INITIAL-BOUNDARY VALUE PROBLEM FOR GENERALIZED STOKES EQUATIONS

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. The paper is concerned with the solvability theory of the generalized Stokes equations arising in the study of the motion of non-Newtonian fluids.

Keywords: Stokes system, non-Newtonian fluids, Schauder estimates

MSC 2000: 35Q30, 76D03

1. Introduction

We consider the initial-boundary value problem

$$\vec{v}_t + A\Big(x,t,\frac{\partial}{\partial x}\Big)\vec{v} + \nabla p = \vec{f}(x,t), \quad \nabla\cdot\vec{v}=0, \quad x\in\Omega\subset\mathbb{R}^3, \quad t\in(0,T), \tag{1.1}$$

$$\vec{v}(x,0)=\vec{v}_0(x), \quad \vec{v}(x,t)\big|_{x\in\partial\Omega}=0 \tag{1.2}$$

where unknown are a vector field $\vec{v}(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))$ and a function $p(x,t)$. By A we mean a matrix-formed differential second order elliptic operator with real coefficients and by A_0 we mean its principal part, i.e. the sum of all terms in A containing derivatives of the second order. We assume that the matrix $A_0(x,t,\mathrm{i}\xi)$ is positive definite for arbitrary $\xi\in\mathbb{R}^3$ and arbitrary fixed $x\in\Omega,\ t\in[0,T]$. The domain Ω is bounded.

When $A=-\nabla^2 I$, then (1.1) is the well known Stokes system.

Our main result is the following existence theorem for the problem (1.1), (1.2).

Theorem 1. *Assume that $\partial\Omega\in C^{2+\alpha}$, $\alpha\in(0,1)$, that the coefficients of A belong to $C^{\alpha,\alpha/2}(\Omega\times(0,T))$, $T>0$, and that the leading coefficients satisfy the Hölder condition with respect to t with an exponent $\alpha_1/2$, $\alpha_1>\alpha$, and belong to*

$W^1_q(\Omega)$, $q > 3$, *for all* $t \in [0,T]$. *Let the data* $\vec{f}(x,t)$, $\vec{v}_0(x)$ *possess the following properties:*

1. $\vec{f} \in C^{\alpha,\alpha/2}(\Omega \times (0,T))$, $\nabla \cdot \vec{f} = 0$ *(in the weak sense)*, $\vec{f} \cdot \vec{n}|_{x\in\partial\Omega} = 0$ *(*$\vec{n}$ *is the unit interior normal to* $\partial\Omega$*),*
2. $\vec{v}_0 \in C^{2+\alpha}(\Omega)$, $\nabla \cdot \vec{v}_0 = 0$,
3. *the following compatibility conditions are satisfied:*

$$\vec{v}_0(x)\big|_{x\in\partial\Omega} = 0, \qquad A\Big(x, 0, \frac{\partial}{\partial x}\Big)\vec{v}_0(x) + \nabla p_0(x) - \vec{f}(x,0)\big|_{x\in\partial\Omega} = 0,$$

where p_0 *is a solution of the Neumann problem*

$$\begin{aligned} \nabla^2 p_0(x) &= -\nabla \cdot A\Big(x, 0, \frac{\partial}{\partial x}\Big)\vec{v}_0(x), \qquad x \in \Omega, \\ \frac{\partial p_0}{\partial n}\big|_{x\in\partial\Omega} &= -\vec{n} \cdot A\Big(x, 0, \frac{\partial}{\partial x}\Big)\vec{v}_0(x)\big|_{x\in\partial\Omega}. \end{aligned}$$

Then the problem (1.1), (1.2) *has a unique solution* $\vec{v} \in C^{2+\alpha,1+\alpha/2}(\Omega \times (0,T))$, $\nabla p \in C^{\alpha,\alpha/2}(\Omega \times (0,T))$, *and the solution satisfies the inequality*

$$\begin{aligned} &|\vec{v}|_{C^{2+\alpha,1+\alpha/2}(\Omega\times(0,T))} + |\nabla p|_{C^{\alpha,\alpha/2}(\Omega\times(0,T))} \\ &\qquad \leqslant c(|\vec{f}|_{C^{\alpha,\alpha/2}(\Omega\times(0,T))} + |\vec{v}_0|_{C^{2+\alpha}(\Omega)}). \end{aligned} \tag{1.3}$$

A similar theorem holds in the two-dimensional case.

We recall the definition of the norms in $C^l(\Omega)$ and in $C^{l,l/2}(\Omega \times (0,T))$ ($l = [l] + \lambda$, $\lambda \in (0,1)$):

$$\begin{aligned} |u|_{C^l(\Omega)} &= [u]^{(l)}_\Omega + \sum_{|j|<l} \sup_\Omega |D^j u(x)|, \quad [u]^{(l)}_\Omega = \sum_{|j|=[l]} \sup_{x,y\in\Omega} \frac{|D^j_x u(x) - D^j_y u(y)|}{|x-y|^{l-[l]}}, \\ |u|_{C^{l,l/2}(\Omega\times(0,T))} &= \sum_{|j|+2k<l} \sup_{\Omega\times(0,T)} |D^k_t D^j_x u(x,t)| + [u]^{(l,l/2)}_{\Omega\times(0,T)}, \\ [u]^{(l,l/2)}_{\Omega\times(0,T)} &= \sup_{t<T} [u(\cdot,t)]^{(l)}_\Omega + \sup_\Omega [u(x,\cdot)]^{(l/2)}_{(0,T)}. \end{aligned}$$

It is known that if $u \in C^{l,l/2}(\Omega \times (0,T))$, then $D^j_x D^k_t u \in C^{r,r/2}(\Omega \times (0,T))$, $r = l - |j| - 2k$, provided that $r > 0$.

Theorem 1 can be used for the study of differentiability properties of solutions of nonlinear equations

$$\vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} - \nabla \mathcal{P}(S(\vec{v})) + \nabla p = \vec{f}, \qquad \nabla \cdot \vec{v} = 0, \tag{1.4}$$

where the tensor $\mathcal{P}$ is a nonlinear function of the rate-of-strain tensor $S(\vec{v}) = \nabla\vec{v} + (\nabla\vec{v})^T$ (see [3–6]). In particular, one of its consequences is the local existence theorem for the Cauchy-Dirichlet problem (1.4), (1.2) (under appropriate assumptions concerning $\mathcal{P}$).

For the Stokes system Theorem 1 is proved in [7–9]. The proof is based on the analysis of the Cauchy and Cauchy-Dirichlet problems for the system with constant coefficients containing only the highest order terms. This analysis is sketched in Sections 2 and 3. Section 4 contains some comments on the Schauder procedure for the problem (1.1), (1.2).

The work was done at the Center of Mathematics and Fundamental Applications of the University of Lisbon and at the Max Planck Institute for Mathematics in Natural Sciences. The author is thankful to both these institutions for hospitality.

2. Cauchy problem and fundamental solution for model equations

The solution of the Cauchy problem

$$\vec{v}_t + A_0\Big(\frac{\partial}{\partial x}\Big)\vec{v} + \nabla p = \vec{f}(x,t), \quad \nabla\cdot\vec{v} = 0, \qquad x\in\mathbb{R}^3,\ t>0, \tag{2.1}$$

$$\vec{v}(x,0) = \vec{v}_0(x) \tag{2.2}$$

can be expressed as the sum of potentials

$$\begin{aligned} v_k(x,t) &= \sum_{m=1}^{3}\int_0^t\int_{\mathbb{R}^3} T_{km}(x-y,t-\tau)f_m(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \sum_{m=1}^{3}\int_{\mathbb{R}^3} T_{km}(x-y,t)v_{0m}(y)\,\mathrm{d}y, \\ p(x,t) &= \sum_{m=1}^{3}\frac{\partial}{\partial x_m}\int_{\mathbb{R}^3} E(x-y)f_m(y,t)\,\mathrm{d}y \\ &\quad + \sum_{m=1}^{3}\int_0^t\int_{\mathbb{R}^3} T'_{4m}(x-y,t-\tau)f_m(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \sum_{m=1}^{3}\int_{\mathbb{R}^3} T'_{4m}(x-y,t)v_{0m}(y)\,\mathrm{d}y \end{aligned} \tag{2.3}$$

where $T_{km}(x,t)$ are elements of the fundamental matrix of solutions of the system (2.1) and $E(x) = -(4\pi|x|)^{-1}$ is a fundamental solution of the Laplace equation. The fundamental matrix is defined in a standard way. Let us write (2.1) in the form

$$\mathcal{L}\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\Big)V = F$$

where $V = (\vec{v}, p)$, $F = (\vec{f}, 0)$ and $\mathcal{L}$ is 4×4 matrix differential operator whose characteristic matrix is given by

$$\mathcal{L}(\mathrm{i}\xi, s) = \begin{pmatrix} s + a_{11}(\mathrm{i}\xi) & a_{12}(\mathrm{i}\xi) & a_{13}(\mathrm{i}\xi) & \mathrm{i}\xi_1 \\ a_{21}(\mathrm{i}\xi) & s + a_{22}(\mathrm{i}\xi) & a_{23}(\mathrm{i}\xi) & \mathrm{i}\xi_2 \\ a_{31}(\mathrm{i}\xi) & a_{32}(\mathrm{i}\xi) & s + a_{33}(\mathrm{i}\xi) & \mathrm{i}\xi_3 \\ \mathrm{i}\xi_1 & \mathrm{i}\xi_2 & \mathrm{i}\xi_3 & 0 \end{pmatrix},$$

where a_{km} are elements of $A_0(\mathrm{i}\xi)$. It is easy to see that

$$\det \mathcal{L}(\mathrm{i}\xi, s) \equiv L(\mathrm{i}\xi, s) = s^2|\xi|^2 + s(|\xi|^2 \mathrm{Sp} A_0(\mathrm{i}\xi) - \xi \cdot A_0(\mathrm{i}\xi)\xi)) + \xi \cdot \hat{A}(\mathrm{i}\xi)\xi,$$

where $\mathrm{Sp} A_0 = a_{11} + a_{22} + a_{33}$ and $\hat{A} = A_0^{-1} \det A_0$ is the adjugate matrix of A_0. The roots of the polynomial $L(\mathrm{i}\xi, s)$ with respect to s have negative real parts for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$:

$$\mathrm{Re}\ s_k \leqslant -\delta|\xi|^2, \qquad \delta > 0,$$

and

$$|L(\mathrm{i}\xi, s)| > c|\xi|^2(|s| + |\xi|^2)^2,$$

if $\xi \in \mathbb{R}^3 \setminus \{0\}$ and

$$\mathrm{Re}\ s > -\kappa|\mathrm{Im}\ s| - \delta|\xi|^2 \tag{2.4}$$

with a small $\kappa > 0$.

The fundamental matrix is defined by

$$T_{km}(x,t) = (FL)^{-1} \frac{\hat{L}_{km}(\mathrm{i}\xi, s)}{L(\mathrm{i}\xi, s)}$$

where FL means the Fourier-Laplace transform (Fourier with respect to x, Laplace with respect to time), and $(FL)^{-1}$ stands for the inverse transformation. $\hat{L}_{km}$ are elements of the adjugate matrix $\hat{\mathcal{L}} = L\mathcal{L}^{-1}$.

If $k, m = 1, 2, 3$, then $T_{km}(x,t)$ satisfy the inequalities

$$|D_x^j T_{km}(x,t)| \leqslant c(|j|)(|x|^2 + t)^{-(3+|j|)/2}, \qquad \forall t > 0, \tag{2.5}$$

and $T_{km}(x,t) = 0$ for $t < 0$. $T_{4m}(x,t)$ contain the Dirac δ-function $\delta(t)$, namely,

$$\begin{gathered} T_{4m}(x,t) = \delta(t)\frac{\partial}{\partial x_m}E(x) + T'_{4m}(x,t), \\ |D_x^j T'_{4m}(x,t)| \leqslant c(|j|)(|x|^2 + t)^{-(4+|j|)/2}, \qquad \forall t > 0, \end{gathered} \tag{2.6}$$

and $T'_{4m}(x,t) = 0$ for $t < 0$.

It follows from (2.5), (2.6) that the solution of the problem (2.1), (2.2) satisfies the inequality

$$\begin{aligned} \sup_{t<T}[\vec{v}_t(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3} + \sup_{t<T}[\vec{v}(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^3} + \sup_{t<T}[p(\cdot,t)]^{(1+\alpha)}_{\mathbb{R}^3} \\ \leqslant c(\sup_{t<T}[\vec{f}(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3} + [\vec{v}_0]^{(2+\alpha)}_{\mathbb{R}^3}), \end{aligned} \tag{2.7}$$

and if $\nabla\cdot\vec{f} = 0$, then

$$[\vec{v}]^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^3\times(0,T)} + [\nabla p]^{(\alpha,\alpha/2)}_{\mathbb{R}^3\times(0,T)} \leqslant c([\vec{f}]^{(\alpha,\alpha/2)}_{\mathbb{R}^3\times(0,T)} + [\vec{v}_0]^{(2+\alpha)}_{\mathbb{R}^3}). \tag{2.8}$$

The number $T > 0$ is arbitrary, and the constants in (2.7), (2.8) are independent of T.

These results are obtained in [11].

3. Model problem in the half space

Let us consider the problem

$$\vec{v}_t + A_0\Big(\frac{\partial}{\partial x}\Big)\vec{v} + \nabla p = 0, \qquad \nabla\cdot\vec{v} = 0, \qquad x\in\mathbb{R}^3_+,\ t>0, \tag{3.1}$$

$$\vec{v}(x,0) = 0, \qquad \vec{v}\big|_{x_3=0} = \vec{b}(x',t) \tag{3.2}$$

in the half space $\mathbb{R}^3_+ = \{x_3 > 0\}$.

Theorem 2. *For arbitrary smooth $\vec{b}(x',t)$, $x' = (x_1,x_2)$, decaying at infinity sufficiently rapidly and satisfying the conditions $\vec{b}(x,0) = \vec{b}_t(x,0) = 0$,*

$$b_{3t}(x't) = \nabla'\cdot\vec{B}'(x',t) \equiv \frac{\partial B_1(x',t)}{\partial x_1} + \frac{\partial B_2(x',t)}{\partial x_2}, \qquad \vec{B}'(x',0) = 0, \tag{3.3}$$

the problem (3.1), (3.2) has a unique solution $\vec{v}\in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+\times(0,T))$, $\nabla p\in C^{\alpha,\alpha/2}(\mathbb{R}^3_+\times(0,T))$, $\forall T>0$, and this solution satisfies the inequality

$$\begin{aligned} [\vec{v}]^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^3_+\times(0,T)} + [\nabla p]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,T)} \\ \leqslant c([\vec{b}']^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^2\times(0,T)} + \sup_{t<T}[b_3(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^2} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)}). \end{aligned} \tag{3.4}$$

The proof of this theorem which is the central point of the analysis of the problem (1.1), (1.2) proceeds in several steps. First of all, we perform the Fourier transform with respect to x' (which we denote by the symbol F') and the Laplace transform with respect to t and reduce the problem (3.1), (3.2) to the boundary value problem for the system of ordinary differential equations:

$$
\begin{gathered}
s\hat{\vec{v}} + \hat{A}\hat{\vec{v}} + \hat{\nabla}\hat{p} = 0, \qquad \hat{\nabla}\cdot\hat{\vec{v}} = 0, \qquad x_3 > 0,\\
\hat{\vec{v}}(x_3),\ \hat{p}(x_3) \to 0, \quad x_3 \to \infty, \quad \hat{\vec{v}}\big|_{x_3=0} = \hat{\vec{b}}
\end{gathered}
$$

where $\hat{u} = F'Lu$, $\hat{\nabla} = (\mathrm{i}\xi_1, \mathrm{i}\xi_2, \frac{\mathrm{d}}{\mathrm{d}x_3})$, $\hat{A} = A_0(\mathrm{i}\xi', \frac{\mathrm{d}}{\mathrm{d}x_3})$. It can be shown that this problem is uniquely solvable for arbitrary $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and arbitrary $s \in \mathbb{C}$ satisfying the condition

$$\mathrm{Re}s + \kappa|\mathrm{Im}s| \geqslant -\delta_1|\xi'|^2 \tag{3.5}$$

with small $\kappa, \delta_1 > 0$. We look for a solution in the form

$$\hat{v}_k(x_3) = \frac{1}{2\pi}\sum_{m=1}^{3}\int_{-\infty}^{\infty}\frac{\hat{L}_{km}(\mathrm{i}\xi', \mathrm{i}\xi_3, s)}{L(\mathrm{i}\xi', \mathrm{i}\xi_3, s)}\mathrm{e}^{\mathrm{i}x_3\xi_3}\,\mathrm{d}\xi_3\ \hat{h}_m, \qquad k = 1, 2, 3, \tag{3.6}$$

$$\hat{p}(x_3) = \frac{1}{2\pi}\sum_{m=1}^{3}\int_{-\infty}^{\infty}\frac{\hat{L}_{4m}(\mathrm{i}\xi', \mathrm{i}\xi_3, s)}{L(\mathrm{i}\xi', \mathrm{i}\xi_3, s)}\mathrm{e}^{\mathrm{i}x_3\xi_3}\,\mathrm{d}\xi_3\ \hat{h}_m. \tag{3.7}$$

Boundary conditions $\hat{\vec{v}}|_{x_3=0} = \hat{\vec{b}}$ lead to a linear algebraic system

$$U\hat{\vec{h}} = \hat{\vec{b}} \tag{3.8}$$

where U is 3×3 matrix with the elements

$$U_{km} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{L}_{km}(\mathrm{i}\xi', \mathrm{i}\xi_3, s)}{L(\mathrm{i}\xi', \mathrm{i}\xi_3, s)}\,\mathrm{d}\xi_3, \qquad k, m = 1, 2, 3.$$

Proposition 3.1. *If $\xi' \in \mathbb{R}^2 \setminus \{0\}$ and $s \in \mathbb{C}$ satisfy the condition* (3.5), *then the system* (3.8) *is uniquely solvable. The elements U^{km}, $k + m < 6$, of the inverse matrix U^{-1} are representable in the form*

$$U^{km}(\xi', s) = u^{km}(\xi', s) + v^{km}(\xi', s)$$

where u^{km} and v^{km} possess the following properties:

(i) *$u^{km}(\xi', s)$ are analytic functions with respect to s and to the first two arguments which can take complex values $\zeta_j = \xi_j + \eta_j$, $j = 1, 2$, provided that condition* (3.5) *is satisfied and*

$$|\eta| \leqslant \delta_2(|\xi'|^2 + |s|)^{1/2} \tag{3.9}$$

with a certain small $\delta_2 > 0$,

(ii) $u^{km}(\xi', s)$ *are homogeneous functions of the first order:*

$$u^{km}(\lambda\xi', \lambda^2 s) = \lambda u^{km}(\xi', s), \qquad \forall\lambda > 0,$$

and they satisfy the inequality

$$|u^{km}(\zeta', s)| \leqslant c(|s| + |\xi'|^2)^{1/2} \tag{3.10}$$

in the domain (3.5), (3.9),

(iii) $u^{k3}(\xi', s) = u^{3k}(\xi', s) = 0, \quad k = 1, 2,$

(iv) $v^{km}(\xi', s)$ *are linear combinations of the functions*

$$\frac{\xi_\beta \xi_\gamma}{|\xi'|} w_{\beta\gamma}(\xi', s), \qquad \beta, \gamma = 1, 2,$$

where $w_{\beta\gamma}(\zeta', s) \equiv w^{km}_{\beta\gamma}$ *are analytic with respect to* s, ζ_1, ζ_2 *if* $s \in C$ *satisfies the condition* (3.5) *and*

$$|\eta| \leqslant \delta_2 |\xi'|, \tag{3.11}$$

(v) $w_{\beta\gamma}(\zeta', s)$ *are homogeneous of order zero:* $w_{\beta\gamma}(\lambda\xi', \lambda^2 s) = w_{\beta\gamma}(\xi', s)$ *and*

$$|w_{\beta\gamma}(\xi', s)| \leqslant c$$

in the domain (3.5), (3.11);

finally,

$$U^{33}(\xi', s) = \frac{1}{|\xi'|}(u^{33}(\xi', s) + v^{33}(\xi', s))$$

where $u^{33}(\xi', s)$, $v^{33}(\xi', s)$ *possess the same properties as* $u^{km}(\xi', s)$, $v^{km}(\xi', s)$, $k, m = 1, 2$, *but* u^{33} *are homogeneous functions of the second order,* $w^{33}_{\beta\gamma}$ *are homogeneous of the first order, and*

$$|u^{33}(\zeta', s)| \leqslant c(|s| + |\xi'|^2),$$
$$|w^{33}_{\beta\gamma}(\zeta', s)| \leqslant c(|s| + |\xi'|^2)^{1/2},$$

in the domains (3.5), (3.9) *and* (3.5), (3.11), *respectively.*

We cannot give the proof of this proposition because of the lack of space.

Formulas (3.6), (3.7) are equivalent to the representation of $(\vec{v}, p)$ in the form of the simple layer potential

$$
\begin{aligned}
v_k(x,t) &= \sum_{m=1}^{3} \int_0^t \int_{\mathbb{R}^2} T_{km}(x'-y', x_3, t-\tau) h_m(y',\tau)\,\mathrm{d}y'\,\mathrm{d}\tau, \qquad k=1,2,3,\\
p(x,t) &= \int_{\mathbb{R}^2} \nabla E(x'-y', x_3) \cdot \vec{h}(y',t)\,\mathrm{d}y'\\
&\quad + \sum_{m=1}^{3} \int_0^t \int_{\mathbb{R}^2} T'_{4m}(x'-y', x_3, t-\tau) h_m(y',\tau)\,\mathrm{d}y'\,\mathrm{d}\tau,
\end{aligned}
$$

for which the following proposition can be proved.

Proposition 3.2. *The estimate*

$$
\begin{aligned}
&\sup_{t<T} [\vec{v}_t(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T} [\vec{v}(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T} [p(\cdot,t)]^{(1+\alpha)}_{\mathbb{R}^3_+}\\
&\leqslant c \sum_{j=1}^{2} \Big([h_j]^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)} + \Big[\frac{\partial h_3}{\partial x_j}\Big]^{(\alpha,\alpha/2)}_{\mathbb{R}^2\times(0,T)} \Big)
\end{aligned}
\tag{3.12}
$$

holds with a constant independent of T.

For $\hat{h}_k$ we have the formula

$$
\hat{h}_k = \sum_{m=1}^{3} U^{km}(\xi', s)\hat{b}_m, \qquad k=1,2,3,
$$

and, as a consequence,

$$
\begin{aligned}
\hat{h}_\beta &= \sum_{\gamma=1}^{2} \Big(\frac{u^{\beta\gamma} + v^{\beta\gamma}}{r^2} r^2 \hat{b}_\gamma + \frac{v^{\beta 3}}{r^2} \mathrm{i}\xi_\gamma \hat{D}_\gamma \Big), \quad \beta = 1,2,\\
\mathrm{i}\xi_\beta \hat{h}_3 &= \sum_{\gamma=1}^{2} \Big(\frac{\mathrm{i}\xi_\beta}{|\xi'|} \frac{v^{3\gamma}}{r^2} r^2 \hat{b}_\gamma - \frac{\xi_\beta \xi_\gamma}{|\xi'|} \frac{u^{33} + v^{33}}{r^3} r \hat{D}_\gamma \Big)
\end{aligned}
$$

where $r = \sqrt{s + |\xi'|^2}$, $\hat{D}_\gamma = \hat{B}_\gamma - \mathrm{i}\xi_\gamma \hat{b}_3$ (we have used the condition (3.3): $s\hat{b}_3 = \mathrm{i}\xi_1 \hat{B}_1 + \mathrm{i}\xi_2 \hat{B}_2$). Making the inverse Fourier-Laplace transformation we represent h_1, h_2, h_{3x_β} in the form of sums of potentials with the kernels

$$
\begin{aligned}
P_{\beta\gamma}(x',t) &= (F'L)^{-1} \frac{u^{\beta\gamma}}{r^2},\\
Q_{km}(x',t) &= (F'L)^{-1} \frac{v^{km}}{r^2}, \ k+m<6\\
\frac{\partial}{\partial x_\gamma} R_\beta(x',t) &= (F'L)^{-1} \frac{\xi_\beta \xi\gamma}{|\xi'|} \frac{u^{33} + v^{33}}{r^3}.
\end{aligned}
$$

We show that

$$
\begin{aligned}
|D_t^k D_x^j P_{\beta\gamma}(x',t)| &\leqslant ct^{-3/2-|j|/2-k}\mathrm{e}^{-b|x'|^2/t},\\
|D_x^j Q_{km}(x',t)| + |D_x^j R_\beta(x',t)| &\leqslant ct^{-1/2}(|x'|^2+t)^{-1-|j|/2},\\
|D_t D_x^j Q_{km}(x',t)| + |D_t D_x^j R_\beta(x',t)| &\leqslant ct^{-3/2}(|x'|^2+t)^{-1-|j|/2}, \qquad \forall t>0,
\end{aligned}
$$

which makes it possible to obtain the inequality

$$
\begin{aligned}
&\sum_{j=1}^{2}\Big([h_j]^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)} + \Big[\frac{\partial h_3}{\partial x_j}\Big]^{(\alpha,\alpha/2)}_{\mathbb{R}^2\times(0,T)}\Big)\\
&\qquad \leqslant c([\vec{b}']^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^2\times(0,T)} + \sup_{t<T}[b_3(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^2} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)}).
\end{aligned}
$$

Together with (3.12), this inequality yields

$$
\begin{aligned}
&\sup_{t<T}[\vec{v}_t(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T}[\vec{v}(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T}[p(\cdot,t)]^{(1+\alpha)}_{\mathbb{R}^3_+}\\
&\qquad \leqslant c([\vec{b}']^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^2\times(0,T)} + \sup_{t<T}[b_3(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^2} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)}).
\end{aligned}
$$

The next step is an estimate of the Hölder constant of ∇p with respect to t. We consider p as a solution of the Neumann problem

$$
\begin{aligned}
&\nabla^2 p = -\nabla\cdot A(\frac{\partial}{\partial x})\vec{v}(x,t), \quad x\in\mathbb{R}^3_+,\\
&\frac{\partial p}{\partial x_3} = -\Big(A\Big(\frac{\partial}{\partial x}\Big)\vec{v}(x,t)\Big)_3 - \nabla'\cdot\vec{B}'(x',t), \quad x_3=0,
\end{aligned}
$$

which is given by the formula

$$
p(x,t) = \int_{\mathbb{R}^3_+}\nabla_y N(x,y)\cdot A\Big(\frac{\partial}{\partial y}\Big)\vec{v}(y,t)\,\mathrm{d}y - \int_{\mathbb{R}^2} N(x,y',0)\nabla'\cdot\vec{B}(y',t)\,\mathrm{d}y'
$$

where $N(x,y)=E(x-y)+E(x-y^*)$, $y^*=(y_1,y_2,-y_3)$. Applying Lemma 8 in [8] and some interpolation inequalities, we obtain

$$
\begin{aligned}
\sup_{R^3_+}[\nabla p(x,\cdot)]^{(\alpha/2)}_{(0,T)} &\leqslant c(\sup_{t<T}[\vec{v}_t(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T}[\vec{v}(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^3_+} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)})\\
&\leqslant c([\vec{b}']^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^2\times(0,T)} + \sup_{t<T}[b_3(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^2} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)})
\end{aligned}
$$

and, in addition,

$$
\begin{aligned}
\langle p\rangle^{(\gamma,1+\alpha)} &\equiv \sup_{t\in(0,T)}\sup_{h\in(0,T-t)}\sup_{x,y\in\mathbb{R}^3_+}\frac{|\vec{p}(y,t+h)-\vec{p}(y,t)-\vec{p}(x,t+h)+\vec{p}(x,t)|}{h^{(1+\alpha-\gamma)/2}|x-y|^\gamma}\\
&\leqslant c([\vec{b}']^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^2\times(0,T)} + \sup_{t<T}[b_3(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^2} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)})
\end{aligned}
$$

where $\gamma \in (0,1)$.

The Hölder constant of $\vec{v}_t$ can now be estimated with help of the equation (3.1). Putting all the estimates together, we arrive at (3.4).

Theorem 2 and results of Sect. 2 make it possible to consider more general half space problems. We present the result for the case of homogeneous initial conditions:

$$\vec{v}_t + A_0\Big(\frac{\partial}{\partial x}\Big)\vec{v} + \nabla p = \vec{f}(x,t), \qquad \nabla\cdot\vec{v} = 0, \qquad x \in \mathbb{R}^3_+,\ t>0, \tag{3.13}$$

$$\vec{v}(x,0) = 0, \qquad \vec{v}\big|_{x_3=0} = \vec{b}(x',t). \tag{3.14}$$

Theorem 3. *If $\vec{f} \in C^{\alpha,\alpha/2}(\mathbb{R}^3_+ \times (0,T))$ decays at infinity sufficiently rapidly and satisfies the conditions $\nabla\cdot\vec{f}(x,t) = 0$, $f_3|_{x_3=0} = 0$, $\vec{f}(x,0) = 0$, and $\vec{b}$ satisfies the hypotheses of Theorem 2, then Problem* (3.13), (3.14) *has a unique solution $\vec{v} \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^3_+ \times (0,T))$, $\nabla p \in C^{\alpha,\alpha/2}(\mathbb{R}^3_+ \times (0,T))$, $\forall T>0$, and this solution satisfies the inequality*

$$\begin{aligned}[\vec{v}]^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^3_+\times(0,T)} &+ [\nabla p]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,T)} + \langle p\rangle^{(\gamma,1+\alpha)} \\ &\leqslant c([\vec{f}]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,T)} + [\vec{b}']^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^2\times(0,T)} + \sup_{t<T}[b_3(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^2} + [\vec{B}']^{(1+\alpha,(1+\alpha)/2)}_{\mathbb{R}^2\times(0,T)}).\end{aligned} \tag{3.15}$$

4. On the estimate (1.3)

In the conclusion, we say a few words about the proof of inequality (1.3). It is obtained by Schauder's method. It is clear that such operations as change of coordinates or multiplication by a cut-off function destroy the solenoidality property, so the results of Sect. 3 are not directly applicable. In what follows we give an idea of how this difficulty should be put under control.

For the Stokes equations Schauder's method was carried out in [8]; here we use the same kind of arguments. Let us estimate $\vec{v}(x,t)$ and $p(x,t)$ in the neighbourhood of an arbitrary point $x^{(0)} \in \partial\Omega$ on a small time interval $(0,t^{(0)})$ assuming for simplicity that $\vec{v}_0 = 0$. Without loss of generality it can be assumed that the point $x^{(0)}$ coincides with the origin of our coordinate system and that the x_3-axis is directed along the interior normal $\vec{n}(x^{(0)}) = \vec{n}(0)$. Let

$$x_3 = F(x_1,x_2) \equiv F(x'), \qquad |x'| = \sqrt{x_1^2+x_2^2} \leqslant d,$$

be the equation of $\partial\Omega$ in the neighbourhood of the origin, and let $\psi_\lambda(x)$, $\lambda \in (0,d/2)$, be a smooth cut-off function equal to one for $|x| \leqslant \lambda$, to zero for $|x| > 2\lambda$ and satisfying the inequalities

$$0 \leqslant \psi_\lambda(x) \leqslant 1, \qquad |D^j\psi_\lambda(x)| \leqslant c(j)\lambda^{-|j|}.$$

We make a change of variables near the origin according to the formula

$$y' = x', \qquad y_3 = x_3 - F(x'),$$

and introduce functions $\vec{u} = \psi_\lambda \vec{v}$, $q = \psi_\lambda p$. They satisfy the relations

$$\vec{u}_t + A_{00}\Big(\frac{\partial}{\partial y}\Big)\vec{u} + \nabla q = \vec{f}\psi_\lambda + (\nabla - \tilde{\nabla})q + (A_{00} - \tilde{A})\vec{u} + \vec{f}_1(y,t) \equiv \vec{h}(y,t), \tag{4.1}$$

$$\nabla \cdot \vec{u} = (\nabla - \tilde{\nabla}) \cdot \vec{u} + \vec{v} \cdot \tilde{\nabla}\psi_\lambda \equiv g(y,t), \tag{4.2}$$

$$\vec{u}|_{t=0} = 0, \qquad \vec{u}|_{y_3=0} = 0$$

where $\tilde{\nabla}$ is the transformed gradient: $\tilde{\nabla} = (\frac{\partial}{\partial y_1} - F'_{y_1}\frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_2} - F'_{y_2}\frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_3})$, $\tilde{A}$ is the transformed operator A, $A_{00}(\frac{\partial}{\partial y}) = A_0(0,0,\frac{\partial}{\partial y}) = \tilde{A}_0(0,0,\frac{\partial}{\partial y})$ and

$$\vec{f}_1 = \tilde{A}(\vec{v}\psi_\lambda) - \psi_\lambda \tilde{A}\vec{v} + p\tilde{\nabla}\psi_\lambda.$$

We extend $\vec{u}$ and q by zero into the domain $|y'| > 2\lambda,\ y_3 > 0$ and consider (4.1), (4.2) as equations in $\mathbb{R}^3_+$.

Now, following [8], we introduce functions $\vec{w}_1$ and Φ as solutions to the problems

$$\vec{w}_{1t} + A_{00}\Big(\frac{\partial}{\partial y}\Big)\vec{w}_1 = \vec{h}^*, \qquad y \in \mathbb{R}^3, \qquad \vec{w}_1|_{t=0} = 0,$$

$$\nabla^2\Phi = g - \nabla \cdot \vec{w}_1, \qquad y \in \mathbb{R}^3_+, \qquad \frac{\partial \Phi}{\partial x_3} = -w_{13}, \qquad y_3 = 0,$$

where $\vec{h}^*$ is an extension of $\vec{h}$ into $\mathbb{R}^3$ with the preservation of class, i.e. such that

$$[\vec{h}^*]^{(\alpha,\alpha/2)}_{\mathbb{R}^3\times(0,t)} \leqslant c[\vec{h}]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,t)}, \qquad \forall t > 0.$$

The first problem is a parabolic Cauchy problem, and $\vec{w}_1$ is given by

$$\vec{w}_1(y,t) = \int_0^t \int_{\mathbb{R}^3} \mathcal{Z}(y-\xi, t-\tau)\vec{h}^*(\xi,\tau)\,\mathrm{d}\tau$$

where $\mathcal{Z}$ is a fundamental matrix of solutions of the parabolic system $\vec{w}_t + A_{00}(\frac{\partial}{\partial y})\vec{w} = 0$. The second Neumann problem has a solution

$$\begin{aligned}
\Phi(y,t) &= \int_{\mathbb{R}^3_+} N(y,z)(g(z,t) - \nabla \cdot \vec{w}_1(z,t))\,\mathrm{d}z - \int_{\mathbb{R}^2} N(y,\xi',0)w_{13}(\xi',t)\,\mathrm{d}\xi' \\
&= \int_{\mathbb{R}^3_+} \nabla_z N(y,z) \cdot \vec{w}_1(z,t)\,\mathrm{d}z - \int_{\mathbb{R}^3_+} \frac{\partial N(y,z)}{\partial z_3}\nabla F(z') \cdot \vec{u}(z,t)\,\mathrm{d}z \\
&\quad + \int_{\mathbb{R}^3_+} N(y,z)\vec{v} \cdot \tilde{\nabla}\psi_\lambda\,\mathrm{d}z.
\end{aligned} \tag{4.3}$$

Further we set

$$\vec{w}_2 = \nabla\Phi, \qquad q_2 = -\Phi_t, \qquad \vec{w}_3 = \vec{u} - \vec{w}_1 - \vec{w}_2, \qquad q_3 = q - q_2.$$

For $\vec{w}_3,\ q_3$ we have

$$\begin{aligned}(4.4)\ \ \vec{w}_{3t} + A_{00}\Big(\frac{\partial}{\partial y}\Big)\vec{w}_3 + \nabla q_3 = -A_{00}\Big(\frac{\partial}{\partial y}\Big)\vec{w}_2, \qquad \nabla\cdot\vec{w}_3 = 0, \qquad y\in\mathbb{R}^3_+,\\ \vec{w}_3|_{t=0} = 0, \qquad \vec{w}_3|_{x_3=0} = -(\vec{w}_1+\vec{w}_2)\big|_{x_3=0} \equiv \vec{b}(y',t).\end{aligned}$$

It is easily seen that $b_3 = 0$.

As for the estimates, for $\vec{w}_1$ they follow from the results of S. D. Eidelman [1]:

$$[\vec{w}_1]^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^3_+\times(0,t)} \leqslant c[\vec{h}^*]^{(\alpha,\alpha/2)}_{\mathbb{R}^3\times(0,t)} \leqslant c[\vec{h}]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,t)},$$

and classical estimates for the solution of the Neumann problem imply

$$\sup_{\tau<t}[\vec{w}_2(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+} = \sup_{\tau<t}[\nabla\Phi(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+} \leqslant c(\sup_{\tau<t}[g(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\vec{w}_1(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+}).$$

The estimate of the Hölder constant of $\vec{w}_{2t} = \nabla\Phi_t$ with respect to t is slightly more complicated. We differentiate (4.3) with respect to t, make use of equations for $\vec{v},\ \vec{u},\ \vec{w}_1$ and observe that the terms containing $\vec{f}$ disappear, because

$$\begin{aligned}\int_{\mathbb{R}^3_+}\nabla_z N(y,z)\cdot\vec{f}(z,t)\psi_\lambda(z)\,\mathrm{d}z - \int_{\mathbb{R}^3_+}\frac{\partial N(y,z)}{\partial z_3}\nabla F(z')\cdot\vec{f}(z,t)\psi_\lambda(z)\,\mathrm{d}z\\ + \int_{\mathbb{R}^3_+}N(y,z)\vec{f}\cdot\tilde{\nabla}\psi_\lambda\,\mathrm{d}z = \int_{\mathbb{R}^3_+}\tilde{\nabla}_z\cdot(N(y,z)\vec{f}(z,t)\psi_\lambda(z))\,\mathrm{d}z\\ = -\int_{\mathbb{R}^2}N(y,z',0)(f_3 - \nabla F\cdot\vec{f})\,\mathrm{d}z' = 0.\end{aligned}$$

The remaining terms contain derivatives of $\vec{v},\ \vec{u},\ \vec{w}_1$ with respect to z_k or derivatives of the cut-off function ψ_λ. Generic principal term in the formula for $\vec{w}_{2t}$ has the form

$$\frac{\partial}{\partial y_k}\int_{\mathbb{R}^3_+}\frac{\partial N(y,z)}{\partial z_j}\vec{d}(z,t)\cdot\nabla H(z,t)\,\mathrm{d}z \equiv \omega(y,t)$$

where H may be equal to q, $\frac{\partial u_m}{\partial z_i}$ or to $\frac{\partial w_{1m}}{\partial z_i}$. Without loss of generality we may assume that $\vec{d}(z,t) = \mathrm{const}$ or $\vec{d}(z,t)$ has a compact support. We estimate such terms using the following proposition.

Proposition 4.1. *Assume that $\vec{d} = \text{const}$ or $\vec{d}$ has a compact support and that $\vec{d}$ belongs to $C^{\alpha}(\mathbb{R}^3_+)$ for arbitrary $t \in (0,T)$ and satisfies the Hölder condition with respect to time with an exponent $\alpha_1/2$, $\alpha_1 > \alpha$. Then*

$$\sup_{\mathbb{R}^3_+}[\omega(y,\cdot)]^{(\alpha/2)}_{(0,T)} \leqslant c_1\big(\sup_{\mathbb{R}^3_+}[\nabla H(y,\cdot)]^{(\alpha/2)}_{(0,T)} + \langle H\rangle^{(\gamma,1+\alpha)}\big) + c_2 \sup_{\tau<T} |\nabla H|_{C^{\beta}(0,T)} \tag{4.5}$$

with arbitrary $\beta \in (0,\alpha)$. The constants c_1 and c_2 are proportional to the norms $\sup\limits_{t<T} |\vec{d}(\cdot,t)|_{C^{\beta}(\mathbb{R}^3_+)}$ and $\sup\limits_{t<T} |\vec{d}(\cdot,t)|_{C^{\alpha}(\mathbb{R}^3_+)} + \sup\limits_{\mathbb{R}^3_+}[\vec{d}(z,\cdot)]^{(\alpha_1/2)}_{(0,T)}$, respectively.

This proposition is an analogue of Lemma 8 in [8] where the case of the constant $\vec{d}$ is considered (see also [11], Proposition 2.3 and Remark).

Let us turn to the problem (4.4). The second term $-A_{00}(\frac{\partial}{\partial y})\vec{w}_2$ is not solenoidal but it can be written as

$$\begin{aligned} -A_{00}\Big(\frac{\partial}{\partial y}\Big)\vec{w}_2 &= \vec{f}_2 + \nabla\chi, \\ \chi(y,t) &= \int_{\mathbb{R}^3_+} \nabla_z N(y,z)\cdot A_{00}\Big(\frac{\partial}{\partial z}\Big)\vec{w}_2(z,t)\,\mathrm{d}z, \end{aligned}$$

so that $\vec{f}_2$ is solenoidal and $f_{23} = 0$ for $z_3 = 0$. By virtue of Proposition 4.1,

$$[\vec{f}_2]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,t)} + [\nabla\chi]^{(\alpha,\alpha/2)}_{\mathbb{R}^3_+\times(0,t)} \leqslant c[\vec{w}_2]^{(2+\alpha,1+\alpha/2)}_{\mathbb{R}^3_+\times(0,t)},$$

hence we may incorporate χ into the pressure p_3 and apply Theorem 3. Putting all the above estimates together and making use of the smallness of λ and of $t^{(0)}$, we estimate higher order norms of $\vec{v}$ and p by the norm of $\vec{f}$ and by some weaker norms of the solution. This may be done in the neighbourhood of an arbitrary point $(x^{(0)}, t^{(0)}) \in \Omega \times (0,T)$, so in the end we obtain

$$\begin{aligned} &|\vec{v}|_{C^{2+\alpha,1+\alpha/2}(\Omega\times(0,T))} + |\nabla p|_{C^{\alpha,\alpha/2}(\Omega\times(0,T))} \\ &\qquad \leqslant c\big(|\vec{f}|_{C^{\alpha,\alpha/2}(\Omega\times(0,T))} + \sup_{\Omega}\sup_{t<T}|\vec{v}(x,t)| + \sup_{\Omega}|p(x,\cdot)|_{C^{\alpha/2}(0,T)}\big). \end{aligned}$$

The last norm we estimate considering p as a solution of the Neumann problem

$$\begin{aligned} \nabla^2 p(x,t) &= -\nabla\cdot A\Big(x,t,\frac{\partial}{\partial x}\Big)\vec{v}, \qquad x \in \Omega, \\ \frac{\partial p}{\partial n}\Big|_{x\in\partial\Omega} &= -\vec{n}\cdot A\Big(x,t,\frac{\partial}{\partial x}\Big)\vec{v}\Big|_{x\in\partial\Omega}. \end{aligned}$$

It can be written in the form

$$p(x,t) = \int_{\Omega} \nabla_y N_{\Omega}(x,y)\cdot A\Big(y,t,\frac{\partial}{\partial y}\Big)\vec{v}(y,t)\,\mathrm{d}y$$

where N_Ω is the Green function for the Neumann problem in Ω studied in [2] (see also [10]). We can prove the following proposition.

Proposition 4.2. *If the hypotheses of Theorem* 1 *are satisfied, then*

$$\begin{aligned} &|p(x,t)| \leqslant c \sum_{|j|\leqslant 2} \sup_{\Omega} |D_x^j \vec{v}(x,t)|, \\ &|p(x,t)-p(x,t')| \leqslant c\Big(|t-t'|^{\alpha/2} \sum_{|j|\leqslant 2} \sup_{\Omega} |D_x^j \vec{v}(x,t)| \\ &\qquad + \sum_{|j|\leqslant 1} \sup_{\Omega} |D_x^j \vec{v}(x,t) - D_x^j \vec{v}(x,t')|\Big). \end{aligned} \tag{4.6}$$

Now, making use of interpolation inequalities and of the Gronwall lemma, we easily arrive at (1.3).

In fact, the assumption that the leading coefficients of A belong to $W_q^1(\Omega)$, $q > 3$, is used only in the proof of (4.6) to prevent the appearence of the differences of the second derivatives of $\vec{v}$ on the right hand side. In the case of the Cauchy problem or of the problem with periodicity conditions these differences can be estimated by inequality (2.7) (also in the case of variable coefficients, see [11]), so the abovementioned assumption is not necessary.

This assumption can be replaced by the requirement that $A_0\vec{v}$ could be written in the form

$$A_0\Big(x,t,\frac{\partial}{\partial x}\Big)\vec{v}(x,t) = \nabla\cdot\mathcal{A}(x,t,\nabla\vec{v}) + \mathcal{A}_1(x,t,\vec{v},\nabla\vec{v})$$

and the coefficients of the operators $\mathcal{A}$, $\mathcal{A}_1$ belong to $C^{\alpha,\alpha/2}(\Omega\times(0,T))$.

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