ON THE THERMAL ASPECT OF DYNAMIC CONTACT PROBLEMS

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Dedicated to Professor J. Nečas on the occasion of his 70th birthday

Abstract. A short survey of available existence results for dynamic contact problems including heat generation and heat transfer is presented.

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1. Introduction

Dynamic contact problems with friction represent an important task of applied mathematics. Up to now, the only efficient method to solve them is due to J. Nečas (first employed in [9]); it uses an approximation of the original problem by certain auxiliary ones and the proof of a certain regularity of solutions which enables to pass to solutions of the original problem. While in the static or quasistatic version the elastic model is suitable for such a procedure, in the dynamic case some kind of viscosity seems to be necessary to dominate the friction term. For the physically well based contact condition in displacements no proof of sufficient regularity of velocities to the "auxiliary" solutions is available, and the problem with given friction was solved only in [5]. Hence, in the sequel we employ the unilateral condition in velocities.

Friction is naturally an important source of heat and should be considered in the problem. Of course, the viscous heat should not be neglected, either. The occurance of these two highly nonlinear terms together with the nonlinear deformation heat in

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the heat equation is the main difficulty in the existence proofs. These problems can be tackled for the following models:

1. a complete linearization of the three terms mentioned;
2. a model including a rapidly growing heat energy;
3. a model including an increasing temperature-dependent diffusion coefficient and heat radiation.

While in the first approach the growth of the three heat sources mentioned is limited, in the remaining models it is compensated by sufficient growth of the inner thermal energy. The constitutive laws employed do not violate any law of thermodynamics, but they need not be suitable for all materials.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ composed of three measurable, mutually disjoint parts Γ_U, Γ_F and Γ_C. Let $I_{\mathscr{T}} \equiv [0, \mathscr{T}]$ be a time interval, let $Q_{\mathscr{T}} \equiv I_{\mathscr{T}} \times \Omega$ denote the time-space domain and let $S_{\mathscr{T}} \equiv I_{\mathscr{T}} \times \Gamma$ be its lateral boundary consisting of the parts $S_{X,\mathscr{T}} \equiv I_{\mathscr{T}} \times \Gamma_X$ for $X = U, F, C$. For $\tau > 0$ we denote $I_\tau \equiv [0, \tau]$ and analogously Q_τ, S_τ etc. We look for a couple $[u, \Theta]$ of a displacement and a temperature such that the following relations are satisfied:

$$\ddot{u}_i - \sigma_{ij,j}(u, \Theta) = f_i, \quad i = 1, \ldots, N, \quad \text{in} \quad Q_{\mathscr{T}}, \tag{1}$$

$$u = U \quad \text{on} \quad S_{U,\mathscr{T}}, \tag{2}$$

$$T(u, \Theta) = h \quad \text{on} \quad S_{F,\mathscr{T}}, \tag{3}$$

$$\left.\begin{aligned} &\dot{u}_n \leqslant 0, \quad T_n \leqslant 0, \quad T_n \dot{u}_n = 0, \\ &\dot{u}_t = 0 \Rightarrow |T_t| \leqslant \mathscr{F}|T_n|, \\ &\dot{u}_t \neq 0 \Rightarrow T_t = -\mathscr{F}|T_n| \frac{\dot{u}_t}{|\dot{u}_t|} \end{aligned}\right\} \text{on} \quad S_{C,\mathscr{T}}, \tag{4}$$

$$u(0, x) = u_0(x), \ \dot{u}(0, x) = u_1(x) \quad \text{for} \quad x \in \Omega, \tag{5}$$

$$\dot{\Theta} - (c_{ij}\Theta_{,j})_{,i} = a^{(1)}_{ijk\ell} e_{ij}(\dot{u}) e_{k\ell}(\dot{u}) - b_{ij}\Theta \dot{u}_{i,j} \quad \text{in} \quad Q_{\mathscr{T}}, \tag{6}$$

$$\Theta = \Theta_0 \quad \text{on} \quad S_{U,\mathscr{T}}, \tag{7}$$

$$c_{ij}\Theta_{,j} n_i = K\ (\Upsilon - \Theta) \quad \text{on} \quad S_{F,\mathscr{T}}, \tag{8}$$

$$c_{ij}\Theta_{,j} n_i = \mathscr{F}|T_n||\dot{u}_t| + K\ (\Upsilon - \Theta) \quad \text{on} \quad S_{C,\mathscr{T}}, \tag{9}$$

$$\Theta(0, x) = \Theta_0(x) \quad \text{for} \quad x \in \Omega. \tag{10}$$

Here and in the sequel, the summation convention is employed. The derivative of a function v with respect to the space variable x_i is denoted by $v_{,i}$, while the time derivatives are denoted by dots. Moreover, n denotes the outer normal vector of the boundary, $T_i = \sigma_{ij} n_j$ the components of the boundary traction; the subscripts $_n$ and $_t$ denote the normal and tangential components of the corresponding vectors. The strain-stress relation is given by a linear thermoviscoelastic law of the Kelvin-Voight type,

$$\sigma_{ij} \equiv \sigma_{ij}(u,\Theta) = a^{(0)}_{ijk\ell} e_{k\ell}(u) + a^{(1)}_{ijk\ell} e_{k\ell}(\dot{u}) - b_{ij}\Theta, \quad i,j = 1,\ldots,N, \tag{11}$$

with $e_{ij}(u) \equiv \frac{1}{2}(u_{i,j} + u_{j,i})$. The tensors $\{a^{(0)}_{ijk\ell}\}$ and $\{a^{(1)}_{ijk\ell}\}$ are assumed to depend Lipschitz-continuously on the space variable and to be symmetric, i.e. $a^{(\iota)}_{ijk\ell} = a^{(\iota)}_{jik\ell} = a^{(\iota)}_{k\ell ij}$, as well as bounded and elliptic, i.e.

$$a^{(\iota)}_0 \xi_{ij}\xi_{ij} \leqslant a^{(\iota)}_{ijk\ell}\xi_{ij}\xi_{k\ell} \leqslant A^{(\iota)}_0 \xi_{ij}\xi_{ij} \tag{12}$$

for all symmetric tensors $\{\xi_{ij}\} \in \mathbb{R}^{N,N}$ with real constants $0 < a^{(\iota)}_0 \leqslant A^{(\iota)}_0$, $\iota = 0,1$. The tensor $\{b_{ij}\}$ of thermal expansion is symmetric, Lipschitz with respect to the space variable and globally bounded. The tensor of thermal conductivity c_{ij} is assumed to be symmetric. Due to its crucial role in the possible compensation of the nonlinear terms on the right hand side of (6) and (9), its further properties differ in the different models employed.

Let us present the weak formulation of the problem. The following notation for function spaces on a domain $M \subset \mathbb{R}^N$, $N > 1$, is employed: For $k \geqslant 0$ and $p \geqslant 1$, $W^k_p(M)$ denotes the Sobolev-Slobodetskii space, $\boldsymbol{W}^k_p(M) \equiv W^k_p(M;\mathbb{R}^N)$ and $H^k(M) \equiv W^k_2(M)$. The duals of these spaces are marked by asterisks. For $\alpha,\beta \geqslant 0$ and an interval I, let $H^{\alpha,\beta}(I \times M) \equiv H^\alpha(I; L_2(M)) \cap L_2(I; H^\beta(M))$. The sets of admissible functions are given by $\mathscr{C} \equiv \big\{v \in L_2\left(I_{\mathscr{T}}; \boldsymbol{H}^1(\Omega)\right);\, v = \dot{U}$ on $S_{U,\mathscr{T}}$, $v_n \leqslant 0$ a.e. on $S_{C,\mathscr{T}}\big\}$ for the contact problem and by $\mathfrak{V}$ for the heat equation. The precise form of the latter space depends on the choice of the model.

A weak solution of the problem shall be a couple $[u,\Theta]$ such that $u \in B_0\big(I_{\mathscr{T}}; \boldsymbol{H}^1(\Omega)\big)$ with $\dot{u} \in \big(B_0 \cap H^{\frac{1}{2}}\big)(I_{\mathscr{T}}; \boldsymbol{L}_2(\Omega)) \cap \mathscr{C}$ and $\ddot{u} \in L_2\big(I_{\mathscr{T}}; \boldsymbol{H}^{-1}(\Omega)\big) \cap H^{\frac{1}{2}}(I_{\mathscr{T}}; \boldsymbol{L}_2(\Omega))^*$ and $\Theta \in \Theta_0 + \mathfrak{V}$ such that the initial conditions (5) and (10) hold and for each $v \in \mathscr{C} \cap \boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}})$ and each $\varphi \in \mathfrak{V}$ the following relations are valid:

$$\begin{aligned}&\langle \ddot{u}_i, v_i - \dot{u}_i\rangle_{Q_{\mathscr{T}}} + \langle \sigma_{ij}(u,\Theta), e_{ij}(v-\dot{u})\rangle_{Q_{\mathscr{T}}}\\ &\qquad + \langle \mathscr{F}\left|T_n(u,\Theta)\right|, |v_t| - |\dot{u}_t|\rangle_{S_{C,\mathscr{T}}} \geqslant \mathscr{L}(v-u),\end{aligned} \tag{13}$$

$$\begin{aligned}&\big\langle \dot{\Theta}, \varphi\big\rangle_{Q_{\mathscr{T}}} + \langle c_{ij}\Theta_{,j}, \varphi_{,i}\rangle_{Q_{\mathscr{T}}} + \langle b_{ij}\,\Theta \dot{u}_{i,j}, \varphi\rangle_{Q_{\mathscr{T}}} + \langle K(\Theta - \Upsilon), \varphi\rangle_{S_{F,\mathscr{T}} \cup S_{C,\mathscr{T}}}\\ &\qquad = \big\langle a^{(1)}_{ijk\ell} e_{ij}(\dot{u}) e_{k\ell}(\dot{u}), \varphi\big\rangle_{Q_{\mathscr{T}}} + \langle \mathscr{F}|T_n(u,\Theta)||\dot{u}_t|, \varphi\rangle_{S_{C,\mathscr{T}}}\end{aligned} \tag{14}$$

with the linear form $\mathscr{L}\colon\ v \mapsto \int_{Q_{\mathscr{T}}} f_i v_i \,\mathrm{d}x + \int_{S_{F,\mathscr{T}}} h_i v_i \,\mathrm{d}x_s$. For a domain M, $\langle\cdot,\cdot\rangle_M$ denotes a generalized $L_2(M)$-duality pairing. The initial and boundary data U and Θ_0 are assumed to be extended onto the whole time-space cylinder $Q_{\mathscr{T}}$.

The models studied in the following sections differ in the heat equation only. The properties of the contact problem with given temperature are the same in all cases considered. This problem will be approximated with the penalty method. Thereby, the first row of (4) is replaced by the condition

$$T_n(u,\Theta) = -\delta^{-1}\dot{u}_n^+,\ \ \delta > 0, \tag{15}$$

where $y^+ = \max\{0,y\}$ for $y \in \mathbb{R}$. For such a modified problem, the cone $\mathscr{C}$ in the variational formulation is replaced by the set $\dot{U} + \mathfrak{U}$ with $\mathfrak{U} \equiv \{v \in \boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}});\ v = 0 \text{ on } S_{U,\mathscr{T}}\}$ and (13) is replaced by the variational inequality

$$\begin{aligned} \langle \ddot{u}_i, v_i - \dot{u}_i\rangle_{Q_{\mathscr{T}}} &+ \langle \sigma_{ij}(u,\Theta), e_{ij}(v-\dot{u})\rangle_{Q_{\mathscr{T}}} + \langle \delta^{-1}\dot{u}_n^+, v_n - \dot{u}_n\rangle_{S_{C,\mathscr{T}}} \\ &+ \langle \mathscr{F}\delta^{-1}\dot{u}_n^+, |v_t| - |\dot{u}_t|\rangle_{S_{C,\mathscr{T}}} \ \geqslant\ \mathscr{L}(v-u) \end{aligned} \tag{16}$$

valid for all $v \in \dot{U} + \mathfrak{U}$. For this problem, the following existence result holds:

Theorem 1. *In addition to the above mentioned conditions on the domain* Ω, *its parts of boundary* Γ_X, $X = U,F,C$ *and the coefficient functions* $a^{(\iota)}_{ijk\ell}$ *and* b_{ij}, *let* $\Gamma_C \in C^{1,1}$, $\mathscr{L} \in L_2\big(I_{\mathscr{T}}; \boldsymbol{H}^{\frac{1}{2}}(\Omega)^*\big)$, $u_0, u_1 \in \boldsymbol{H}^{\frac{3}{2}}(\Omega)$ *and* $U \in \boldsymbol{H}^2(Q_{\mathscr{T}})$. *Let* U *satisfy the compatibility conditions* $U = 0$ *on* $S_{C,\mathscr{T}}$, $U(0,\cdot) = u_0$ *and* $\dot{U}(0,\cdot) = u_1$ *on* Ω. *Let* $\mathscr{F}$ *be a nonnegative function of the space variable satisfying* $\operatorname{supp}\mathscr{F} \subset \Gamma_{C,\omega} \equiv \{x \in \Gamma_C;\ \operatorname{dist}(x,\partial\Gamma_C) \geqslant \omega\}$ *for some* $\omega > 0$ *and* $\|\mathscr{F}\|_{L_\infty(\Gamma_C)} < C_{\mathscr{F}}$, *where the constant* $C_{\mathscr{F}}$ *is given in* [6], *Proposition* 4 *and formula* (4.23) *for an anisotropic material*[2] *and in* [2] *for an isotropic material in two dimensions. Then the penalized contact problem* (16) *with given* $\Theta \equiv \Theta^{(0)} \in L_2\big(I_{\mathscr{T}}; H^{\frac{1}{2}}(\Omega)\big)$ *has a unique solution which satisfies the a-priori estimates*

$$\|\dot{u}\|^2_{L_\infty(I_{\mathscr{T}};\boldsymbol{L}_2(\Omega))} + \|\dot{u}\|^2_{\boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}})} \leqslant \check{c}_1\|\Theta^{(0)}\|^2_{L_2(Q_{\mathscr{T}})} + \check{c}_2, \tag{17}$$

$$\|\dot{u}\|_{\boldsymbol{H}^{\frac{1}{2},1}(S_{C,\omega,\mathscr{T}})} + \|\delta^{-1}\dot{u}_n^+\|_{L_2(S_{C,\omega,\mathscr{T}})} \leqslant \check{c}_3\|\Theta^{(0)}\|_{L_2(I_{\mathscr{T}};H^{\frac{1}{2}}(\Omega))} + \check{c}_4 \tag{18}$$

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[2] This constant is given by $C_{\mathscr{F}} \equiv \sqrt{\frac{a_0^{(1)}}{2A_0^{(1)}}}\cdot\begin{cases} \sqrt{z/\big((1+\sqrt{8z})(z+1)\big)}, & z \geqslant 1, \\ \sqrt{\sqrt{z}/\big(2(1+\sqrt{8z})\big)}, & z \leqslant 1, \end{cases}$ with $z = \frac{\pi\sqrt{a_0^{(1)}A_0^{(1)}}}{\sqrt{2}c^2_{N-1}(\frac{1}{2})}$, where $c_{N-1}(\frac{1}{2}) = 2\int_{\mathbb{R}} s^{-2}\sin^2 s\,\mathrm{d}s \int_{\mathbb{R}^{N-2}}(1+|s|^2)^{-N/2}\,\mathrm{d}s$ and $a_0^{(1)}, A_0^{(1)}$ are the constants of ellipticity and boundedness, respectively, of the viscous part from formula (12).

with $S_{C,\omega,\mathscr{T}} \equiv I_{\mathscr{T}} \times \Gamma_{C,\omega}$ *and* $\check{c}_i$, $i = 1, \ldots, 4$, *independent of* δ.

The proof of this result follows from [6] and [2] with a small modification described in [7] in order to prove estimate (18).

3. A linearized model

The problem studied here is solved in [3] in detail. We assume the tensor $\{c_{ij}\}$ to be independent of the temperature and to be elliptic and bounded,

$$c_0\xi_i\xi_i \leqslant c_{ij}\xi_i\xi_j \leqslant C_0\xi_i\xi_i,\ \xi \in \mathbb{R}^N,\ x \in \Omega, \quad \text{with } 0 < c_0 \leqslant C_0 < +\infty. \tag{19}$$

The frictional heat $\mathscr{F}|T_n(u,\Theta)|\,|\dot{u}_t|$ in (14) is replaced by $J(x, \mathscr{F}|T_n(u,\Theta)|, |\dot{u}_t|)$ with a measurable function J being monotone in the second and third argument and satisfying the growth condition

$$J(x,y,z) \leqslant \check{c}_5(1 + |y| + |z|) \tag{20}$$

with a constant $\check{c}_5 \in \mathbb{R}$. Furthermore, the operator $\mathscr{J}\colon (f,g) \mapsto J(\cdot, f, g)$ shall satisfy the continuity relation

$$\mathscr{J}(f_n, g_n) \rightharpoonup \mathscr{J}(f,g) \text{ in } L_\alpha(S_{C,\mathscr{T}}) \tag{21}$$

for some $\alpha > \frac{8}{5}$ if $f_n \rightharpoonup f$ in $L_2(S_{C,\mathscr{T}})$ and $g_n \to g$ in $L_2(S_{C,\mathscr{T}})$. If both these convergences are strong then the convergence in (21) is assumed to be strong, too. A function having such a continuity property is e.g. given by $J\colon [x,f,g] \mapsto G(x,g)f + H(x,g)$, where H and G satisfy the usual Carathéodory condition, G is uniformly bounded and $|H(x,y)| \leqslant \check{c}_6|y| + \check{c}_7$, $\check{c}_j \in \mathbb{R}$, $j = 6, 7$. Instead of the deformation heat $b_{ij}\Theta$ we use $\tilde{b}_{ij} = b_{ij}\widetilde{\Theta}$, where $\widetilde{\Theta}$ is a fixed reference temperature. The linearization of the viscous heat $a^{(1)}_{ijk\ell}e_{ij}(\dot{u})e_{k\ell}(\dot{u})$ yields a term of the same structure as such a linearized deformation heat, hence we neglect it. The problem to be considered is then given by variational inequality (13) and the modified heat equation

$$\begin{aligned}\big\langle \dot{\Theta}, \varphi \big\rangle_{Q_{\mathscr{T}}} + \langle c_{ij}\Theta_{,j}, \varphi_{,i}\rangle_{Q_{\mathscr{T}}} + \big\langle \tilde{b}_{ij}\,\dot{u}_{i,j}, \varphi\big\rangle_{Q_{\mathscr{T}}} + \langle K(\Theta - \Upsilon), \varphi\rangle_{S_{F,\mathscr{T}} \cup S_{C,\mathscr{T}}} \\ = \langle \mathscr{J}(\mathscr{F}|T_n|, |\dot{u}_t|), \varphi\rangle_{S_{C,\mathscr{T}}}\end{aligned} \tag{22}$$

being valid for functions φ from the space $\mathfrak{V} \equiv \big\{\varphi \in H^{\frac{1}{2},1}(Q_{\mathscr{T}});\ \varphi = 0 \text{ on } S_{U,\mathscr{T}}\big\}$.

To solve this problem the penalty approximation of the contact condition is employed. Here the term $|T_n|$ must be replaced also in the heat equation (22). The resulting problem is solved by means of the fixed-point technique: the temperature

Θ in (16) is replaced by a given temperature $\Theta^{(0)}$, the properties of the solution operators $\Phi_1\colon\ \Theta^{(0)} \mapsto u$ of the penalized contact problem and $\Phi_2\colon\ u \mapsto \Theta$ of the heat conduction problem are studied. Fixed points of the operator $\Phi \equiv \Phi_2 \circ \Phi_1$ define solutions of the penalized problem. In the sequel, the constants $\check{c}_i,\ i = 8, \dots$ are dependent on the input data only, their possible dependence on some other parameters will be explicitly written.

For the operator Φ_1 the following proposition holds:

Proposition 1. *Let the assumptions of Theorem 1 be satisfied. Then the solution operator $\Phi_1\colon\ L_2(Q_{\mathscr{T}}) \ni \Theta^{(0)} \to \dot{u} \in \boldsymbol{H}^{\frac12,1}(Q_{\mathscr{T}})$ is Lipschitz. If, moreover, $\mathscr{L} \in H^{\frac14}(I_{\mathscr{T}}; \boldsymbol{H}^1(\Omega)^*)$ and $\Theta^{(0)} \in H^{\frac14,\frac12}(Q_{\mathscr{T}})$, then*

$$\|\dot{u}\|_{H^{\frac14}(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))} \leqslant \check{c}_9(\|\Theta^{(0)}\|_{H^{\frac14,\frac12}(Q_{\mathscr{T}})} + 1). \tag{23}$$

The operator Φ_1 is then also continuous from $H^{\frac14,\frac12}(Q_{\mathscr{T}})$ into $H^{\frac14-\varepsilon}(I_{\mathscr{T}}; H^1(\Omega))$ for any $\varepsilon \in \left(0, \frac14\right)$.

The proof of this proposition is given in [3]; estimate (23) is proved with a time shift technique, [7].

An analogous result for the heat equation with a given displacement field is proved by a standard Galerkin technique, [3]:

Proposition 2. *Let the assumptions concerning Ω and the parts of boundary Γ_X mentioned in Section 2 be valid, let $\{c_{ij}\}$ be symmetric, bounded and elliptic as described in (19), $\tilde{b}_{ij} \in L_\infty(\Omega)$, $\Theta_0 \in \mathfrak{V}$, $0 \leqslant K \in L_\infty(\Gamma)$, $\Upsilon \in L_2(S_{\mathscr{T}})$ and $0 \leqslant \mathscr{F} \in L_\infty(\Gamma_C)$. Then, for a fixed $\dot{u} \in \boldsymbol{H}^{\frac12,1}(Q_{\mathscr{T}})$, the penalized version of (22) has a unique solution Θ satisfying the a priori estimate*

$$\|\Theta\|_{H^{\frac12,1}(Q_{\mathscr{T}})} \leqslant \check{c}_{10}\|\dot{u}\|_{L_2(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))} + \check{c}_{11}\|\Theta_0\|_{L_2(\Omega)} + \check{c}_{12} \tag{24}$$

with constants $\check{c}_i = \check{c}_i(\delta)$, $i = 10, \dots, 12$, independent of Θ_0. The solution operator $\Phi_2\colon\ \boldsymbol{H}^{\frac12,1}(Q_{\mathscr{T}}) \ni \dot{u} \to \Theta \in \mathfrak{V}$ is continuous.

Due to the just established continuity of Φ_1 and Φ_2, Φ is continuous from $L_2(Q_{\mathscr{T}})$ into $\boldsymbol{H}^{\frac12,1}(Q_{\mathscr{T}})$. For a small time interval an estimate

$$\|\Phi(\Theta^{(0)})\|_{L_2(Q_{\mathscr{T}})} \leqslant \varkappa\|\Theta^{(0)}\|_{L_2(Q_{\mathscr{T}})} + \check{c}_{13}$$

with $\varkappa < 1$ can be derived and the Schauder fixed point theorem can be employed for this time interval. The estimate is valid with $\varkappa$ independent of both the initial time and the initial temperature Θ_0, hence a successive extension of the solution is possible and finally the existence of a solution to the penalized problem is proved on the whole time interval $I_{\mathscr{T}}$.

All estimates in Theorem 1 are δ-independent, but those in Proposition 2 are not. However, from (22) and from the assumption concerning the operator $\mathscr{J}$ it is easy to derive the estimate

$$\begin{aligned}&\|\Theta\|^2_{L_\infty(I_{\mathscr{T}};L_2(\Omega))}+\|\Theta\|^2_{H^{\frac{1}{2},1}(Q_{\mathscr{T}})}\\&\qquad\leqslant \check{c}_{14}\Big(\|\dot{u}\|^2_{L_2(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))}+\|\delta^{-1}\dot{u}_n^+\|^2_{L_2(S_{C,\omega,\mathscr{T}})}+\|\dot{u}\|_{H^{\frac{1}{2},1}(S_{C,\omega,\mathscr{T}})}+1\Big)\end{aligned}$$

with a constant independent of δ. By interpolation we get $\|\Theta\|_{H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}})}\leqslant \varepsilon\|\Theta\|_{H^{\frac{1}{2},1}(Q_{\mathscr{T}})}+\check{c}_{15}(\varepsilon)\|\Theta\|_{L_2(Q_{\mathscr{T}})}$, where $\varepsilon>0$ can be arbitrarily small. This combined with estimate (18) and the Gronwall lemma leads to the estimate

$$\text{(25)}\quad \|\dot{u}\|_{\boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}})}+\|\dot{u}\|_{\boldsymbol{H}^{\frac{1}{2},1}(S_{C,\omega,\mathscr{T}})}+\|\delta^{-1}\dot{u}_n^+\|_{L_2(S_{C,\omega,\mathscr{T}})}+\|\Theta\|_{H^{\frac{1}{2},1}(Q_{\mathscr{T}})}\leqslant \check{c}_{16}.$$

As a consequence, there exists a sequence $\delta_k\to 0$ of penalty parameters and a corresponding sequence of solutions (u_k,Θ_k) to the penalized problems such that the latter converges weakly to some limit (u,Θ) in the spaces mentioned in (25). Performing the limit procedure $k\to+\infty$ in the penalized problem proves that u and Θ are solutions of the original problem (13, 22).

Theorem 2. *Let the assumptions of Theorem* 1 *and Proposition* 2 *be satisfied. Then there exists at least one weak solution to problem* (13, 22).

4. A model with rapidly growing thermal energy

In this section we assume the tensor of thermal conductivity c_{ij} to be symmetric and to depend locally Lipschitz-continuously on the temperature gradient such that for a positive parameter γ the growth condition

$$\text{(26)}\quad \check{c}_{17}\,(1+|\nabla\Theta|^\gamma)\,\xi_i\xi_i\leqslant c_{ij}(\nabla\Theta)\xi_i\xi_j\leqslant \check{c}_{18}\,(1+|\nabla\Theta|^\gamma)\,\xi_i\xi_i,\ \ \xi\in\mathbb{R}^N,$$

is satisfied, the strong monotonicity

$$\text{(27)}\quad\begin{aligned}&\langle c_{ij}(\nabla\Theta)\Theta_{,j}-c_{ij}(\nabla\Xi)\Xi_{,j},\Theta_{,i}-\Xi_{,i}\rangle_{Q_{\mathscr{T}}}\\&\qquad\geqslant \check{c}_{19}\|\nabla(\Theta-\Xi)\|^{\gamma+2}_{\boldsymbol{L}_{\gamma+2}(Q_{\mathscr{T}})}+\check{c}_{20}\|\nabla(\Theta-\Xi)\|^2_{\boldsymbol{L}_2(Q_{\mathscr{T}})}\end{aligned}$$

holds for each $\Theta,\Xi\in L_{\gamma+2}\big(I_{\mathscr{T}};W^1_{\gamma+2}(\Omega)\big)$, and the continuity relation

$$\text{(28)}\quad c_{ij}\big(\nabla\Theta^{(k)}\big)\Theta^{(k)}_{,j}\to c_{ij}(\nabla\Theta)\Theta_{,j}\ \text{in}\ L_{\frac{\gamma+2}{\gamma+1}}(Q_{\mathscr{T}}),\ \ i=1,\dots,N,$$

is valid for $\Theta^{(k)} \to \Theta$ strongly in $L_{\gamma+2}\big(I_{\mathscr{T}}; W^1_{\gamma+2}(\Omega)\big)$. Here, $0 < \check{c}_i < +\infty$ for $i = 17, \ldots, 20$. An example for a matrix-valued function satisfying (26)–(28) is $c_{ij}(x;\Xi) = \delta_{ij}(d_0(x) + d_1(x)|\Xi|^{\gamma})$ with Kronecker symbol δ_{ij} and measurable functions d_0 and d_1 such that $q_1 \leqslant d_i \leqslant q_2$, $i = 0, 1$, with constants $0 < q_1, q_2 < +\infty$. With these assumptions the original problem (13, 14) is solvable for a suitable choice of γ. We will also give the (weaker) requirements for γ sufficient for the model without viscous heat. The set of admissible functions for the heat equation is $\mathfrak{V} \equiv \{w \in L_{2+\gamma}(I_{\mathscr{T}}; W^1_{2+\gamma}(\Omega)); w|_{S_{U,\mathscr{T}}} = 0\}$.

Here the fixed point approach and Proposition 1 in combination with the penalization of the contact condition can be again employed, but the behaviour of the operator Φ_2 is more complicated.

Proposition 3. *Let $\dot{u} \in H^{\frac{1}{4}}\big(I_{\mathscr{T}}; \boldsymbol{H}^1(\Omega)\big) \cap \boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}}) \cap \boldsymbol{H}^{\frac{1}{2},1}(S_{C,\omega,\mathscr{T}})$. Let the assumptions of Theorem 1 concerning Ω, Γ_X for $X = U, F, C$, b_{ij} and $\mathscr{F}$, and the above mentioned assumptions for the tensor of heat conductivity $\{c_{ij}\}$ be valid. Let $\gamma > 1, N \leqslant 3$ and $\operatorname{mes} \Gamma_U > 0$. Let, moreover, $\Theta_0 \in L_{2+\gamma}(I_{\mathscr{T}}; W^1_{2+\gamma}(\Omega))$ be continuous at $\tau = 0$ with respect to the space $W^1_{2+\gamma}(\Omega)$, let K be bounded and non-negative and $\Upsilon \in L_2(S_{\mathscr{T}})$. Then problem (14) (with $|T_n(u,\Theta)|$ replaced by $\delta^{-1}\dot{u}_n^+$) has a unique solution which satisfies the a priori estimate*

$$
\begin{aligned}
&\|\Theta\|^2_{L_\infty(I_{\mathscr{T}};L_2(\Omega))} + \|\Theta\|^{\gamma+2}_{L_{\gamma+2}(I_{\mathscr{T}};W^1_{\gamma+2}(\Omega))} + \|\Theta\|^2_{H^{\alpha,1}(Q_{\mathscr{T}})} \\
&\qquad \leqslant \check{c}_{21}\big[\|\dot{u}\|^{1+\frac{2}{\gamma}}_{\boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}})} + \|\dot{u}\|^{1+\frac{1}{\gamma+1}}_{H^{\frac{1}{3}}(I_{\mathscr{T}};\boldsymbol{L}_2(\Gamma_{C,\omega}))}\|\delta^{-1}\dot{u}_n^+\|^{1+\frac{1}{\gamma+1}}_{L_2(S_{C,\omega,\mathscr{T}})} \\
&\qquad\quad + \|\dot{u}\|^{2+\frac{2}{\gamma+1}}_{H^{\frac{1}{2(\gamma+2)}}(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))}\big] + \check{c}_{22}
\end{aligned} \tag{29}
$$

for any $\alpha \in \big(0, \frac{1}{2}\big)$. If the viscous heat is neglected in the model, then this estimate is valid without the term $\|\dot{u}\|^{2+\frac{2}{\gamma+1}}_{H^{\frac{1}{2(\gamma+2)}}(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))}$ on the right hand side. The mapping $\Phi_2\colon \dot{u} \mapsto \Theta$ is strongly continuous from $\boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}}) \cap H^{\frac{1}{4}}\big(I_{\mathscr{T}}; \boldsymbol{H}^1(\Omega)\big) \cap \boldsymbol{H}^{\frac{1}{2},1}(S_{C,\omega,\mathscr{T}})$ to $L_{2+\gamma}\big(I_{\mathscr{T}}; W^1_{2+\gamma}(\Omega)\big) \cap H^{\beta}(I_{\mathscr{T}}; L_2(\Omega))$ for any $\beta \in \big(0, \frac{1}{2}\big)$.

Proof. The proof starts from the Galerkin approximation. Let $\{\mathscr{V}_m\}$ be an increasing sequence of m-dimensional subspaces of $\mathscr{V} \equiv \big\{v \in W^1_{2+\gamma}(\Omega);\ v = 0 \text{ on } \Gamma_U\big\}$ such that $\bigcup_{m\in\mathbb{N}} \mathscr{V}_m$ is dense in $\mathscr{V}$. Then for $\mathfrak{V}_m \equiv \Big\{w\colon Q_{\mathscr{T}} \to \mathbb{R};\ \exists c_i \in L_\infty(I_{\mathscr{T}}),\ i = 1, \ldots, m\colon w(t,x) = \sum_{i=1}^{m} c_i(t) v_{im}(x)\Big\}$ with an $L_2(\Omega)$-orthogonal basis $\{v_{im}\}_{i=1}^m$ of $\mathscr{V}_m$ the union $\mathfrak{V}_0 = \bigcup_{m\in\mathbb{N}} \mathfrak{V}_m$ is dense in $\mathfrak{V}$. A Galerkin solution Θ_m of the heat conduction problem is a function from $\Theta_0 + \mathfrak{V}_m$ which satisfies for all test functions $\varphi \in \mathscr{V}_m$ and almost every $\tau \in I_{\mathscr{T}}$ the Galerkin equations

$$\begin{aligned}\langle\dot{\Theta}_m,\varphi\rangle_\Omega + \big\langle c_{ij}(\nabla\Theta_m)\Theta_{m,j},\varphi_{,i}\big\rangle_\Omega + \langle b_{ij}\Theta_m\dot{u}_{i,j},\varphi\rangle_\Omega + \langle K\left(\Theta_m-\Upsilon\right),\varphi\rangle_{\Gamma_F\cup\Gamma_C}\\ = \big\langle\mathscr{F}\delta^{-1}\dot{u}_n^+|\dot{u}_t|,\varphi\big\rangle_{\Gamma_C} + \big\langle a^{(1)}_{ijk\ell}\,e_{ij}(\dot{u})e_{k\ell}(\dot{u}),\varphi\big\rangle_{Q_{\mathscr{T}}}\end{aligned} \tag{30}$$

and the initial condition $\Theta_m(0)=\Theta_0$. The existence of Θ_m follows as usual from the theory of ordinary differential equations.

In order to obtain estimates of the type (29) for Θ_m independent of $m\in\mathbb{N}$, the approximate solution is put into the finite-dimensional version of (14). The viscous heat we estimate by

$$\begin{aligned}&\check{c}_{23}\|\nabla\dot{u}\|^2_{L_{\frac{2(\gamma+2)}{\gamma+1}}(I_{\mathscr{T}};\boldsymbol{L}_2(\Omega))}\|\Theta_m\|_{L_{\gamma+2}(I_{\mathscr{T}};L_\infty(\Omega))}\\ &\qquad\leqslant \check{c}_{24}\|\nabla\dot{u}\|^2_{H^{\frac{1}{2\gamma+4}}(I_{\mathscr{T}};\boldsymbol{L}_2(\Omega))}\|\Theta_m\|_{L_{\gamma+2}(I_{\mathscr{T}};W^1_{\gamma+2}(\Omega))}.\end{aligned} \tag{31}$$

The deformation heat is estimated after the application of the Green formula by

$$\check{c}_{25}\|\Theta\|^2_{L_{\gamma+2}(I_{\mathscr{T}};W^1_{\gamma+2}(\Omega))}\|\dot{u}\|_{H^{\frac{1}{2},1}(Q_{\mathscr{T}})}, \tag{32}$$

and the frictional heat by

$$\check{c}_{26}\|\mathscr{F}\|_{L_\infty(\Gamma_C)}\|T_n\|_{L_2(S_{C,\omega,\mathscr{T}})}\|\dot{u}\|_{L_{2+\frac{4}{\gamma}}(I_{\mathscr{T}};L_2(\Gamma_{C,\omega}))}\|\Theta_m\|_{L_{\gamma+2}(I_{\mathscr{T}};L_\infty(\Gamma))}. \tag{33}$$

The time regularity is proved as follows: For $0\leqslant s_1,\ s_2\leqslant\mathscr{T}$, $s_1\neq s_2$, we put the test function $\varphi=\Theta_m(s_2)-\Theta_m(s_1)-(\Theta_0(s_2)-\Theta_0(s_1))$ into the Galerkin equations at time τ, multiply the result by $|s_2-s_1|^{-1-2\alpha}$ with $\alpha\in(0,\frac{1}{2})$ and integrate the result both with respect to τ from s_1 to s_2 and with respect to $s=(s_1,s_2)$ over $I^2_{\mathscr{T}}$. By this procedure, the term $\langle\dot{\Theta},\varphi\rangle_\Omega$ yields the norm $\|\Theta\|^2_{H^\alpha(I_{\mathscr{T}};L_2(\Omega))}$; the remaining terms are estimated by $\|\Theta\|^{\gamma+2}_{L_{\gamma+2}(I_{\mathscr{T}};W^1_{\gamma+2}(\Omega))}$ and the right hand side of relation (29). Here, the inequality

$$\int_{I^2_{\mathscr{T}}}\int_{s_1}^{s_2}\frac{|f(\tau)||g(s_2)-g(s_1)|}{|s_2-s_1|^{1+2\alpha}}\,\mathrm{d}\tau\,\mathrm{d}s_1\,\mathrm{d}s_2\leqslant\check{c}_{27}\|f\|_{L_p(I_{\mathscr{T}})}\|g\|_{L_q(I_{\mathscr{T}})}$$

is essential. It is valid for functions $f\in L_p(I_{\mathscr{T}})$, $g\in L_q(I_{\mathscr{T}})$ with $1/p+1/q=1$, $1<p,q<+\infty$ with a constant $\check{c}_{27}$ independent of f, g, provided $\alpha<\frac{1}{2}$. This completes the proof of (29) for Θ_m.

After some series of interpolations (based on [1] and [10]), we prove that a suitable sequence $\Theta_k\equiv\Theta_{m_k}$ of solutions to Galerkin parameters $m_k\to 0$ converges in $\mathfrak{V}$ to a limit Θ which solves the penalized version of problem (14).

In order to prove uniqueness, we consider two solutions $\Theta^{(1)}$, $\Theta^{(2)}$ of the heat equation with the same displacement field u. Let $\Xi \equiv \Theta^{(1)} - \Theta^{(2)}$ and let a function ψ be defined by $\psi = \Xi$ for $\tau \leqslant \tau_0$ and $\psi = 0$ for $\tau \geqslant \tau_0$. We put ψ into the equation with the solution $\Theta^{(1)}$, $-\psi$ into the equation with the solution $\Theta^{(2)}$ and add the results. Then, since the viscous heat term and the frictional heat term cancel, we arrive at the inequality

$$
\begin{aligned}
\|\Xi(\tau_0)\|^2_{L_2(\Omega)} + \|\Xi\|^2_{L_2(0,\tau_0\,;\,H^1(\Omega))} &\leqslant \check{c}_{28} \int_0^{\tau_0} \int_\Omega |\nabla\dot{u}||\Xi|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \check{c}_{29} \int_0^{\tau_0} \|\nabla\dot{u}\|_{L_2(\Omega;\mathbb{R}^{N^2})} \|\Xi\|^{\frac{1}{2}}_{L_2(\Omega)} \|\Xi\|^{\frac{3}{2}}_{L_6(\Omega)} \,\mathrm{d}\tau \\
&\leqslant \check{c}_{30}(\varepsilon) \int_0^{\tau_0} \|\nabla\dot{u}\|^4_{L_2(\Omega;\mathbb{R}^{N^2})} \|\Xi\|^2_{L_2(\Omega)} \,\mathrm{d}\tau + \varepsilon\|\Xi\|^2_{L_2(0,\tau_0;L_6(\Omega))},
\end{aligned}
\tag{34}
$$

valid for any parameter $\varepsilon > 0$. For dimension $N \leqslant 3$ the imbeddings

$$
H^{\frac{1}{4}}(I_{\mathscr{T}}; H^1(\Omega)) \hookrightarrow L_4(I_{\mathscr{T}}; H^1(\Omega)) \quad \text{and} \quad H^1(\Omega) \hookrightarrow L_6(\Omega)
$$

are valid, hence by means of the Gronwall lemma the equation $\|\Xi(\tau_0)\|_{L_2(\Omega)} = 0$ holds for all $\tau_0 \geqslant 0$.

The continuity of Φ_2 is based on estimate (29). For a sequence of displacement fields $u_k \to u$ converging in the sense required in Proposition 3, and the corresponding solutions $\Theta_k = \Phi_2(u_k)$, we may extract a subsequence such that Θ_k converges weakly in the spaces $L_{2+\gamma}\big(I_{\mathscr{T}}; W^1_{2+\gamma}(\Omega)\big)$ and in $H^\beta\big(I_{\mathscr{T}}; L_2(\Omega)\big)$, $\beta \in (0, \frac{1}{2})$ and $\dot{\Theta}_k$ weakly in $\mathfrak{V}^*$. Using the monotonicity technique based on (27) for the heat equation (14) we prove with help of some interpolations that the convergence of Θ_k is in fact strong first in $L_{2+\gamma}\big(I_{\mathscr{T}}; W^1_{2+\gamma}(\Omega)\big)$ and then, again by interpolation and compact embedding results, also in $H^\beta\big(I_{\mathscr{T}}; L_2(\Omega)\big)$. Passing to the limit in the heat equation proves that the limit function Θ must be equal to $\Phi_2(u)$, hence it is unique, and the whole sequence Θ_k converges to $\Phi_2(u)$. □

Proposition 4. *Let the assumptions of Theorem* 1 *and Propositions* 1 *and* 3 *be valid. If the viscous heat is neglected, let* $\gamma > 1$, *otherwise let* $\gamma > 2$. *Then the penalized version of the thermoviscoelastic contact problem* (13, 14) *has a solution* (u, Θ) *which satisfies for all* $\alpha < \frac{1}{2}$ *the a-priori estimate*

$$
\begin{aligned}
&\|\dot{u}\|_{H^{\frac{1}{4}}(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))} + \|\dot{u}\|_{\boldsymbol{H}^{\frac{1}{2},1}(Q_{\mathscr{T}})} + \|\dot{u}\|_{\boldsymbol{H}^{\frac{1}{2},1}(S_{C,\omega,\mathscr{T}})} + \|\Theta\|_{H^{\alpha,1}(Q_{\mathscr{T}})} \\
&\qquad + \|\Theta\|_{L_{\gamma+2}\left(I_{\mathscr{T}};W^1_{\gamma+2}(\Omega)\right)} + \|\dot{u}\|_{L_\infty(I_{\mathscr{T}};\boldsymbol{L}_2(\Omega))} + \|\Theta\|_{L_\infty(I_{\mathscr{T}};L_2(\Omega))} \leqslant \check{c}_{31}.
\end{aligned}
\tag{35}
$$

P r o o f. The proof is done by verifying the requirements of the Schauder fixed point theorem for the operator $\Phi\colon H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}}) \ni \Theta^{(0)} \mapsto \Theta \in H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}})$. First, we observe that Propositions 1 and 3 yield the (weak) continuity of the operator Φ from $H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}})$ into $H^{\beta,1}(Q_{\mathscr{T}})$ with an arbitrary $\beta \in \left(0, \frac{1}{2}\right)$. The choice $\beta > \frac{1}{4}$ proves that $\Phi\colon H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}}) \to H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}})$ is completely continuous. Let

$$\mathscr{M}(R) \equiv \left\{ \|\varphi\|^{1+\frac{\gamma}{2}}_{L_2(I_{\mathscr{T}};H^{\frac{1}{2}}(\Omega))} + \|\varphi\|_{H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}})} \leqslant R \right\}$$

and let the viscous heat be neglected. A combination of the estimates (17), (18) and (29) (without the term $\|\dot{u}\|^{2+\frac{2}{\gamma+1}}_{H^{\frac{1}{2(\gamma+2)}}(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))}$ coming from the viscous heat) and the fact that for $\gamma > 1$ the inequality $1 + \frac{2}{\gamma} < 2 + \frac{2}{1+\gamma}$ holds, yield

$$\|\Theta\|^{2+\gamma}_{L_2(I_{\mathscr{T}};H^{\frac{1}{2}}(\Omega))} + \|\Theta\|^2_{H^{\frac{1}{4},1}(Q_{\mathscr{T}})} \leqslant \check{c}_{32}\|\Theta^{(0)}\|^{2+\frac{2}{1+\gamma}}_{L_2(I_{\mathscr{T}};H^{\frac{1}{2}}(\Omega))} + \check{c}_{33}.$$

From this we easily see that $\Phi\colon \mathscr{M}(R) \to \mathscr{M}\ (\check{c}_{34}(R^{\frac{2}{\gamma+1}} + 1))$ and for any $\gamma > 1$ an R_0 can be found such that $\Phi\colon \mathscr{M}(R_0) \to \mathscr{M}(R_0)$.

If the viscous heat is not neglected, then estimate (31) is replaced by

$$\begin{aligned} &\check{c}_{35}\|\dot{u}\|^2_{H^{\frac{1}{2\gamma+4}}(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))}\|\Theta\|_{L_{2+\gamma}(I_{\mathscr{T}};L_\infty(\Omega))} \\ &\qquad \leqslant \check{c}_{36}\|\Theta^{(0)}\|^{2\lambda}_{H^{\frac{1}{4},\frac{1}{2}}(Q_{\mathscr{T}})}\|\Theta^{(0)}\|^{2-2\lambda}_{L_2(I_{\mathscr{T}};H^{\frac{1}{2}}(\Omega))}\|\Theta\|_{L_{2+\gamma}\left(I_{\mathscr{T}};W^1_{2+\gamma}(\Omega)\right)} \end{aligned} \tag{36}$$

with $\lambda = \frac{2}{\gamma+2}$. Hence, after the use of the Hölder inequality we get from the viscous term

$$\check{c}_{37}\left[\|\Theta^{(0)}\|^{2-\varepsilon}_{H^{\frac{1}{4},\frac{1}{2}}}(Q_{\mathscr{T}}) + \|\Theta^{(0)}\|^{2\gamma/(\gamma-1)+\varepsilon_1}_{L_2(I_{\mathscr{T}};H^{1/2}(\Omega))}\right],$$

where $\varepsilon_1 \equiv \varepsilon_1(\varepsilon)$ and $\varepsilon_1 \to 0$ for $\varepsilon \to 0$. In order to satisfy the estimate $2\gamma/(\gamma-1) < 2+\gamma$, condition $\gamma > 2$ is necessary and sufficient. We obtain again $\Phi(\mathscr{M}(R)) \subset \mathscr{M}\left(\mathrm{const}(R^{1-\tilde{\varepsilon}} + 1)\right)$ for some $\tilde{\varepsilon} > 0$. Then there is again some $R_0 > 0$ such that $\Phi\colon \mathscr{M}(R_0) \to \mathscr{M}(R_0)$. A fixed point of Φ whose existence is ensured by the Schauder theorem is the solution of the penalized thermoviscoelastic contact problem. $\square$

The limit procedure $\delta \to 0$ exploits estimate (35). A sequence $\delta_k \to 0$ with the corresponding solutions (u_k, Θ_k) is taken such that the latter converges weakly in the spaces mentioned in (35) to a limit u, Θ. The limit procedure is done first for the contact problem, which in combination with a standard monotonicity technique also proves the strong convergence of the displacements. Then the convergence in the heat equation for any prescribed test function can be verified.

Theorem 3. *Let the assumptions listed in Theorem* 1 *and Propositions* 3 *and* 4 *be satisfied. Then there exists a weak solution to problem* (13, 14).

A more detailed proof is given in [7].

In the model studied now the growth of the viscous heat and the deformation heat is compensated by a growth of the temperature-dependent diffusion coefficients formulated by the condition

$$c_0\big(1+|\Theta|^\gamma\big)\xi_i\xi_i \leqslant c_{ij}(\Theta)\xi_i\xi_j \leqslant C_0\big(1+|\Theta|^\gamma\big)\xi_i\xi_i \tag{37}$$

for all $\{\xi_i\} \in \mathbb{R}^N$ and $x \in \Omega$ with $0 < c_0 \leqslant C_0 < +\infty$. Furthermore, the components $c_{ij}(x, \Theta)$ are continuous in the sense of Carathéodory. The growth of the frictional heat is compensated by a heat radiation law: condition (9) is replaced by

$$c_{ij}\partial_j\Theta n_i = \mathscr{F}|T_n(u,\Theta)||\dot u_t| + K(\Upsilon-\Theta) + R(\Upsilon) - R(\Theta) \text{ on } S_{C,\mathscr{T}}. \tag{38}$$

The heat radiation function R depends on the space variable and on the temperature. It shall be continuous in the sense of Carathéodory, monotone in the variable Θ and satisfy $R(\Theta) = 0$ for $\Theta \leqslant 0$ as well as the growth condition

$$\check c_{38}|\Theta|^4 \leqslant R(\Theta) \leqslant \check c_{39}|\Theta|^4 + \check c_{40} \text{ for } \Theta \geqslant 0$$

with constants $\check c_{38}, \check c_{39}, \check c_{40} > 0$. The exponent 4 is motivated by the classical Stefan Boltzmann radiation law.

In this section we assume that Θ^+ stands for Θ in the strain-stress relation (11). This is no restrictive change of the model, because the temperature is later proved to be positive. For simplicity we assume, moreover, $\Gamma = \Gamma_C$ which ensures that estimate (23) can be avoided and

$$\|\dot u\|_{\boldsymbol{H}^{\frac34,\frac32}(Q_{\mathscr{T}})} \leqslant \check c_{41}\big(\|\Theta\|_{L_2(I_{\mathscr{T}};H^{\frac12}(\Omega))} + 1\big) \tag{39}$$

is valid. This follows from (18) with $S_{C,\omega,\mathscr{T}}$ replaced by $S_{\mathscr{T}}$ by the usual regularity theory for linear parabolic initial-boundary value problems. Its validity can be extended to a more general situation with different boundary conditions in different parts of $S_{\mathscr{T}}$, but some special conditions for the relative boundaries of these parts must be required.

The variational formulation of the problem is again (13, 14) with the additional term $\langle R(\Theta)-R(\Upsilon), \varphi\rangle_{S_{\mathscr{T}}}$ on the left-hand side of the heat equation (14) and the space of admissible functions for the heat equation $\mathfrak{V} \equiv H^{\beta,1}(Q_{\mathscr{T}})$ for any $\beta \in \big(0, \frac{2}{4+N}\big)$. Let us refer to the modified heat equation by (14_R). In the linear functional $\mathscr{L}$ for the contact problem the boundary term disappears.

Theorem 4. *Let us assume the validity of the above mentioned assumptions concerning the tensor of heat conductivity $\{c_{ij}\}$ and the heat radiation function R, as well as the requirements of Theorem* 1. *Let $\Gamma \in C^{1,1}$, $0 \leqslant K \in L_\infty(\Gamma)$, $f \in L_2(Q_{\mathscr{T}})$, $0 \leqslant \Upsilon \in L_5(S_{\mathscr{T}})$ and $1 - \frac{1}{N} - \frac{1}{N+2} < \gamma < 1$. Then Problem $(13, 14_R)$ has at least one weak solution. The temperature field of this solution is non-negative. Every solution satisfies the a priori estimate*

$$\begin{aligned} \|\dot{u}\|_{L_\infty(I_{\mathscr{T}};\boldsymbol{L}_2(\Omega))} &+ \|\nabla \dot{u}\|_{\boldsymbol{H}^{\frac14,\frac12}(Q_{\mathscr{T}})} \\ &+ \|\Theta\|_{L_\infty(I_{\mathscr{T}};L_{2-\gamma}(\Omega))} + \|\Theta\|_{H^{\beta,1}(Q_{\mathscr{T}})} + \|\Theta\|_{L_{5-\gamma}(S_{\mathscr{T}})} \leqslant \check{c}_{42} \end{aligned} \tag{40}$$

with $\beta \in \left(0, \frac{2}{4+N}\right)$ arbitrary but fixed.

P r o o f. In the proof of this result three successive approximation steps for the whole problem are employed. First, the contact condition is relaxed by the usual penalty approximation. The resulting heat equation will be refered to by $(14_R)_\delta$. Second, the growth of the "mixed" terms $b_{ij}\Theta^+$, $\mathscr{F}\delta^{-1}\dot{u}_n^+|\dot{u}_t|$ and $a^{(1)}_{ijk\ell}e_{ij}(\dot{u})e_{k\ell}(\dot{u})$ in the heat equation is cut. Therefore these terms are replaced by $b_{ij}H_M(\Theta)$, $H_M\big(\mathscr{F}\delta^{-1}\dot{u}_n^+|\dot{u}_t|\big)$ and $H_M\big(a^{(1)}_{ijk\ell}e_{ij}(\dot{u})e_{k\ell}(\dot{u})\big)$ with a continuous function $H_M\colon \mathbb{R} \to \mathbb{R}$ defined by $H_M(x) = x^+$ for $x \leqslant M$ and $H_M(x) = M$ for $x > M$. Observe that the viscous heat and the frictional heat terms are non-negative. Later we will prove that the solution Θ of the problem obtained is also non-negative, therefore the restriction to non-negative values of the temperature here is justified, too. Let us refer to the approximate problems obtained—whose form is obvious—by $(16)_M$ and $(14_R)_{\delta,M}$. Finally, the norm $|\cdot|$ in the friction law is replaced by a suitable smooth, convex function Ψ_η such that $\Psi_\eta = |\cdot|$ on $\{x \in \mathbb{R}^N\,;\ |x| \geqslant \eta\}$. The problem resulting from these approximations is given as follows

Find a couple $[u,\Theta] \in \mathfrak{U} \times \mathfrak{V}$ satisfying the initial conditions (5) and (10) such that for each $v \in \mathfrak{U}$ and each $\varphi \in \mathfrak{V}$ the following relations are valid:

$$\begin{aligned} \langle \ddot{u}_i, v_i\rangle_{Q_{\mathscr{T}}} &+ \langle \sigma_{ij}(u, H_M(\Theta)), e_{ij}(v)\rangle_{Q_{\mathscr{T}}} \\ &+ \big\langle \delta^{-1}\dot{u}_n^+, v_n\big\rangle_{S_{\mathscr{T}}} + \big\langle \mathscr{F}\delta^{-1}\dot{u}_n^+\nabla\Psi_\eta(\dot{u}_t), v_t\big\rangle_{S_{\mathscr{T}}} = \mathscr{L}(v) \end{aligned} \tag{41}$$

$$\begin{aligned} \langle \dot{\Theta}, \varphi\rangle_{Q_{\mathscr{T}}} &+ \langle c_{ij}\Theta_{,j}, \varphi_{,i}\rangle_{Q_{\mathscr{T}}} + \langle b_{ij}\, H_M(\Theta)\dot{u}_{i,j}, \varphi\rangle_{Q_{\mathscr{T}}} \\ &+ \langle K(\Theta - \Upsilon) + R(\Theta) - R(\Upsilon), \varphi\rangle_{S_{\mathscr{T}}} \\ &= \Big\langle H_M\big(a^{(1)}_{ijk\ell}e_{ij}(\dot{u})e_{k\ell}(\dot{u})\big), \varphi\Big\rangle_{Q_{\mathscr{T}}} + \big\langle H_M\big(\mathscr{F}\delta^{-1}\dot{u}_n^+\Psi_\eta(\dot{u}_t)\big), \varphi\big\rangle_{S_{\mathscr{T}}}. \end{aligned} \tag{42}$$

The solvability of this problem is proved by a standard Galerkin procedure. Then, besides the limit procedure for the penalty parameter $\delta \to 0$, we have to perform those for $\eta \to 0$ and for $M \to +\infty$. If we fix δ and M, there is no problem for the procedure $\eta \to 0$ due to the limited growth of the mixed terms. Thus, the solvability of problem $(16)_M$ and $(14_R)_{\delta,M}$ (the non-smoothed version of (41, 42)) is proved. For the next two limit procedures, *a priori* estimates independent of the parameters M and δ must be derived. The first is an energy estimate corresponding to the physical nature of the problem. By putting $\Theta^- = \min\{0, \Theta\}$ into the heat equation we obtain $\Theta^- = 0$, hence the temperature is positive. Then, putting $v = 0$ for $\tau \leqslant \tau_0$, $v = \dot{u}$ for $\tau \in (\tau_0, \mathscr{T})$ into $(16)_M$ and simultaneously $\varphi = 1$ for $\tau \leqslant \tau_0$, $\varphi = 0$ for $\tau > \tau_0$ into the heat equation $(14_R)_{\delta,M}$ and adding the resulting relations we arrive at the estimate

$$\|\Theta\|_{L_\infty(I_{\mathscr{T}};L_1(\Omega))} + \|\dot{u}\|_{L_\infty(I_{\mathscr{T}};\boldsymbol{L}_2(\Omega))} + \|u\|_{L_\infty(I_{\mathscr{T}};\boldsymbol{H}^1(\Omega))} \leqslant \check{c}_{43}, \tag{43}$$

as the mixed terms cancel. Starting from this result it is possible to derive better *a priori* estimates. Like in the previous section, the estimates derived from the heat equation play the key role, while the estimate of the Lamé system is based on (17), (18) and (39). We put $\varphi = \Theta^{1-\gamma}$ as the test function. Then, on the left hand side of the desired inequality to be established, we obtain the term

$$\|\Theta\|^{2-\gamma}_{L_\infty(I_{\mathscr{T}};L_{2-\gamma}(\Omega))} + \|\nabla\Theta\|^2_{L_2(Q_{\mathscr{T}})} + \|\Theta\|^{5-\gamma}_{L_{5-\gamma}(S_{\mathscr{T}})}. \tag{44}$$

All the other terms must be estimated uniformly in M and δ by this one. Here the viscous heat and the deformation heat are those creating the most substantial difficulties. We use interpolation and embedding theorems for Sobolev spaces, in particular the relations

$$\|\nabla\dot{u}\|_{L_p(I_{\mathscr{T}};L_q(\Omega;\mathbb{R}^{N^2}))} \leqslant \check{c}_{44}\|\dot{u}\|_{\boldsymbol{H}^{\frac{3}{4},\frac{3}{2}}(Q_{\mathscr{T}})}$$

valid for $p, q \geqslant 2$ and $\frac{4}{p} + \frac{2N}{q} \geqslant N+1$,

$$\|\Theta\|_{L_p(I_{\mathscr{T}};L_q(\Omega))} \leqslant \|\Theta\|^{\lambda}_{L_\infty(I_{\mathscr{T}};L_1(\Omega))}\|\Theta\|^{1-\lambda}_{L_2(I_{\mathscr{T}};H^1(\Omega))}$$

valid for $1 - \frac{2}{p} \leqslant \lambda \leqslant \frac{1/q-1/p_0}{1-1/p_0}$ with $p_0 = \frac{2N}{N-2}$ in the case $N \geqslant 3$ and $p_0 < +\infty$ in the case $N = 2$, and with $1 \leqslant q \leqslant p_0$ (the parameter p_0 is calculated from the embedding $H^1(\Omega) \hookrightarrow L_{p_0}(\Omega)$), and

$$\|\Theta\|_{L_2(I_{\mathscr{T}};H^{\frac{1}{2}}(\Omega))} \leqslant \|\Theta\|^{\frac{1}{N+2}}_{L_\infty(I_{\mathscr{T}};L_1(\Omega))}\|\Theta\|^{1-\frac{1}{N+2}}_{L_{2-\frac{2}{N+2}}(I_{\mathscr{T}};H^1(\Omega))}.$$

Employing these inequalities and estimate (39), it is possible to prove the relations

$$\int_{Q_{\mathscr{T}}} |\nabla\dot{u}|^2\Theta^{1-\gamma}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \check{c}_{45}\|\Theta\|^{1-\gamma+\varepsilon}_{L_\infty(I_{\mathscr{T}};L_1(\Omega))}\|\Theta\|^{2-\varepsilon}_{L_2(I_{\mathscr{T}};H^1(\Omega))} + \check{c}_{46}$$

with $\varepsilon > 0$ for the viscous heat term, provided $\gamma > 1 - \frac{1}{N} - \frac{1}{N+2}$, and

$$\int_{Q_{\mathscr{T}}} |\nabla \dot{u}| \Theta^{2-\gamma} \, \mathrm{d}x \, \mathrm{d}\tau \leqslant \check{c}_{47} \|\Theta\|^{1-\gamma+\varepsilon}_{L_\infty(I_{\mathscr{T}};L_1(\Omega))} \|\Theta\|^{2-\varepsilon}_{L_2(I_{\mathscr{T}};H^1(\Omega))} + \check{c}_{48}$$

for the deformation heat term, provided $1 - \frac{3}{2N} - \frac{1}{2(N+2)} < \gamma < 1$. The estimate of the frictional heat term is easy here, since the term $\|\Theta\|^{5-\gamma}_{L_{5-\gamma}(S_{\mathscr{T}})}$ is available. This finally proves the uniform boundedness of (44) with respect to the approximation parameters λ and M.

In the limit procedures $M \to +\infty$ and $\lambda \to 0$ it is necessary to have strong convergences of Θ in $L_2(Q_{\mathscr{T}})$ and in $L_2(S_{\mathscr{T}})$. This can be proved by using compact embedding theorems for Sobolev spaces, if some time regularity of Θ is available. In order to obtain this, a dual estimate of $\dot{\Theta}$ is derived by using an arbitrary test function $\varphi \in L_2(I_{\mathscr{T}}; \mathring{H}^{1+\frac{N}{2}+\varepsilon}(\Omega))$ in the heat equation. Here the order of the space derivatives is due to the imbedding $\mathring{H}^{1+\frac{N}{2}+\varepsilon}(\Omega) \hookrightarrow W^1_\infty(\Omega)$. After some estimates similar to those described above, we obtain a uniform bound first for $\|\dot{\Theta}\|_{L_2(I_{\mathscr{T}};\mathring{H}^{1+\frac{N}{2}+\varepsilon}(\Omega)^*)}$ and then, by interpolation with $\Theta \in L_2\big(I_{\mathscr{T}}; H^1(\Omega)\big)$, for $\|\Theta\|_{H^\beta(I_{\mathscr{T}};L_2(\Omega))}$ with $\beta < \frac{2}{4+N}$. Passing to the limits $M_k \to +\infty$ and $\delta_k \to 0$ for suitable sequences of approximation parameters and the corresponding sequences of solutions to the appropriate approximate problems Theorem 4 is proved. The details of the proof can be found in [4]. $\square$

Remark. 1. With the use of more sophisticated test functions in the heat equation after the derivation of the energy estimate, the interval for the magnitude of γ for which Theorem 4 holds was remarkably extended. For the case with and without radiation the sufficiency of its lower bound $1 - \frac{2}{N}$, $1 - \frac{1}{N} - \frac{1}{N+1}$, respectively, was proved in the final version of [4].

2. This technique using the validity of estimate (39) can also improve remarkably the results of Theorem 3. In [7] it is proved that for $N = 2, 3$, the requirement $\gamma > \big(\sqrt{12.2} - 3\big)/4$, $1/2$, respectively, is sufficient if no radiation is assumed. With radiation the bound for $N = 2$ is 0.

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