OSCILLATION OF A NONLINEAR DIFFERENCE EQUATION WITH SEVERAL DELAYS

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Abstract. In this paper we consider the nonlinear difference equation with several delays

$$(ax_{n+1}+bx_n)^k-(cx_n)^k+\sum_{i=1}^{m}p_i(n)x^k_{n-\sigma_i}=0$$

where $a,b,c\in(0,\infty)$, $k=q/r, q, r$ are positive odd integers, m, σ_i are positive integers, $\{p_i(n)\}$, $i=1,2,\dots,m$, is a real sequence with $p_i(n)\geqslant 0$ for all large n, and $\liminf\limits_{n\to\infty}p_i(n)=p_i<\infty$, $i=1,2,\dots,m$. Some sufficient conditions for the oscillation of all solutions of the above equation are obtained.

Keywords: nonlinear difference equtions, oscillation, eventually positive solutions, characteristic equation

MSC 2000: 39A10

1. Introduction

Consider the nonlinear difference equation

$$(ax_{n+1}+bx_n)^k-(cx_n)^k+\sum_{i=1}^{m}p_i(n)x^k_{n-\sigma_i}=0 \tag{1}$$

where $a,b,c\in(0,\infty)$, $c>b$, $k=q/r, q, r$ are positive odd integers, m,σ_i are positive integers, $\{p_i(n)\}$ are real sequences with $p_i(n)\geqslant 0$ for all large n, and $\liminf\limits_{n\to\infty}p_i(n)=$

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$p_i < \infty$, $i = 1, 2, \ldots, m$. It is easy to see that if $c < b$ then (1) cannot have an eventually positive solution. The corresponding "limiting" equation of (1) is

$$(ax_{n+1} + bx_n)^k - (cx_n)^k + \sum_{i=1}^{m} p_i x_{n-\sigma_i}^k = 0 \tag{2}$$

with the characteristic equation

$$(a\lambda + b)^k - c^k + \sum_{i=1}^{m} p_i \lambda^{-k\sigma_i} = 0. \tag{3}$$

For the special case where $k = a = 1, c = b + 1$, equation (1) reduces to the linear difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-\sigma_i} = 0.$$

There have been a lot of activities concerning the oscillation of solutions of linear difference equations. But there have been few results for the oscillation of solutions of the nonlinear equation (1). Under the condition that $0 < \frac{c-b}{a} \leqslant 1$, a sufficient condition of nonexistence of eventually positive solutions for (1) was obtained in [5], [6]. In this paper we obtain several new sufficient conditions for oscillation of all solutions of (1) by removing the condition $\frac{c-b}{a} \leqslant 1$. A sufficient and necessary condition for oscillation of all solutions of (2) is obtained as well.

A solution $\{x_n\}$ of equation (1) is said to oscillate about zero or simply to oscillate if the terms x_n of the sequence $\{x_n\}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

2. Main results

Lemma 1 [3]. *If x, y are positive numbers and $x \neq y$, then*

$$rx^{r-1}(x-y) \geqslant x^r - y^r \geqslant ry^{r-1}(x-y) \quad \text{for} \quad r \geqslant 1.$$

Theorem 1. *If* (3) *has no positive roots, then every solution of* (1) *oscillates.*

P r o o f. By way of contradiction, assume that $\{x_n\}$ is an eventually positive solution of (1). By (1) we have

$$\left(a\frac{x_{n+1}}{x_n} + b\right)^k - c^k + \sum_{i=1}^{m} p_i(n) \left(\frac{x_{n-\sigma_i}}{x_n}\right)^k = 0. \tag{4}$$

For sufficiently large n, set $\frac{x_{n+1}}{x_n} = \beta_n$. Then eventually

$$0 < \beta_n \leqslant \frac{c-b}{a},$$

and (4) yields

$$(a\beta_n + b)^k - c^k + \sum_{i=1}^{m} p_i(n)\bigg(\prod_{j=1}^{\sigma_i} \beta_{n-j}^{-1}\bigg)^k = 0. \tag{5}$$

Set

$$\limsup_{n\to\infty} \beta_n = \beta.$$

It follows from (5) that $0 < \beta \leqslant \frac{c-b}{a}$. We also claim that

$$(a\beta + b)^k - c^k + \sum_{i=1}^{m} p_i \beta^{-k\sigma_i} \leqslant 0.$$

Indeed, by virtue of $\liminf\limits_{n\to\infty} p_i(n) = p_i < \infty$, for every $\varepsilon \in (0,1)$ there exists an $n_\varepsilon > 0$ such that

$$p_i(n) \geqslant (1-\varepsilon)p_i \quad \text{for} \quad i = 1, 2, \dots, m \quad \text{and} \quad n \geqslant n_\varepsilon.$$

Therefore,

$$(a\beta_n + b)^k - c^k + (1-\varepsilon)\sum_{i=1}^{m} p_i \bigg(\prod_{j=1}^{\sigma_i} \beta_{n-j}^{-1}\bigg)^k \leqslant 0 \quad \text{for} \quad n \geqslant n_\varepsilon.$$

Let $N_\varepsilon > n_\varepsilon$ be such that

$$\beta_n < (1+\varepsilon)\beta \quad \text{for} \quad n \geqslant N_\varepsilon.$$

Then

$$(a\beta_n + b)^k - c^k + (1-\varepsilon)\sum_{i=1}^{m} p_i (1+\varepsilon)^{-k\sigma_i}\beta^{-k\sigma_i} \leqslant 0 \quad \text{for} \quad n \geqslant N_\varepsilon + \sigma_i,$$

i.e.,

$$(a\beta_n + b)^k \leqslant c^k - (1-\varepsilon)\sum_{i=1}^{m} p_i (1+\varepsilon)^{-k\sigma_i}\beta^{-k\sigma_i},$$

which implies

$$(a\beta + b)^k \leqslant c^k - (1-\varepsilon)\sum_{i=1}^{m} p_i (1+\varepsilon)^{-k\sigma_i}\beta^{-k\sigma_i}.$$

As this is true for every $\varepsilon \in (0,1)$, it follows that

$$(6) \qquad (a\beta + b)^k \leqslant c^k - \sum_{i=1}^{m} p_i \beta^{-k\sigma_i},$$

which proves our claim. Set

$$(7) \qquad F(\lambda) = (a\lambda + b)^k - c^k + \sum_{i=1}^{m} p_i \lambda^{-k\sigma_i}.$$

Then $F(0+) = +\infty$ and $F(\beta) \leqslant 0$. It follows that (3) has a positive root. This contradiction completes the proof. $\square$

Corollary 1. *Every solution of* (2) *oscillates if and only if* (3) *has no positive roots.*

P r o o f. Sufficiency can be directly derived from Theorem 1. So it suffices to prove Necessity. Suppose that (3) has a positive root λ.

Let

$$x_n = \lambda^n, \ n = 1, 2, \ldots.$$

Then we have

$$(ax_{n+1} + bx_n)^k - (cx_n)^k + \sum_{i=1}^{m} p_i x_{n-\sigma_i}^k = \lambda^{kn}\left[(a\lambda + b)^k - c^k + \sum_{i=1}^{m} p_i \lambda^{-k\sigma_i}\right] = 0.$$

Hence, $\{x_n\}$ is an eventually positive solution of (2). This contradiction completes the proof. $\square$

Corollary 2. *Assume that* $\liminf_{n\to\infty} p_i(n) = p_i < \infty$, $i = 1, 2, \ldots, m$, $k > 1$, *and that*

$$\sum_{i=1}^{m} \frac{p_i a^{k\sigma_i}(k\sigma_i + 1)^{k\sigma_i+1}}{kc^{k-1}(c-b)^{k\sigma_i+1}(k\sigma_i)^{k\sigma_i}} > 1.$$

Then every solution of (1) *oscillates.*

P r o o f. Let

$$F(\lambda) = (a\lambda + b)^k - c^k + \sum_{i=1}^{m} p_i \lambda^{-k\sigma_i}.$$

Then

$$F(\lambda) > 0 \quad \text{for} \quad \lambda \geqslant \frac{c-b}{a}.$$

For $\lambda < \frac{c-b}{a}$, by Lemma 1, we have

$$F(\lambda) \geqslant \{c^k - (a\lambda + b)^k\}\bigg\{ -1 + \sum_{i=1}^{m} \frac{p_i \lambda^{-k\sigma_i}}{kc^{k-1}(c - a\lambda - b)}\bigg\}.$$

Since

$$\min_{0<\lambda<\frac{c-b}{a}} \left\{ \frac{\lambda^{-k\sigma_i}}{kc^{k-1}(c - a\lambda - b)} \right\} = \frac{a^{k\sigma_i}(k\sigma_i + 1)^{k\sigma_i+1}}{kc^{k-1}(c-b)^{k\sigma_i+1}(k\sigma^i)^{k\sigma_i}},$$

by the condition of Corollary 2 we get

$$F(\lambda) \geqslant \{c^k - (a\lambda+b)^k\}\bigg\{ -1+\sum_{i=1}^{m} \frac{p_i a^{k\sigma_i}(k\sigma_i + 1)^{k\sigma_i+1}}{kc^{k-1}(c-b)^{k\sigma_i+1}(k\sigma^i)^{k\sigma_i}}\bigg\} > 0 \quad \text{for} \quad \lambda < \frac{c-b}{a}.$$

By Theorem 1, every solution of (1) oscillates. The proof is completed. □

Define a sequence $\{\lambda_l\}$ by

$$\lambda_1 = \frac{c-b}{a},\ \lambda_{l+1} = \frac{1}{a}\bigg[\bigg(c^k - \sum_{i=1}^{m} p_i \lambda_l^{-k\sigma_i}\bigg)^{\frac{1}{k}} - b\bigg],\ l = 1, 2, \dots \tag{8}$$

Lemma 2. *Suppose that $\{\lambda_l\}$ is defined by* (8). *Then $0 < \lambda_* \leqslant \lambda_l \leqslant \frac{c-b}{a}$ and $\lim\limits_{l\to\infty} \lambda_l = \lambda_*$, where λ_* is the largest root of equation* (3) *on* $(0, \frac{c-b}{a}]$.

P r o o f. First, we prove the sequence $\{\lambda_l\}$ is nonincreasing. Since

$$\lambda_2 = \frac{1}{a}\bigg[\bigg(c^k - \sum_{i=1}^{m} p_i \lambda_1^{-k\sigma_i}\bigg)^{\frac{1}{k}} - b\bigg] < \frac{c-b}{a} = \lambda_1,$$

hence, by induction, supposing that $\lambda_l \leqslant \lambda_{l-1}$, we have

$$\lambda_{l+1} = \frac{1}{a}\bigg[\bigg(c^k - \sum_{i=1}^{m} p_i \lambda_l^{-k\sigma_i}\bigg)^{\frac{1}{k}} - b\bigg] \leqslant \frac{1}{a}\bigg[\bigg(c^k - \sum_{i=1}^{m} p_i \lambda_{l-1}^{-k\sigma_i}\bigg)^{\frac{1}{k}} - b\bigg] = \lambda_l.$$

Hence, the sequence $\{\lambda_l\}$ is nonincreasing. From (3) it is obvious that $\lambda_1 \geqslant \lambda_*$, and by induction

$$\lambda_{l+1} = \frac{1}{a}\bigg[\bigg(c^k - \sum_{i=1}^{m} p_i \lambda_l^{-k\sigma_i}\bigg)^{\frac{1}{k}} - b\bigg] \geqslant \frac{1}{a}\bigg[\bigg(c^k - \sum_{i=1}^{m} p_i \lambda_*^{-k\sigma_i}\bigg)^{\frac{1}{k}} - b\bigg] = \lambda_*.$$

Hence, $\{\lambda_l\}$ is nonincreasing and bounded. Therefore, $\lim\limits_{l\to\infty} \lambda_l$ exists. Letting $l \to \infty$ in (8) and noting that λ_* is the largest root of (3), we conclude $\lim\limits_{l\to\infty} \lambda_l = \lambda_*$. The proof is completed. □

Theorem 2. *Assume that* (3) *has positive roots,* λ_* *is the largest root of* (3) *and*

$$\limsup_{n\to\infty}\left\{\left[a\left(\frac{1}{c^k-b^k}\sum_{i=1}^{m}p_i(n+1)\lambda_*^{k(1-\sigma_i)}\right)^{\frac{1}{k}}+b\right]^k+\sum_{i=1}^{m}p_i(n)\lambda_*^{-k\sigma_i}\right\}>c^k. \tag{9}$$

Then every solution of (1) *oscillates.*

Proof. Suppose that $\{x_n\}$ is an eventually positive solution of (1). Then there exists $n_1>0$ such that $x_n>0$ and $x_{n-\sigma_i}>0$, $i=1,2,\ldots,m$, for $n\geqslant n_1$. From (1) we have

$$(ax_{n+1}+bx_n)^k-(cx_n)^k=-\sum_{i=1}^{m}p_i(n)x_{n-\sigma_i}^k\leqslant 0,$$

i.e.,

$$(ax_{n+1}+bx_n)^k\leqslant(cx_n)^k.$$

Since $k=q/r,q,r$ are positive odd integers, we have

$$x_{n+1}\leqslant\theta x_n\quad\text{for}\quad n\geqslant n_1, \tag{10}$$

where $\theta=\frac{c-b}{a}$. By virtue of $\liminf\limits_{n\to\infty}p_i(n)=p_i<\infty$, for every $\varepsilon\in(0,1)$ there exists an $n_\varepsilon>n_1$ such that

$$p_i(n)\geqslant(1-\varepsilon)p_i\quad\text{for}\quad i=1,2,\ldots,m\quad\text{and}\quad n\geqslant n_\varepsilon. \tag{11}$$

Define a sequence $\{\mu_l(\varepsilon)\}$ by

$$\mu_1(\varepsilon)=\theta,\ \mu_{l+1}(\varepsilon)=\frac{1}{a}\left\{\left[c^k-(1-\varepsilon)\sum_{i=1}^{m}p_i(\mu_l(\varepsilon))^{-k\sigma_i}\right]^{\frac{1}{k}}-b\right\},\ l=1,2,\ldots.$$

From (10) we get

$$x_{n-\sigma_i}^k\geqslant(\mu_1(\varepsilon))^{-k\sigma_i}x_n^k\quad\text{for}\quad n\geqslant n_\varepsilon+\sigma_i. \tag{12}$$

From (1), (11), and (12) we have

$$(ax_{n+1}+bx_n)^k\leqslant(cx_n)^k-(1-\varepsilon)\sum_{i=1}^{m}p_i(\mu_1(\varepsilon))^{-k\sigma_i}x_n^k,$$

i.e.,

$$x_{n+1}\leqslant\frac{1}{a}\left\{\left[c^k-(1-\varepsilon)\sum_{i=1}^{m}p_i(\mu_1(\varepsilon))^{-k\sigma_i}\right]^{\frac{1}{k}}-b\right\}x_n.$$

That is

$$x_{n+1} \leqslant \mu_2(\varepsilon)x_n,\ n \geqslant n_\varepsilon + \sigma_i,$$

which gives

$$x_n \leqslant \mu_2(\varepsilon)x_{n-1} \leqslant (\mu_2(\varepsilon))^2 x_{n-2} \leqslant \ldots \leqslant (\mu_2(\varepsilon))^{\sigma_i} x_{n-\sigma_i},\ n \geqslant n_\varepsilon + 2\sigma_i,$$

i.e.,

$$x_{n-\sigma_i} \geqslant (\mu_2(\varepsilon))^{-\sigma_i} x_n.$$

Repeating the above process, we obtain

$$x_{n+1} \leqslant \frac{1}{a}\left\{\left[c^k - \sum_{i=1}^{m} p_i(\mu_{l-1}(\varepsilon))^{-k\sigma_i}\right]^{\frac{1}{k}} - b\right\}x_n,$$

i.e.,

$$x_{n+1} \leqslant \mu_l(\varepsilon)x_n,\ n \geqslant n_\varepsilon + (l-1)\sigma_i. \tag{13}$$

Since $\lim\limits_{\varepsilon\to 0} \mu_l(\varepsilon) = \lambda_l$ and $\lim\limits_{l\to\infty} \lambda_l = \lambda_*$, for a sequence $\{\varepsilon_l\}$ with $\varepsilon_l > 0$ and $\varepsilon_l \to 0$ as $l \to \infty$, by (13) there exists a sequence $\{n_l\}$ such that $n_l \to \infty$ as $l \to \infty$ and

$$x_{n+1} \leqslant (\lambda_* + \varepsilon_l)x_n,\ n > n_l, \tag{14}$$

and

$$x_{n-\sigma_i} \geqslant (\lambda_* + \varepsilon_l)^{-\sigma_i} x_n,\ n > n_l + \sigma_i. \tag{15}$$

On the other hand, from (1) we have

$$(b^k - c^k)x_n^k + \sum_{i=1}^{m} p_i(n)x_{n-\sigma_i}^k \leqslant 0,$$

i.e.,

$$(c^k - b^k)x_n^k \geqslant \sum_{i=1}^{m} p_i(n)x_{n-\sigma_i}^k. \tag{16}$$

From (15) and (16) we obtain

$$(c^k - b^k)x_n^k \geqslant \sum_{i=1}^{m} p_i(n)(\lambda_* + \varepsilon_l)^{k(1-\sigma_i)} x_{n-1}^k,$$

i.e.,

$$\frac{x_n}{x_{n-1}} \geqslant \left[\frac{1}{c^k-b^k}\sum_{i=1}^{m} p_i(n)(\lambda_*+\varepsilon_l)^{k(1-\sigma_i)}\right]^{\frac{1}{k}},\ n > n_l+\sigma_i. \tag{17}$$

From (1) we have

$$c^k = \left(\frac{ax_{n+1}+bx_n}{x_n}\right)^k + \sum_{i=1}^{m} p_i(n)\frac{x^k_{n-\sigma_i}}{x^k_n}. \tag{18}$$

From (15), (17) and (18) we obtain

$$\begin{aligned} c^k \geqslant \Bigg\{ \Bigg[a\Bigg(\frac{1}{c^k-b^k}\sum_{i=1}^{m} p_i(n+1)(\lambda_*+\varepsilon_l)^{k(1-\sigma_i)}\Bigg)^{\frac{1}{k}} + b\Bigg]^k \\ + \sum_{i=1}^{m} p_i(n)(\lambda_*+\varepsilon_l)^{-k\sigma_i}\Bigg\}. \end{aligned} \tag{19}$$

Let $l \to \infty$, then (19) implies

$$\limsup_{n\to\infty}\Bigg\{\Bigg[a\Bigg(\frac{1}{c^k-b^k}\sum_{i=1}^{m} p_i(n+1)\lambda_*^{k(1-\sigma_i)}\Bigg)^{\frac{1}{k}} + b\Bigg]^k + \sum_{i=1}^{m} p_i(n)\lambda_*^{-k\sigma_i}\Bigg\} \leqslant c^k,$$

which contradicts (9) and completes the proof. □

R e m a r k s 1. Theorems 1 and 2 extend the results on linear difference equations in [2], [9].

E x a m p l e 1. Consider equation (1). Let

$$\begin{gathered} p_1(0) = \frac{4}{3},\ p_1(1) = \frac{3}{4},\ p_1(n+2) = p_1(n) \quad \text{for} \quad n = 0,1,2,\dots, \\ p_2(0) = \frac{3}{4},\ p_2(1) = \frac{1}{2},\ p_2(n+2) = p_2(n) \quad \text{for} \quad n = 0,1,2,\dots, \\ k=3,\ a=1,\ b=1,\ c=2,\ \sigma_1=1,\ \sigma_2=2,\ m=2. \end{gathered}$$

Then

$$\liminf_{n\to\infty} p_1(n) = \frac{3}{4},\ \liminf_{n\to\infty} p_2(n) = \frac{1}{2},$$

and

$$\sum_{i=1}^{2} \frac{p_i a^{k\sigma_i}(k\sigma_i+1)^{k\sigma_i+1}}{kc^{k-1}(c-b)^{k\sigma_i+1}(k\sigma_i)^{k\sigma_i}} = 1.32\ldots > 1.$$

Thus according to Corollary 2, every solution of (1) oscillates.

E x a m p l e 2. Consider equation (1). Let

$$p_1(0)=1,\ p_1(1)=7,\ p_1(n+2)=p_1(n) \quad \text{for} \quad n=0,1,2,\ldots,$$
$$k=3,\ a=1,\ b=1,\ c=2,\ \sigma_1=1,\ m=1.$$

Then

$$\liminf_{n\to\infty} p_1(n)=1.$$

Hence, the characteristic equation (3) has the largest root $\lambda_*=0.8586$. Therefore,

$$\limsup_{n\to\infty}\left\{\left[a\left(\frac{1}{c^k-b^k}\sum_{i=1}^{m}p_i(n+1)\lambda_*^{k(1-\sigma_i)}\right)^{\frac{1}{k}}+b\right]^k+\sum_{i=1}^{m}p_i(n)\lambda_*^{-k\sigma_i}\right\}=9.5799>8.$$

Thus according to Theorem 2, every solution of (1) oscillates.

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