. Bourhim*: Essentially spectrally bounded linear maps. Proc. Amer. Math. Soc. *92* (2009), 257–265. zbl
- [4] *M. Bendaoud, M. Sarih*: Linear maps preserving the local spectral radius, preprint.
- [5] *A. Torgašev*: On operators with the same local spectra. Czech. Math. J. *48* (1998), 77–83. zbl
- [6] *A. Bourhim, V. G. Miller*: Linear maps on $M_n(\mathbb{C})$ preserving the local spectral radius. Studia Math. *188* (2008), 67–75. zbl
- [7] *J. Bračič, V. Müller*: Local spectrum and local spectral radius at a fixed vector. Studia Math. *194* (2009), 155–162. zbl
- [8] *P. R. Chernoff*: Representations, automorphisms, and derivations of some operator algebras. J. Funct. Anal. *12* (1973), 257–289. zbl

- [9] *M. González, M. Mbekhta*: Linear maps on $M_n(\mathbb{C})$ preserving the local spectrum. *Linear Algebra Appl.* 427 (2007), 176–182. zbl
- [10] *I. N. Herstein*: Jordan homomorphisms. *Trans. Amer. Math. Soc.* 81 (1956), 331–341. zbl
- [11] *K. B. Laursen, M. M. Neumann*: *An Introduction to Local Spectral Theory*. Oxford University Press, New York, 2000. zbl
- [12] *C. E. Richart*: *General Theory of Banach Algebras*. Van Nostrand, Princeton, 1960. zbl
- [13] *P. Šemrl*: Spectrally bounded linear maps on $B(H)$. *Quart. J. Math. Oxford* 49 (1998), 87–92. zbl

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