Scaling and singular limits in fluid mechanics

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

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Recommended literature

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- N. Masmoudi: Examples of singular limits in hydrodynamics. *In Handbook of Differential Equations, III, C. Dafermos, E. Feireisl Eds., Elsevier, Amsterdam, 2006*
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Balance laws

General balance law

$$\partial_t d + \mathrm{div}_x(\mathbf{F}) = S, \ d \text{ density}, \ \mathbf{F} \text{ flux}, \ S \text{ source (sink)}$$

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \ \varrho \text{ mass density}, \ \mathbf{u} \text{ velocity field}$$

Balance law - Eulerian form

$$\partial_t(\varrho s) + \underbrace{\mathrm{div}_x(\varrho s \mathbf{u})}_{\text{convective flux}} + \underbrace{\mathrm{div}_x \mathbf{q}}_{\text{diffusive flux}} = S$$

Material derivative

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \equiv \varrho \Big[\partial_t s + \mathbf{u} \cdot \nabla_x s \Big] \equiv \varrho \frac{\mathrm{d}}{\mathrm{d}t} s(t, \mathbf{X}(t, x))$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X} = \mathbf{u}(t, \mathbf{X})$$

Characteristic values and scaling

Geometry

$$t \to \frac{t}{T_{\rm char}},\ x \to \frac{x}{L_{\rm char}},\ \mathbf{u} \to \frac{\mathbf{u}}{U_{\rm char}}$$

$$\partial_t \to \frac{1}{T_{\rm char}}\partial_t,\ \partial_x \to \frac{x}{L_{\rm char}}\partial_x$$

Re-scaled (dimensionless) balance law

$$\boxed{\frac{L_{\rm char}}{T_{\rm char}U_{\rm char}}}\partial_t(\varrho s) + {\rm div}_x(\varrho s\mathbf{u}) + \frac{q_{\rm char}}{\varrho_{\rm char}s_{\rm char}U_{\rm char}}{\rm div}_x\mathbf{q}$$

$$= \frac{S_{\rm char}}{\varrho_{\rm char}s_{\rm char}U_{\rm char}L_{\rm char}}S$$

Strouhal number

$$[{\rm Sr}] \equiv \frac{L_{\rm char}}{T_{\rm char}U_{\rm char}}$$

Compressible viscous rotating fluids

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \underbrace{\varrho \mathbf{f} \times \mathbf{u}}_{\text{Coriolis force}} + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x G$$

Newton's rheological law - viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Geo-potential

$$G = \underbrace{g}_{\text{gravity}} + \underbrace{\frac{1}{2} |\mathbf{f} \times x|^2}_{\text{centrifugal force}}$$

Scaled system

Field equations

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$[\mathrm{Sr}]\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{[\mathrm{Ro}]}\varrho \mathbf{f} \times \mathbf{u} + \frac{1}{[\mathrm{Ma}]^2}\nabla_x p(\varrho)$$

$$= \frac{1}{[\mathrm{Re}]}\mathrm{div}_x \mathbb{S} + \frac{1}{[\mathrm{Fr}]^2}\varrho \nabla_x G$$

Characteristic numbers

Rossby number . . . . $\mathrm{Ro} = \frac{U_{\mathrm{char}}}{L_{\mathrm{char}} f_{\mathrm{char}}}$

Mach number . . . . $\mathrm{Ma} = \frac{U_{\mathrm{char}}}{\sqrt{p_{\mathrm{char}}/\varrho_{\mathrm{char}}}}$

Reynolds number . . . . $\mathrm{Re} = \frac{\varrho_{\mathrm{char}} L_{\mathrm{char}} U_{\mathrm{char}}}{\mu_{\mathrm{char}}}$

Froude number I . . . . $\mathrm{Fr}_{\mathrm{I}} = \frac{U_{\mathrm{char}}}{\sqrt{g_{\mathrm{char}}}}$

Froude number II . . . . $\mathrm{Fr}_{\mathrm{II}} = \frac{U_{\mathrm{char}}}{L_{\mathrm{char}} f_{\mathrm{char}}}$

Incompressible limit

Primitive system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Formal singular limit - target problem

$$\nabla_x p(\varrho) = 0 \ \Rightarrow \varrho = \varrho_{\mathrm{char}} \equiv \overline{\varrho} \Rightarrow \mathrm{div}_x \mathbf{u} = 0$$

$$\overline{\varrho}\left[\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u})\right] + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \mathrm{div}_x \left[\mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}\right)\right]$$

Stability condition

$$p'(\varrho) > 0 \text{ for all } \varrho > 0$$

Fast rotation

Primitive system - incompressible Navier-Stokes equations

$$\mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \mathbf{f} \times \mathbf{u} + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Target problem

$$\mathbf{f} = [0,0,1] \Rightarrow \mathbf{f} \times \mathbf{u} \equiv \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix} = \nabla_x \Psi \Rightarrow \mathbf{u} = [\mathbf{u}_h(x_1, x_2), 0]$$

$$\mathrm{div}_x \mathbf{u}_h = 0$$

$$\partial_t \mathbf{u}_h + \mathrm{div}_x (\mathbf{u}_h \otimes \mathbf{u}_h) + \nabla_h \Pi = \mu \Delta_h \mathbf{u}_h$$

Inviscid limit

Primitive system - incompressible Navier-Stokes equations

$$\mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \boxed{\varepsilon} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Target problem - incompressible Euler system

$$\mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = 0$$

Strongly stratified limit

Primitive system - compressible Navier-Stokes equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \varrho \nabla_x g$$

Stationary density profile

$$\nabla_x p(\tilde{\varrho})) = \tilde{\varrho} \nabla_x g \Rightarrow P(\tilde{\varrho}) = g + \mathrm{const}, \ \ P'(\varrho) = \frac{p'(\varrho)}{\varrho}$$

Target problem - anelastic system

$$\mathrm{div}_x(\tilde{\varrho}\mathbf{u}) = 0$$

$$\tilde{\varrho}\Big[\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u}\Big] + \tilde{\varrho} \nabla_x \Pi = \mu \Delta \mathbf{u}$$

Fundamental issues

Solvability of the primitive system

The primitive system should admit (global) in time solutions for any choice of the scaling parameters and any admissible initial data

Solvability of the target system

The target system should admit solutions, at least locally in time; the solutions are regular

Stability

The family of solutions to the primitive system should be stable with respect to the scaling parameters

Control of the "oscillatory" component of solutions

The component of solutions to the primitive system that "disappears" in the singular limit must be controlled

Analysis of singular limits

Primitive system

$$\partial_t U + \frac{1}{\varepsilon}\mathcal{A}[U] + \mathcal{B}[U] = 0, \ \ U(0,\cdot) = U_0$$

• Existence of solutions on a time interval $(0, T)$, T independent of ε

Identifying the limit system

$$\mathcal{A}[U] = 0, \ \ U_{\rm limit} \in {\rm Ker}[\mathcal{A}], \ \ U_{\rm osc} \in {\rm Range}[\mathcal{A}], \ \ U = U_{\rm osc} + U_{\rm limit}$$

Uniform bounds

• Find uniform bounds $\|U_\varepsilon\|_X < c$ independent of $\varepsilon \to 0$, prepared initial data

Equations for the limit and oscillatory components

Compactness of the "limit" component

$$\partial_t U_{\rm lim} + \mathcal{B}[U_{\rm lim}] = 0$$

- Convergence via standard compactness arguments or "stability" of the system

Equation for the oscillatory component

$$\varepsilon \partial_t U_{\rm osc} + \mathcal{A}[U_{\rm osc}] \approx 0, \ \ U_{\rm osc} \approx V\left(\frac{t}{\varepsilon}\right), \ \ \partial_t V + \mathcal{A}[V] = 0$$

- Goal is to show

$$U_{\rm osc} \to 0 \text{ in some sense}$$

- Convergence via dispersive estimates

Global-in-time solutions to the primitive system

Navier-Stokes system (in rotating frame)

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho \mathbf{f} \times \mathbf{u} + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x G$$

Slip boundary and far field conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0$$

$$\varrho \to \overline{\varrho}, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Initial data

$$\varrho(0,\cdot) = \varrho_0 > 0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \text{ in } \Omega \subset R^3$$

Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

Lyapunov function - energy inequality

Stationary solutions

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \nabla_x G, \ \ \tilde{\varrho} \to \overline{\varrho} \text{ as } |x| \to \infty$$

Potential energy

$$H(\varrho) = \varrho \int_{\overline{\varrho}}^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z$$

Energy inequality

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - H(\tilde{\varrho}) \right) \, \mathrm{d}x$$

$$+ \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_{\Omega} \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H(\varrho_0) - H'(\tilde{\varrho})(\varrho_0 - \tilde{\varrho}) - H(\tilde{\varrho}) \right) \, \mathrm{d}x$$

A priori bounds

Energy bounds

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3))$$

$$\varrho - \tilde{\varrho} \in L^\infty(0,T;L^2 + L^\gamma(\Omega)) \text{ provided } p(\varrho) \approx \varrho^\gamma$$

Bounds due to dissipation

$$\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \in L^1((0,T) \times \Omega)$$

$$\nabla_x \mathbf{u} \in L^2(0,T;L^2(\Omega;R^3)),\ \mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;R^3))$$

Existence for the primitive system

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

$$b(\varrho) = \varrho \log(\varrho)$$

Compactness of (approximate) solutions

$$\varrho_n \to \varrho \text{ weakly in } L^p$$

$$\partial_t \int (\varrho \log(\varrho)) + \int (\varrho \mathrm{div}_x \mathbf{u}) = 0$$

$$\partial_t \int \left(\overline{\varrho \log(\varrho)} \right) + \int (\overline{\varrho \mathrm{div}_x \mathbf{u}}) = 0$$

Effective viscous flux

Renormalized equation indentity

$$\int \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right)(\tau) + \int_0^\tau \int \left(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u} \right)$$

$$= \int \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right)(0) = 0, \ p(\varrho) \approx \varrho^\gamma, \ \gamma > \frac{3}{2}$$

Lions' identity

$$\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u} \boxed{=} \overline{p(\varrho)\varrho} - \overline{p(\varrho)} \ \varrho \geq 0$$

Strong convergence

$$\int \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right)(\tau) = 0$$

$$\Rightarrow$$

$\varrho_n \to \varrho$ strongly in L^1

Relative entropy (energy)

Relative entropy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) \right) \, \mathrm{d}x$$

Potential energy

$$H(\varrho) = \varrho \int_{\overline{\varrho}}^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z$$

Coercivity of the elastic energy

$\varrho \mapsto p(\varrho)$ strictly increasing $\Rightarrow \varrho \mapsto H(\varrho)$ strictly convex

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U} \right) \right]_{t=0}^{\tau} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\mathbf{U}$ satisfying relevant boundary and far field conditions

Dissipative solutions

Dissipative solution is a weak solution satisfying the relative entropy inequality

Remainder

$$\boxed{\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r) - p(\varrho) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r) \right] \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r) + \mathbf{U} \cdot \nabla_x p(r) \Big) \ \mathrm{d}x$$

$$+ \int_\Omega (\varrho \mathbf{f} \times \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x + \int_\Omega \varrho \nabla_x G \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x$$

Applications of the relative entropy inequality

Weak strong uniqueness

Weak and strong solutions emanating from the same initial data coincide as long as the latter exists

Regularity criterion

Suppose that a weak solution to the compressible Navier-Stokes system emanating from *regular* initial data admits a bound $\|\varrho\|_{L^\infty} < c$. Then the solution is smooth.

Dimension reduction

Solutions of the compressible Navier-Stokes system on "thin" domains converge to the solutions of the limit problem

Stability

Stability in the singular limits problems without compactness, e.g. the inviscid limit

Weak solutions - summary

Stability hypothesis (not strictly necessary for existence)

$$p \in C[0,\infty) \cap C^2(0,\infty),\ \ p(0) = 0,\ \ p'(\varrho) > 0 \text{ for all } \varrho > 0$$

$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\ \ \gamma > \frac{3}{2}$$

Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

Weak-strong uniqueness

Weak and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

General strategy - Step I

Ansatz

$$\varrho = \varrho_\varepsilon, \ \mathbf{u} = \mathbf{u}_\varepsilon - \text{a dissipative weak solution}$$

$$r = \varrho_{\rm limit} + \varrho_{{\rm osc},\varepsilon,\delta}, \ \mathbf{U} = \mathbf{u}_{\rm limit} + \mathbf{u}_{{\rm osc},\varepsilon,\delta}$$

Initial data

$$\varrho_{{\rm osc},\varepsilon,\delta}(0,\cdot) = \varrho_{0,{\rm osc},\delta}, \ \mathbf{u}_{{\rm osc},\varepsilon,\delta}(0\cdot) = \mathbf{u}_{0,{\rm osc},\delta}$$

$$\mathcal{E}_\varepsilon \left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \ \Big| \varrho_{0,{\rm limit}} + \varrho_{0,{\rm osc},\delta}, \mathbf{u}_{0,{\rm limit}} + \mathbf{u}_{0,{\rm osc},\delta} \right) \to h(\delta) \text{ as } \varepsilon \to 0$$

$$h(\delta) \to 0 \text{ as } \delta \to 0$$

General strategy - Step 2

Vanishing oscillatory components

$$\|\varrho_{\mathrm{osc},\varepsilon,\delta}(\tau,\cdot)\|_{L^\infty} \to 0, \ \|\mathbf{u}_{\mathrm{osc},\varepsilon,\delta}\|_{L^\infty} \to 0 \text{ as } \varepsilon \to 0$$

for any any fixed $\delta > 0$ and any $\tau > 0$

Gronwall type argument

$$\mathcal{E}_\varepsilon \left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \ \Big| \varrho_{0,\mathrm{limit}} + \varrho_{\mathrm{osc},\varepsilon,\delta}, \mathbf{u}_{0,\mathrm{limit}} + \mathbf{u}_{\mathrm{osc},\varepsilon,\delta} \right)(\tau)$$

$$\leq \mathcal{E}_\varepsilon \left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \ \Big| \varrho_{0,\mathrm{limit}} + \varrho_{0,\mathrm{osc},\delta}, \mathbf{u}_{0,\mathrm{limit}} + \mathbf{u}_{0,\mathrm{osc},\delta} \right) K(T)$$

for a.a. $\tau \in (0, T)$

Limit passage

Taking the limits: first $\varepsilon \to 0$ then $\delta \to 0$

Example: Inviscid, incompressible limit

Primitive system

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho_\varepsilon)$$

$$= \mu_\varepsilon \mathrm{div}_x \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \right)$$

$$\mu_\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

$$\varrho \to \overline{\varrho},\ \mathbf{u} \to 0 \text{ for } |x| \to \infty$$

Target system

$$\varrho_{\rm limit} = \overline{\varrho},\ \mathbf{u}_{\rm limit} = \mathbf{v}$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

Energy inequality and initial data

Scaled energy inequality

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \Big(H(\varrho_\varepsilon) - H'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) - H(\overline{\varrho}) \Big) \right] \ \mathrm{d}x$$

$$+\mu_\varepsilon \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \Big(H(\varrho_{0,\varepsilon}) - H'(\overline{\varrho})(\varrho_{0,\varepsilon} - \overline{\varrho}) - H(\overline{\varrho}) \Big) \right] \ \mathrm{d}x$$

Ill prepared initial data

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3), \ \ \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)}$$

$$\|\varrho_{0,\varepsilon}^{(1)}\|_{L^\infty(\Omega)} \leq c, \quad \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ in } L^2(\Omega)$$

Uniform bounds

Bounds in L^p

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \text{ bounded in } L^\infty(0,T;(L^2 + L^\gamma)(\Omega))$$

Dissipative bounds

$$\mu_\varepsilon \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \approx \mu_\varepsilon \|\nabla_x \mathbf{u}_\varepsilon\|^2_{L^2} < c$$

Relative entropy and the initial data

Scaled relative entropy

$$\mathcal{E}_\varepsilon \left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \;\Big|\; r(0,\cdot), \mathbf{U}(0,\cdot) \right)$$

$$= \int_\Omega \frac{1}{2} \varrho_{0,\varepsilon} \left| \mathbf{u}_{0,\varepsilon} - \mathbf{v}_0 - \mathbf{u}_{0,\mathrm{osc},\delta} \right|^2 \, \mathrm{d}x$$

$$+ \frac{1}{\varepsilon^2} \int_\Omega \Big(H(\varrho_{0,\varepsilon})$$

$$- H' \left(\overline{\varrho} + \varepsilon s_{\varepsilon,\delta} \right) \left(\varrho_{0,\varepsilon} - \overline{\varrho} - \varepsilon s_{\varepsilon,\delta} \right) - H \left(\overline{\varrho} + \varepsilon s_{\varepsilon,\delta} \right) \Big) \, \mathrm{d}x$$

Initial data

$$\mathbf{u}_0 = \mathbf{H}[\mathbf{u}_0] + \mathbf{H}^\perp[\mathbf{u}_0] = \mathbf{v}_0 + \nabla_x \Psi_0, \;\; \mathbf{u}_{0,\mathrm{osc},\delta} \approx \nabla_x \Psi_0$$

$$\varrho_{\mathrm{osc},\varepsilon,\delta} = \varepsilon s_{\varepsilon,\delta}$$

$$\varrho_{0,\mathrm{osc},\delta} = \varepsilon s_{0,\delta}, \;\; s_{0,\delta} \approx \varrho_0^{(1)}$$

Acoustic analogy

Lighthill's acoustic analogy

$$\varepsilon\partial_t\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + p'(\overline{\varrho})\nabla_x\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) = -\varepsilon\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \varepsilon\mu_\varepsilon \mathrm{div}_x \mathbb{S}$$

$$-\frac{1}{\varepsilon}\nabla_x\left(p(\varrho_\varepsilon) - p'(\overline{\varrho})\nabla_x\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right) - p(\overline{\varrho})\right) \approx o(\varepsilon)$$

Oscillatory part

$$\varrho_{\mathrm{osc},\varepsilon,\delta} = \varepsilon s_{\varepsilon,\delta}, \ \ s_{\varepsilon,\delta} \approx \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \ \mathbf{u}_{\mathrm{osc},\varepsilon,\delta} = \nabla_x \Psi_{\varepsilon,\delta} \approx \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$$

H - Helmholtz projection

Acoustic waves

Acoustic (wave) equation

$$\varepsilon \partial_t s_{\varepsilon,\delta} + \overline{\varrho} \Delta_x \Psi_{\varepsilon,\delta} = 0$$

$$\varepsilon \overline{\varrho} \partial_t \Psi_{\varepsilon,\delta} + p'(\overline{\varrho}) s_{\varepsilon,\delta} = 0$$

Initial data

$$s_{\varepsilon,\delta}(0,\cdot) = s_{0,\delta} \to \varrho_0^{(1)} \text{ in } L^2(\Omega), \ \|s_{0,\delta}\|_{L^\infty(\Omega)} \leq c(\delta)$$

$$\Psi_{\varepsilon,\delta}(0,\cdot) = \Psi_{0,\delta}, \ \nabla_x \Psi_{0,\delta} \to \mathbf{H}^\perp[\mathbf{u}_0] \text{ in } L^2(\Omega; R^3)$$

$$\text{as } \delta \to 0$$

Boundary conditions

$$\nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dispersive estimates

$$\Omega = R^3$$

Total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}|\nabla_x \Psi_{\varepsilon,\delta}|^2 + \frac{p'(\overline{\varrho})}{2\overline{\varrho}^2}|s_{\varepsilon,\delta}|^2 \right) \ \mathrm{d}x = 0$$

$$\int_\Omega \left(\frac{1}{2}|\nabla_x D^m \Psi_{\varepsilon,\delta}|^2 + \frac{p'(\overline{\varrho})}{2\overline{\varrho}^2}|D^m s_{\varepsilon,\delta}|^2 \right) \ \mathrm{d}x$$

$$= \int_\Omega \left(\frac{1}{2}|\nabla_x D^m \Psi_{0,\delta}|^2 + \frac{p'(\overline{\varrho})}{2\overline{\varrho}^2}|D^m s_{0,\delta}|^2 \right) \ \mathrm{d}x, \ \ m = 0, 1, \dots$$

$L^1 - L^\infty$-estimates

$$\|\Psi_{\varepsilon,\delta}(t,\cdot)\|_{L^\infty(\Omega)} + \|s_{\varepsilon,\delta}(t,\cdot)\|_{L^\infty(\Omega)}$$

$$\leq c(\overline{\varrho}) \left(\frac{\varepsilon}{t} \right) \left(\|\Psi_{0,\delta}\|_{W^{k,1}(\Omega)} + \|s_{0,\delta}\|_{W^{k,1}(\Omega)} \right), \ \ k = k(N)$$

Dispersive estimates - elementary approach

Approximation

$$s_{0,\delta} = \mathcal{F}^{-1}_{\xi\to x}\left[\psi_\delta(|\xi|)\varrho_0^{(1)}\right], \;\; \psi_\delta \in C_c^\infty(0,\infty), \;\; \psi_\delta \nearrow 1$$

Typical terms in the wave equation

$$Z(\tau, x) = \mathcal{F}^{-1}_{\xi\to x}\left[\exp\left(\pm \mathrm{i}|\xi|\tau\right)\psi_\delta(|\xi|)h(\xi)\right], \;\; \boxed{\tau = \frac{t}{\varepsilon}}$$

$$\|Z(\tau,\cdot)\|_{L^\infty(R^3)} \le \left\|\mathcal{F}^{-1}_{\xi\to x}\left[\exp\left(\pm \mathrm{i}|\xi|\tau\right)\psi_\delta(|\xi|)h(\xi)\right]\right\|_{L^\infty(R^3)}$$

$$\le \left\|\mathcal{F}^{-1}_{\xi\to x}\left[\exp\left(\pm \mathrm{i}|\xi|\tau\right)\psi_\delta(|\xi|)\right]\right\|_{L^\infty(R^3)} \|h\|_{L^1(R^3)}$$

Fourier transform of radially symmetric functions

$$\mathcal{F}^{-1}_{\xi\to x}\left[\exp\left(\pm \mathrm{i}|\xi|\tau\right)\right](x)$$

$$= \int_0^\infty \exp\left(\pm \mathrm{i}\tau r\right)\psi_\delta(r) r^{3/2}|x|^{-1/2} J_{1/2}(r|x|)\,\mathrm{d}r$$

van der Corput's lemma

Lemma

Let $\Lambda = \Lambda(z)$ *be a smooth function away from the origin,*

$$\partial_z \Lambda(z) \text{ monotone}, \ |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

for all $z \in [a, b]$, $0 < a < b < \infty$. *Let* Φ *be a smooth function on* $[a, b]$.
Then

$$\left| \int_a^b \exp\left(\mathrm{i} \Lambda(z) \tau \right) \Phi(z) \ \mathrm{d}z \right| \leq c \frac{1}{\tau \Lambda_0} \left[|\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \ \mathrm{d}z \right],$$

where c *is an absolute constant* *independent* *of the specific shape* Λ *and* Φ.

Convergence - application of relative entropy

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \\
&= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \boxed{\mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u})} + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \, \mathrm{d}x \\
&+ \int_\Omega \left[\Big(p(r) - p(\varrho) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r) \right] \, \mathrm{d}x \\
&+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r) + \mathbf{U} \cdot \nabla_x p(r) \Big) \, \mathrm{d}x \\
&+ \int_\Omega (\varrho \mathbf{f} \times \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x + \int_\Omega \varrho \nabla_x G \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x
\end{aligned}
$$

Convergence - example

$$\int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x(\mathbf{v} + \nabla_x \Psi_\varepsilon) \cdot (\mathbf{v} + \nabla_x \Psi_\varepsilon - \mathbf{u}_\varepsilon) \ \mathrm{d}x$$

$$\approx \int_\Omega \varrho_\varepsilon \Big| \nabla_x(\mathbf{v} + \nabla_x \Psi_\varepsilon) \Big| \Big| \mathbf{v} + \nabla_x \Psi_\varepsilon - \mathbf{u}_\varepsilon \Big|^2 \ \mathrm{d}x$$

$$+ \int_\Omega \varrho_\varepsilon \Big(\mathbf{v} + \nabla_x \Psi_\varepsilon \Big) \cdot \nabla_x(\mathbf{v} + \nabla_x \Psi_\varepsilon) \cdot (\mathbf{v} + \nabla_x \Psi_\varepsilon - \mathbf{u}_\varepsilon) \ \mathrm{d}x$$

$$\approx \int_\Omega \varrho_\varepsilon \Big| \nabla_x(\mathbf{v} + \nabla_x \Psi_\varepsilon) \Big| \Big| \mathbf{v} + \nabla_x \Psi_\varepsilon - \mathbf{u}_\varepsilon \Big|^2 \ \mathrm{d}x$$

$$+ \int_\Omega \overline{\varrho} \mathbf{v} \cdot \nabla_x \mathbf{v} \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) \ \mathrm{d}x$$

- First integral “absorbed” by Gornwall type argument
- Second integral forms a part of the limit system

Rotating fluids

Primitive system

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \boxed{\frac{1}{\varepsilon}} \varrho_\varepsilon \mathbf{f} \times \mathbf{u}_\varepsilon + \boxed{\frac{1}{\varepsilon^{2m}}} \nabla_x p(\varrho_\varepsilon)$$

$$= \boxed{\varepsilon^\alpha} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) + \boxed{\frac{1}{\varepsilon^{2n}}} \varrho_\varepsilon \nabla_x G$$

Infinite strip

$$\mathbf{f} = [0, 0, -1],\ \Omega = R^2 \times (0, 1),\ \mathbf{u} \cdot \mathbf{n} = [\mathbb{S} \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0$$

$$\mathbf{u} \to 0,\ \varrho \to \tilde{\varrho}_\varepsilon \approx \overline{\varrho} \text{ as } |x| \to \infty,\ \nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2m-2n} \tilde{\varrho}_\varepsilon \nabla_x G$$

Multiscale limit

$$\alpha > 0,\ \frac{m}{2} > n \geq 1$$

Expected limit for $\varepsilon \to 0$

Low Mach number

Mach number $\approx \varepsilon^m$:

compressible $\to$ incompressible

Low Rossby number

Rossby number $\approx \varepsilon$:

3D flow $\to$ 2D flow

High Reynolds number

Reynolds number $\approx \varepsilon^{-\alpha}$:

viscous (Navier-Stokes) $\to$ inviscid (Euler)

Target system

Limit density deviation

$$\mathrm{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\|_{L^1_{\mathrm{loc}}(\Omega)} \leq \varepsilon^m c$$

Limit velocity

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\mathbf{v} \begin{cases} \text{weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)), \\ \boxed{\text{strongly in } L^1_{\mathrm{loc}}((0,T)\times\Omega;R^3)}, \end{cases}$$

Euler system

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x \mathbf{v} + \nabla_x \Pi = 0 \text{ in } (0,T)\times R^2$$

$$\mathbf{v}_0 = \mathbf{H}\left[\int_0^1 \mathbf{u}_0 \, \mathrm{d}x_3\right]$$

Oscillatory - vanishing part

Poincaré waves

$$\varepsilon^m \partial_t s_{\varepsilon,\delta} + \mathrm{div}_x \mathbf{V}_{\varepsilon,\delta} = 0$$

$$\varepsilon^m \partial_t \mathbf{V}_{\varepsilon,\delta} + \boxed{\omega \mathbf{f} \times \mathbf{V}_{\varepsilon,\delta}} + \nabla_x s_{\varepsilon,\delta} = 0, \ \omega = \varepsilon^{m-1}$$

Antisymmetric acoustic propagator

$$\mathcal{B}(\omega) : \left[\begin{array}{c} s \\ \mathbf{V} \end{array} \right] \mapsto \left[\begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \omega \mathbf{f} \times \mathbf{V} + \nabla_x s \end{array} \right].$$

Fourier representation

Poincaré waves

$$\varepsilon^m \partial_t \begin{bmatrix} s_\varepsilon(\xi, k, \omega) \\ \mathbf{V}_\varepsilon(\xi, k, \omega) \end{bmatrix} = \mathrm{i}\mathcal{A}(\xi, k, \omega) \begin{bmatrix} s_\varepsilon(\xi, k, \omega) \\ \mathbf{V}_\varepsilon(\xi, k, \omega) \end{bmatrix}$$

Hermitian matrix

$$\mathrm{i}\mathcal{B}(\omega) \approx \mathcal{A}(\xi, k, \omega) = \begin{bmatrix} 0 & \xi_1 & \xi_2 & k \\ \xi_1 & 0 & \omega \mathrm{i} & 0 \\ \xi_2 & -\omega \mathrm{i} & 0 & 0 \\ k & 0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues

$$\lambda_{1,2}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 + \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

$$\lambda_{3,4}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 - \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

Fourier analysis

Frequency cut-off

$$k \text{ fixed}, \ \psi \in C_c^\infty(0,\infty), \ 0 \le \psi \le 1, \ h \approx \hat{h}(\xi, k)$$

$$Z(\tau, x_h, k, \omega) = \mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\Big)\psi(|\xi|)\hat{h}(\xi, k)\right], \ \tau = \frac{t}{\varepsilon^m}$$

Fourier transform of radially symmetric function

$$\|Z(\tau, \cdot, k, \omega)\|_{L^\infty(R^2)}$$

$$\le \left\| \mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\Big)\psi(|\xi|)\right]\right\|_{L^\infty(R^2)} \|h\|_{L^1(R^2)}$$

$$\mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\Big)\psi(|\xi|)\right](x_h)$$

$$= \int_0^\infty \exp\Big(\pm \mathrm{i}\lambda_j(r, k, \omega)\tau\Big)\psi(r) r J_0(r|x_h|)\, \mathrm{d}r,$$

van der Corput's lemma

Lemma

Let $\Lambda = \Lambda(z)$ *be a smooth function away from the origin,*

$$\partial_z \Lambda(z) \ \textit{monotone}, \ |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

for all $z \in [a,b]$, $0 < a < b < \infty$. *Let* Φ *be a smooth function on* $[a,b]$.
Then

$$\left| \int_a^b \exp\left({\rm i} \Lambda(z) \tau \right) \Phi(z) \ {\rm d}z \right| \leq c \frac{1}{\tau \Lambda_0} \left[|\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \ {\rm d}z \right],$$

where c *is an absolute constant* *independent* *of the specific shape* Λ *and* Φ.

Decay estimates

$L^p - L^q$ estimates

$$\|Z(\tau,\cdot,k,\omega)\|_{L^p(R^2)} \le c(\psi,p,k)\max\left\{\frac{1}{\omega\tau^{1-\beta/2}};\frac{1}{\tau^{\beta/2}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$

$$\text{for } p\ge 2,\ \frac{1}{p}+\frac{1}{p'}=1,\ \beta>0,\ \lambda_j\ne 0.$$

Scaling

$$\omega\approx\varepsilon^{m-1},\ \tau\approx t/\varepsilon^m$$

Dispersive decay

$$\left\|Z\left(\frac{t}{\varepsilon^m},\cdot,k,\omega\right)\right\|_{L^p(R^2)} \le c\,\varepsilon^{\frac{1}{2}-\frac{1}{p}}\max\left\{\frac{1}{t^{1-1/2m}};\frac{1}{t^{1/2m}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$

Navier-Stokes-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Internal energy balance

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Constitutive relations

Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamics stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Local well posedness

Initial data

$$\varrho(0,\cdot) = \varrho_0 > 0, \ \vartheta(0,\cdot) = \vartheta_0 > 0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

Regularity

$$\varrho, \ \vartheta, \ \mathbf{u} \in W^{m,2}, \ m \geq 3$$

Local existence for viscous fluids - Navier-Stokes-Fourier system

A. Valli, W.Zajaczkowski [1982] - local existence for large data, A.Matsumura, T.Nishida [1980,1983] - global existence for small data

Local existence for ideal (inviscid) fluids - Euler-Fourier system

T. Alazard [2006] - local existence for large data

Several "equivalent" forms of energy balance

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \boxed{p \mathrm{div}_x \mathbf{u}}$$

Entropy production

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \boxed{=} \frac{1}{\vartheta} \left(\boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Total energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) \mathbf{u} + p \mathbf{u} \right] + \mathrm{div}_x \mathbf{q}$$

$$= -\boxed{\mathrm{div}_x (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u})}$$

Weak formulation

Second law - entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

First law - total energy balance

$$\partial_t \int \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)\ \mathrm{d}x = 0$$

Relative entropy (energy)

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)\Big)$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Weak solutions - summary

Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes-Fourier system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

Compatibility

Regular weak solutions are strong solutions

Weak-strong uniqueness

Weak and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

Conditional regularity

Sufficient condition for regularity

Suppose that a dissipative weak solution to the Navier-Stokes-Fourier system emanating from regular initial data satisfies

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} < \infty.$$

Then the solution is regular in $(0, T)$.

Example of a singular limit for the full system

Oberbeck-Boussinesq approximation

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = \mu \Delta \mathbf{v} + r \nabla_x G$$

$$\partial_t \Theta + \mathbf{v} \cdot \nabla_x \Theta - \alpha \mathrm{div}_x (\mathbf{v} G) = \kappa \Delta \Theta$$

Boussinesq relation

$$r + \beta \Theta = 0$$

Primitive system

Full Navier-Stokes-Fourier system

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \boxed{\frac{1}{\varepsilon^2}} p(\varrho_\varepsilon, \vartheta_\varepsilon) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \varrho \nabla_x G$$

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}_x(\varrho s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon)}{\vartheta_\varepsilon} \right)$$

$$\geq \frac{1}{\vartheta_\varepsilon} \left(\boxed{\varepsilon^2} \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon)}{\vartheta_\varepsilon} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\boxed{\frac{\varepsilon^2}{2}} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) - \boxed{\varepsilon} \varrho_\varepsilon G \right) \, \mathrm{d}x = 0$$

Spatial domain

Gravitational potential

$$\Omega \subset R^3, \text{ unbounded}$$

$$-\Delta G = m \text{ in } R^3, \ \mathrm{supp}[m] \subset R^3 \setminus \Omega$$

Entropy inequality and uniform bounds

Entropy production equation

$$\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)) + \mathrm{div}_x(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon)}{\vartheta_\varepsilon}\right) = \sigma_\varepsilon$$

$$\sigma_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\left(\varepsilon^2 \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right)$$

Relative entropy inequality

$$\left[\mathcal{E}_\varepsilon\left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \Theta, \mathbf{U}\right)\right]_{t=0}^{\tau} + \frac{1}{\varepsilon^2}\sigma_\varepsilon|_{[0,\tau]}$$

$$\boxed{=} \int_0^\tau \mathcal{R}_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon, r, \Theta, \mathbf{U})\ \mathrm{d}t$$

Ill prepared initial data

Density

$$\varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \boxed{\varrho^{(1)}_{0,\varepsilon}}$$

Temperature

$$\vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \boxed{\vartheta^{(1)}_{0,\varepsilon}}$$

Velocity

$$\mathbf{u}_{0,\varepsilon} = \mathbf{v}_{0,\varepsilon} + \boxed{\nabla_x \Psi_\varepsilon}$$

Acoustic analogy

Acoustic analogy

$$\varepsilon \partial_t \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right) + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(p_\varrho(\overline{\varrho}, \overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + p_\vartheta(\overline{\varrho}, \overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \overline{\varrho} F \right) = o(\varepsilon)$$

$$\varepsilon \partial_t \left(s_\varrho(\overline{\varrho}, \overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + s_\vartheta(\overline{\varrho}, \overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right) = \sigma_\varepsilon + o(\varepsilon)$$

Thermodynamics and acoustic equation

$$\frac{\partial s(\varrho, \vartheta)}{\partial \varrho} = - \frac{1}{\varrho^2} \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta}$$

$$\varepsilon \partial_t Z + \mathrm{div}_x \mathbf{V} = 0, \ \ \varepsilon \partial_t \mathbf{V} + \nabla_x Z = 0$$

$$Z = p_\varrho(\overline{\varrho}, \overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + p_\vartheta(\overline{\varrho}, \overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \overline{\varrho} F$$

Heat-entropy equation

Heat equation

$$\partial_t \left(s_\varrho(\overline{\varrho},\overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + s_\vartheta(\overline{\varrho},\overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right)$$

$$+ \mathrm{div}_x \left[\left(s_\varrho(\overline{\varrho},\overline{\vartheta}) \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + s_\vartheta(\overline{\varrho},\overline{\vartheta}) \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right) \mathbf{u} \right]$$

$$- \tilde{\kappa}(\overline{\vartheta}) \Delta \left(\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right) = o(\varepsilon)$$

Asymptotic limit

Velocity

$$\mathbf{u}_\varepsilon \to \mathbf{v}$$

Density deviation

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r$$

Temperature deviation

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta$$

Other problems...

Related issues

- stratified fluid, acoustic equations, problems with "vacuum"
- limit passage "weak $\rightarrow$ weak"
- bounded (periodic) domains
- boundary conditions, no-slip and the boundary layer problems

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