ON THE DIFFERENCE EQUATION $x_{n+1} = \dfrac{a_0x_n + a_1x_{n-1} + \ldots + a_kx_{n-k}}{b_0x_n + b_1x_{n-1} + \ldots + b_kx_{n-k}}$

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Abstract. In this paper we investigate the global convergence result, boundedness and periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{a_0x_n + a_1x_{n-1} + \ldots + a_kx_{n-k}}{b_0x_n + b_1x_{n-1} + \ldots + b_kx_{n-k}}, \quad n = 0, 1, \ldots$$

where the parameters a_i and b_i for $i = 0, 1, \ldots, k$ are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$ are arbitrary positive numbers.

Keywords: stability, periodic solution, difference equation

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1. Introduction

Our goal in this paper is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{a_0x_n + a_1x_{n-1} + \ldots + a_kx_{n-k}}{b_0x_n + b_1x_{n-1} + \ldots + b_kx_{n-k}}, \tag{1}$$

where the parameters a_i and b_i for $i = 0, 1, \ldots, k$ are positive real numbers and the initial conditions are arbitrary positive numbers.

Suppose that $A = \sum\limits_{i=0}^{k} a_i$, $B = \sum\limits_{i=0}^{k} b_i$, $A^r = \sum\limits_{\substack{i=0\\ i\neq r}}^{k} a_i$, $B^r = \sum\limits_{\substack{i=0\\ i\neq r}}^{k} b_i$.

The case when $k = 1$ was investigated in [11]. Other nonlinear rational difference equations were investigated in [8]–[12]. See also [1]–[4].

The study of these equations is quite challenging and rewarding and still at its infancy.

Definition 1. A solution of the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \tag{2}$$

is said to be persistent if there exist numbers m and M with $0 < m \leqslant M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leqslant x_n \leqslant M \quad \text{for all } n \geqslant N.$$

Definition 2 (Stability).
(i) An equilibrium point $\overline{x}$ of Eq. (2) is locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta$$

we have

$$|x_n - \overline{x}| < \varepsilon \quad \text{for all } n \geqslant -k.$$

(ii) An equilibrium point $\overline{x}$ of Eq. (2) is locally asymptotically stable if $\overline{x}$ is a locally stable solution of Eq. (2) and there exists $\gamma > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) An equilibrium point $\overline{x}$ of Eq. (2) is a global attractor if for all x_{-k}, $x_{-k+1}, \dots,$ $x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) An equilibrium point $\overline{x}$ of Eq. (2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq. (2).
(v) An equilibrium point $\overline{x}$ of Eq. (2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq. (2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, \dots, \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [7]**.** *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \;\; n = 0, 1, \ldots.$$

Remark 1. Theorem A can be easily extended to general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, \ldots \tag{4}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. Then Eq. (4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

The following theorem (which we state and prove for the convenience of the reader) treats the method of Full Limiting Sequences which was developed by Karakostas (see [5] and [6]).

Theorem B. *Let $F \in C[I^{k+1}, I]$ for an interval I of real numbers and for a non-negative integer k. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (2), and suppose that there exist constants $A \in I$ and $B \in I$ such that*

$$A \leqslant x_n \leqslant B \quad \text{for all } n \geqslant -k.$$

Let $\mathcal{L}_0$ be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.

(i) *There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq. (2), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = \mathcal{L}_0$ and that for every $N \in \{\ldots, -1, 0, 1, \ldots\}$, L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.*

(ii) *For every $i_0 \leqslant -k$, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that*

$$L_N = \lim_{i\to\infty} x_{r_i+N} \quad \text{for every } N \geqslant i_0.$$

Proof. We first show that there exists a solution $\{l_n\}_{n=-k-1}^{\infty}$ of Eq. (2) such that $l_0 = \mathcal{L}_0$ and that for every $N \geqslant -k-1$, l_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.

To this end, observe that there exists a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i} = \mathcal{L}_0.$$

Now the subsequence $\{x_{n_i-1}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ also lies in the interval $[A,B]$ and so it has a limit point, which we denote by $\mathcal{L}_{-1}$. It follows that there exists another subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ of $\{x_{n_i}\}_{i=0}^{\infty}$ such that $\lim\limits_{j\to\infty} x_{n_j-1} = \mathcal{L}_{-1}$.

Thus we see that

$$\lim_{j\to\infty} x_{n_j-1} = \mathcal{L}_{-1} \qquad \text{and} \qquad \lim_{j\to\infty} x_{n_j} = \mathcal{L}_0.$$

It follows similarly to the above that after re-labelling, if necessary, we may assume that

$$\lim_{j\to\infty} x_{n_j-k-1} = \mathcal{L}_{-k-1},\ \ \lim_{j\to\infty} x_{n_j-k} = \mathcal{L}_{-k}, \dots,\ \lim_{j\to\infty} x_{n_j} = \mathcal{L}_0.$$

Consider the solution $\{l_n\}_{n=-k-1}^{\infty}$ of Eq. (2) with the initial conditions

$$l_{-1} = \mathcal{L}_{-1},\ l_{-2} = \mathcal{L}_{-2}, \dots,\ l_{-k-1} = \mathcal{L}_{-k-1}.$$

Then

$$\begin{aligned} F(\mathcal{L}_{-1}, \mathcal{L}_{-2}, \dots, \mathcal{L}_{-k-1}) &= \lim_{j\to\infty} F(x_{n_j-1}, x_{n_j-2}, \dots, x_{n_j-k-1}) \\ &= \lim_{j\to\infty} x_{n_j} = \mathcal{L}_0 = l_0. \end{aligned}$$

It follows by induction that the solution $\{l_n\}_{n=-k-1}^{\infty}$ of Eq. (2) has the desired property that $l_0 = \mathcal{L}_0$, and that l_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$ for every $N \geqslant -k-1$.

Let S be the set of all solutions $\{\mathcal{L}_n\}_{n=-m}^{\infty}$ of Eq. (2) such that the following statements are true.

(i) $-\infty \leqslant -m \leqslant -k-1$.

(ii) $\mathcal{L}_n = l_n$ for all $n \geqslant -k-1$.

(iii) For every $j_0 \in$ domain $\{\mathcal{L}_n\}_{n=-m}^{\infty}$, there exists a subsequence $\{x_{n_l}\}_{l=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$\mathcal{L}_N = \lim_{l\to\infty} x_{n_l+N} \qquad \text{for all} \qquad N \geqslant j_0.$$

Clearly $\{l_n\}_{n=-k-1}^{\infty} \in S$, and so $S \neq \varphi$. Given $y, z \in S$, we say that $y \preceq z$ if $y \subset z$. It follows that $(S, \preceq)$ is a partially ordered set which satisfies the hypotheses of Zorn's Lemma, and so we see that S has a maximal element which clearly is the desired solution $\{L_n\}_{n=-\infty}^{\infty}$.

2. Linearized stability analysis

In this section we study the local stability character of the solutions of Eq. (1). Eq. (1) has a unique positive equilibrium point and it is given by

$$\overline{x} = \sum_{i=0}^{k} a_i \Big/ \sum_{i=0}^{k} b_i = \frac{A}{B}.$$

Let $f\colon (0,\infty)^{k+1} \longrightarrow (0,\infty)$ be a function defined by

$$f(u_0, u_1, \ldots, u_k) = \frac{a_0u_0 + a_1u_1 + \ldots + a_ku_k}{b_0u_0 + b_1u_1 + \ldots + b_ku_k}. \tag{5}$$

Then it follows that

$$\begin{aligned}
f_{u_0}(u_0, u_1, \ldots, u_k) &= \frac{(a_0b_1 - a_1b_0)u_1 + (a_0b_2 - a_2b_0)u_2 + \ldots + (a_0b_k - a_kb_0)u_k}{(b_0u_0 + b_1u_1 + \ldots + b_ku_k)^2} \\
&= \Bigl(a_0 \sum_{i=1}^{k} b_iu_i - b_0 \sum_{i=1}^{k} a_iu_i\Bigr) \Big/ \Bigl(\sum_{i=0}^{k} b_iu_i\Bigr)^2, \\
f_{u_1}(u_0, u_1, \ldots, u_k) &= \frac{(a_1b_0 - a_0b_1)u_0 + (a_1b_2 - a_2b_1)u_2 + \ldots + (a_1b_k - a_kb_1)u_k}{(b_0u_0 + b_1u_1 + \ldots + b_ku_k)^2} \\
&= \Bigl(a_1 \sum_{\substack{i=0,\\ i\neq 1}}^{k} b_iu_i - b_1 \sum_{\substack{i=0,\\ i\neq 1}}^{k} a_iu_i\Bigr) \Big/ \Bigl(\sum_{i=0}^{k} b_iu_i\Bigr)^2, \\
&\ \vdots \\
f_{u_k}(u_0, u_1, \ldots, u_k) &= \frac{(a_kb_0 - a_0b_k)u_0 + (a_kb_1 - a_1b_k)u_1 + \ldots + (a_kb_{k-1} - a_{k-1}b_k)u_{k-1}}{(b_0u_0 + b_1u_1 + \ldots + b_ku_k)^2} \\
&= \Bigl(a_k \sum_{i=0}^{k-1} b_iu_i - b_k \sum_{i=0}^{k-1} a_iu_i\Bigr) \Big/ \Bigl(\sum_{i=0}^{k} b_iu_i\Bigr)^2.
\end{aligned}$$

Now we see that

$$\begin{aligned}
f_{u_0}(\overline{x}, \overline{x}, \ldots, \overline{x}) &= \frac{(a_0b_1 - a_1b_0) + (a_0b_2 - a_2b_0) + \ldots + (a_0b_k - a_kb_0)}{AB} \\
&= \frac{a_0B^0 - b_0A^0}{AB}, \\
f_{u_1}(\overline{x}, \overline{x}, \ldots, \overline{x}) &= \frac{(a_1b_0 - a_0b_1) + (a_1b_2 - a_2b_1) + \ldots + (a_1b_k - a_kb_1)}{AB} \\
&= \frac{a_1B^1 - b_1A^1}{AB}, \\
&\ \vdots
\end{aligned}$$

$$
\begin{aligned}
f_{u_k}(\overline{x},\overline{x},\ldots,\overline{x}) &= \frac{(a_k b_0 - a_0 b_k) + (a_k b_1 - a_1 b_k) + \ldots + (a_k b_{k-1} - a_{k-1} b_k)}{AB} \\
&= \frac{a_k B^k - b_k A^k}{AB}.
\end{aligned}
$$

The linearized equation of Eq. (1) about $\overline{x}$ is

$$
y_{n+1} + \sum_{i=0}^{k} d_i y_{n-i} = 0,
$$

where $d_i = -f_{u_i}(\overline{x},\overline{x},\ldots,\overline{x})$ for $i = 0,1,\ldots,k$, whose characteristic equation is

$$
\lambda^{k+1} + \sum_{i=0}^{k} d_i \lambda^i = 0.
$$

Theorem 2.1. *Assume that*

$$
\sum_{i=0}^{k} \left| a_i B^i - b_i A^i \right| < AB.
$$

Then the positive equilibrium point of Eq. (1) *is locally asymptotically stable.*

Proof. The proof is a direct consequence of Theorem A. □

3. Boundedness of solutions

Here we study the persistence of Eq. (1).

Theorem 3.1. *Every solution of Eq.* (1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$
\begin{aligned}
x_{n+1} &= \frac{a_0 x_n + a_1 x_{n-1} + \ldots + a_k x_{n-k}}{b_0 x_n + b_1 x_{n-1} + \ldots + b_k x_{n-k}} \\
&= \frac{a_0 x_n}{b_0 x_n + b_1 x_{n-1} + \ldots + b_k x_{n-k}} + \frac{a_1 x_{n-1}}{b_0 x_n + b_1 x_{n-1} + \ldots + b_k x_{n-k}} + \ldots \\
&\quad + \frac{a_k x_{n-k}}{b_0 x_n + b_1 x_{n-1} + \ldots + b_k x_{n-k}} \\
&\leqslant \frac{a_0 x_n}{b_0 x_n} + \frac{a_1 x_{n-1}}{b_1 x_{n-1}} + \ldots + \frac{a_k x_{n-k}}{b_k x_{n-k}}.
\end{aligned}
$$

Hence

$$x_n \leqslant \sum_{i=0}^{k} \frac{a_i}{b_i} = M \quad \text{for all} \quad n \geqslant 1. \tag{6}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \geqslant m \quad \text{for all} \quad n \geqslant 1.$$

The transformation

$$x_n = \frac{1}{y_n}$$

will reduce Eq. (1) to the equivalent form

$$\begin{aligned}
y_{n+1} &= \frac{b_0 \prod\limits_{i=1}^{k} y_{n-i} + b_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i} + \ldots + b_k \prod\limits_{i=1}^{k-1} y_{n-i}}{a_0 \prod\limits_{i=1}^{k} y_{n-i} + a_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i} + \ldots + a_k \prod\limits_{i=1}^{k-1} y_{n-i}} \\
&= \frac{b_0 \prod\limits_{i=1}^{k} y_{n-i}}{a_0 \prod\limits_{i=1}^{k} y_{n-i} + a_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i} + \ldots + a_k \prod\limits_{i=1}^{k-1} y_{n-i}} \\
&\quad + \frac{b_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i}}{a_0 \prod\limits_{i=1}^{k} y_{n-i} + a_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i} + \ldots + a_k \prod\limits_{i=1}^{k-1} y_{n-i}} + \ldots \\
&\quad + \frac{b_k \prod\limits_{i=1}^{k-1} y_{n-i}}{a_0 \prod\limits_{i=1}^{k} y_{n-i} + a_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i} + \ldots + a_k \prod\limits_{i=1}^{k-1} y_{n-i}},
\end{aligned}$$

which implies that

$$y_{n+1} \leqslant \frac{b_0 \prod\limits_{i=1}^{k} y_{n-i}}{a_0 \prod\limits_{i=1}^{k} y_{n-i}} + \frac{b_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i}}{a_1 \prod\limits_{i=0, i\neq 1}^{k} y_{n-i}} + \ldots + \frac{b_k \prod\limits_{i=1}^{k-1} y_{n-i}}{a_k \prod\limits_{i=1}^{k-1} y_{n-i}} = \frac{b_0}{a_0} + \frac{b_1}{a_1} + \ldots + \frac{b_k}{a_k}.$$

Hence

$$\frac{1}{x_{n+1}} \leqslant \sum_{i=0}^{k} \frac{b_i}{a_i}.$$

It follows that

$$\frac{1}{x_{n+1}} \leqslant \sum_{i=0}^{k} \frac{b_i}{a_i} = H \quad \text{for all} \quad n \geqslant 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geqslant \frac{1}{H} = m \quad \text{for all } n \geqslant 1. \tag{7}$$

From (6) and (7) we see that

$$m \leqslant x_n \leqslant M \quad \text{for all } n \geqslant 1.$$

Therefore every solution of Eq. (1) is bounded and persists.

4. Periodicity of solutions

In this section we study the existence of a prime period two solutions of Eq. (1). Let α, β, γ and δ be defined as follows:
If k is odd, then

$$\alpha = \sum_{i=0}^{(k-1)/2} a_{2i}, \quad \beta = \sum_{i=0}^{(k-1)/2} a_{2i+1},$$
$$\gamma = \sum_{i=0}^{(k-1)/2} b_{2i}, \quad \delta = \sum_{i=0}^{(k-1)/2} b_{2i+1},$$

if k is even, then

$$\alpha = \sum_{i=0}^{k/2} a_{2i}, \quad \beta = \sum_{i=0}^{k/2-1} a_{2i+1},$$
$$\gamma = \sum_{i=0}^{k/2} b_{2i}, \quad \delta = \sum_{i=0}^{k/2-1} b_{2i+1}.$$

Theorem 4.1. *Eq. (1) has a positive prime period two solution if and only if*

$$4\delta\alpha < (\gamma - \delta)(\beta - \alpha). \tag{8}$$

P r o o f. First suppose that there exists a prime period two solution

$$\ldots, p, q, p, q, \ldots$$

of Eq. (1). We will prove that condition (8) holds.

We see from Eq. (1) that

$$p = \frac{\alpha q + \beta p}{\gamma q + \delta p}$$

and

$$q = \frac{\alpha p + \beta q}{\gamma p + \delta q}.$$

Then

$$\gamma pq + \delta p^2 = \alpha q + \beta p \tag{9}$$

and

$$\gamma pq + \delta q^2 = \alpha p + \beta q. \tag{10}$$

Subtracting (9) from (10) gives

$$\delta(p^2 - q^2) = (\beta - \alpha)(p - q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{\beta - \alpha}{\delta}. \tag{11}$$

Also, since p and q are positive, $(\beta - \alpha)$ should be positive.

Again, adding (9) and (10) yields

$$2\gamma pq + \delta(p^2 + q^2) = (p + q)(\alpha + \beta). \tag{12}$$

It follows by (11), (12) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{for all } p, q \in \mathbb{R}$$

that

$$2(\gamma - \delta)pq = \frac{2\alpha(\beta - \alpha)}{\delta}.$$

Again, since p and q are positive and $\beta > \alpha$, we see that $\gamma > \delta$.

Thus

$$pq = \frac{\alpha(\beta - \alpha)}{\delta(\gamma - \delta)}. \tag{13}$$

Now it is clear from Eq. (11) and Eq. (13) that p and q are the two positive distinct roots of the quadratic equation

$$t^2 - \frac{\beta - \alpha}{\delta} t + \frac{\alpha(\beta - \alpha)}{\delta(\gamma - \delta)} = 0, \tag{14}$$

and so

$$\left[\frac{\beta-\alpha}{\delta}\right]^2 - \frac{4\alpha(\beta-\alpha)}{\delta(\gamma-\delta)} > 0.$$

Since $\gamma - \delta$ and $\beta - \alpha$ have the same sign,

$$\frac{\beta-\alpha}{\delta} > \frac{4\alpha}{\gamma-\delta},$$

which is equivalent to

$$4\delta\alpha < (\gamma-\delta)(\beta-\alpha).$$

Therefore inequality (8) holds.

Second suppose that inequality (8) is true. We will show that Eq. (1) has a prime period two solution.

Assume that

$$p = \frac{\frac{\beta-\alpha}{\delta} - \sqrt{\left[\frac{\beta-\alpha}{\delta}\right]^2 - \frac{4\alpha(\beta-\alpha)}{\delta(\gamma-\delta)}}}{2}$$

and

$$q = \frac{\frac{\beta-\alpha}{\delta} + \sqrt{\left[\frac{\beta-\alpha}{\delta}\right]^2 - \frac{4\alpha(\beta-\alpha)}{\delta(\gamma-\delta)}}}{2}.$$

We see from inequality (8) that

$$(\gamma-\delta)(\beta-\alpha) > 4\delta\alpha$$

or

$$[\beta-\alpha]^2 > \frac{4\delta\alpha(\beta-\alpha)}{\gamma-\delta},$$

which is equivalent to

$$\left[\frac{\beta-\alpha}{\delta}\right]^2 > \frac{4\alpha(\beta-\alpha)}{\delta(\gamma-\delta)}.$$

Therefore p and q are distinct positive real numbers.

If k is odd, then we set (the case when k is even is similar and will be omitted) $x_{-k} = q$, $x_{-k+1} = p, \ldots,$ and $x_0 = p$. We wish to show that $x_1 = x_{-1} = q$ and $x_2 = x_0 = p$. It follows from Eq. (1) that

$$x_1 = \frac{\alpha p + \beta q}{\gamma p + \delta q},$$

where p and q are as given above.

It follows that

$$x_1 = \frac{\alpha\left[1-\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}\right]+\beta\left[1+\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}\right]}{\gamma\left[1-\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}\right]+\delta\left[1+\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}\right]}$$

$$= \frac{(\alpha+\beta)+(\beta-\alpha)\left[\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}\right]}{(\gamma+\delta)+(\delta-\gamma)\left[\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}\right]}.$$

Hence

$$x_1 = \frac{(\alpha+\beta)(\gamma+\delta)-(\beta-\alpha)(\delta-\gamma)\left[1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}\right]}{(\gamma+\delta)^2-(\delta-\gamma)^2\left[1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}\right]}$$

$$+\frac{\{(\beta-\alpha)(\gamma+\delta)-(\alpha+\beta)(\delta-\gamma)\}\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}}{(\gamma+\delta)^2-[\delta-\gamma]^2\left[1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}\right]}$$

$$= \frac{[2\beta\gamma-2\delta\alpha]+[2\beta\gamma-2\delta\alpha]\sqrt{1-\dfrac{4\delta\alpha}{(\beta-\alpha)(\gamma-\delta)}}}{4\delta\gamma-\dfrac{4\delta\alpha[\delta-\gamma]}{\beta-\alpha}}$$

$$= \frac{\dfrac{\beta-\alpha}{\delta}+\sqrt{\left[\dfrac{\beta-\alpha}{\delta}\right]^2-\dfrac{4\alpha(\beta-\alpha)}{\delta(\gamma-\delta)}}}{2} = q.$$

Similarly to the above one can show that

$$x_2 = p.$$

Then it follows by induction that

$$x_{2n} = p \quad \text{and} \quad x_{2n+1} = q \quad \text{for all} \quad n \geqslant -1.$$

Thus Eq. (1) has the positive prime period two solution

$$\dots, p, q, p, q, \dots$$

where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

5. Global stability of eq. (1)

In this section we investigate the global asymptotic stability of Eq. (1).

Theorem 5.1. *If the function $f(u_0, u_1, \dots, u_k)$ defined by Eq. (5) is non decreasing in u_i, non increasing in u_j and*

$$A^j B^i \leqslant a_j(2b_i + B^i), \quad i, j = 0, 1, \dots, k, \tag{15}$$

then the equilibrium point $\overline{x}$ is a global attractor of Eq. (1).

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1) and let f be the function defined by Eq. (5) which is non decreasing in u_i if $a_i/b_i \geqslant a_j/b_j$, and non increasing in u_i if $a_i/b_i \leqslant a_j/b_j$, $i, j = 0, 1, \dots, k$.

From Eq. (1) we see that

$$\begin{aligned}
x_{n+1} &= \frac{a_0x_n + a_1x_{n-1} + \ldots + a_jx_{n-j} + \ldots + a_kx_{n-k}}{b_0x_n + b_1x_{n-1} + \ldots + b_jx_{n-j} + \ldots + b_kx_{n-k}} \\
&\leqslant \frac{a_0x_n + a_1x_{n-1} + \ldots + a_j(0) + \ldots + a_kx_{n-k}}{b_0x_n + b_1x_{n-1} + \ldots + b_j(0) + \ldots + b_kx_{n-k}} \\
&\leqslant \frac{a_0x_n}{b_0x_n} + \frac{a_1x_{n-1}}{b_1x_{n-1}} + \ldots + \frac{a_{j-1}x_{n-(j-1)}}{b_{j-1}x_{n-(j-1)}} + \frac{a_{j+1}x_{n-(j+1)}}{b_{j+1}x_{n-(j+1)}} + \ldots + \frac{a_kx_{n-k}}{b_kx_{n-k}} \\
&= \sum_{\substack{i=0 \\ i\neq j}}^{k} \frac{a_i}{b_i} = M.
\end{aligned}$$

Hence

$$x_n \leqslant M \quad \text{for all} \quad n \geqslant 1. \tag{16}$$

In the other hand,

$$\begin{aligned}
x_{n+1} &\geqslant \frac{a_0x_n + a_1x_{n-1} + \ldots + a_j(M) + \ldots + a_i(0) + \ldots + a_kx_{n-k}}{b_0x_n + b_1x_{n-1} + \ldots + b_j(M) + \ldots + b_i(0) + \ldots + b_kx_{n-k}} \\
&\geqslant \frac{a_jM}{b_0M + b_1M + \ldots + b_j(M) + \ldots + b_i(0) + \ldots + b_kM} \\
&= \frac{a_jM}{B^iM} = \frac{a_j}{B^i} = m.
\end{aligned} \tag{17}$$

From Eqs. (16) and (17) we see that

$$m=\frac{a_j}{B^i}\leqslant x_n\leqslant\sum_{\substack{i=0\\ i\neq j}}^{k}\frac{a_i}{b_i}=M\quad\text{for all } n\geqslant 1. \tag{18}$$

It follows by the Method of Full Limiting Sequences that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq. (1) with

$$I=I_0=\lim_{n\to\infty}\inf x_n\leqslant\lim_{n\to\infty}\sup x_n=S_0=S,$$

where

$$I_n,S_n\in[I,S],\quad n=0,-1,\ldots.$$

It suffices to show that $I=S$.

It follows from Eq. (1) that

$$\begin{aligned}I&=\frac{a_0I_{-1}+a_1I_{-2}+\ldots+a_jI_{-j-1}+\ldots+a_iI_{-i-1}+\ldots+a_kI_{-k-1}}{b_0I_{-1}+b_1I_{-2}+\ldots+b_jI_{-j-1}+\ldots+b_iI_{-i-1}+\ldots+b_kI_{-k-1}}\\&\geqslant\frac{a_0I_{-1}+a_1I_{-2}+\ldots+a_j(S)+\ldots+a_i(I)+\ldots+a_kI_{-k-1}}{b_0I_{-1}+b_1I_{-2}+\ldots+b_j(S)+\ldots+b_i(I)+\ldots+b_kI_{-k-1}}\\&\geqslant\frac{a_0I+a_1I+\ldots+a_j(S)+\ldots+a_i(I)+\ldots+a_kI}{b_0S+b_1S+\ldots+b_j(S)+\ldots+b_i(I)+\ldots+b_kS}=\frac{A^jI+a_jS}{B^iS+b_iI},\end{aligned}$$

and so

$$B^iSI\geqslant A^jI+a_jS-b_iI^2. \tag{19}$$

Similarly, we see from Eq. (1) that

$$\begin{aligned}S&=\frac{a_0S_{-1}+a_1S_{-2}+\ldots+a_jS_{-j-1}+\ldots+a_iS_{-i-1}+\ldots+a_kS_{-k-1}}{b_0S_{-1}+b_1S_{-2}+\ldots+b_jS_{-j-1}+\ldots+b_iS_{-i-1}+\ldots+b_kS_{-k-1}}\\&\leqslant\frac{a_0S_{-1}+a_1S_{-2}+\ldots+a_j(I)+\ldots+a_i(S)+\ldots+a_kS_{-k-1}}{b_0S_{-1}+b_1S_{-2}+\ldots+b_j(I)+\ldots+b_i(S)+\ldots+b_kS_{-k-1}}\\&\leqslant\frac{a_0S+a_1S+\ldots+a_j(I)+\ldots+a_i(S)+\ldots+a_kS}{b_0I+b_1I+\ldots+b_j(I)+\ldots+b_i(S)+\ldots+b_kI}=\frac{A^jS+a_jI}{B^iI+b_iS},\end{aligned}$$

and so

$$B^iSI\leqslant A^jS+a_jI-b_iS^2. \tag{20}$$

Therefore it follows from (19) and (20) that

$$A^jI+a_jS-b_iI^2\leqslant A^jS+a_jI-b_iS^2$$

or

$$A^j(S-I)+a_j(I-S)+b_i(I^2-S^2)\geqslant 0,$$
$$(I-S)\{a_j+b_i(I+S)-A^j\}\geqslant 0,$$

and so

$$I\geqslant S \quad \text{if } a_j+b_i(I+S)-A^j\geqslant 0.$$

Now, we know by (15) that

$$A^jB^i\leqslant a_j(2b_i+B^i).$$

Hence

$$A^jB^i\leqslant a_jB^i\Big(\frac{2b_i}{B^i}+1\Big)$$

or

$$A^j\leqslant b_i\Big(\frac{a_j}{B^i}+\frac{a_j}{B^i}\Big)+a_j.$$

It follows from Eq. (18) that

$$A^j\leqslant b_i(I+S)+a_j,$$

and so it follows that

$$I\geqslant S.$$

Therefore

$$I=S.$$

This completes the proof. □

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