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ON FREDHOLM ALTERNATIVE FOR CERTAIN QUASILINEAR BOUNDARY VALUE PROBLEMS

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Abstract. We study the Dirichlet boundary value problem for the p-Laplacian of the form

$$-\Delta_p u - \lambda_1 |u|^{p-2} u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geqslant 1$, $p > 1$, $f \in C(\overline{\Omega})$ and $\lambda_1 > 0$ is the first eigenvalue of Δ_p. We study the geometry of the energy functional

$$E_p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \frac{\lambda_1}{p} \int_\Omega |u|^p - \int_\Omega f u$$

and show the difference between the case $1 < p < 2$ and the case $p > 2$. We also give the characterization of the right hand sides f for which the above Dirichlet problem is solvable and has multiple solutions.

Keywords: p-Laplacian, variational methods, PS condition, Fredholm alternative, upper and lower solutions

MSC 2000: 35J60, 35P30, 35B35, 49N10

1. Statement of the results

Our aim is to study the solvability of the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u - \lambda_1 |u|^{p-2} u = f & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $p > 1$ is a real number, Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ [1], $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian and $f \in C(\overline{\Omega})$. By λ_1 we

[1] We assume that if $N \geqslant 2$ then $\partial\Omega$ is a compact connected manifold of class C^2.

denote the first eigenvalue of the related homogeneous eigenvalue problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = 0 \text{ in } \Omega, \\ \qquad\qquad\qquad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

In this paper, the function u is said to be a (*weak*) *solution* of (1.1) if $u \in W_0^{1,p}(\Omega)$ and the integral identity

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda_1 \int_\Omega |u|^{p-2} uv = \int_\Omega fv \tag{1.3}$$

holds for all $v \in W_0^{1,p}(\Omega)$.

As for the properties of λ_1 (see e.g. [1], [15]), let us mention that λ_1 is positive, simple and isolated and the corresponding eigenfunction φ_1 (associated with λ_1) satisfies $\varphi_1 > 0$ in $\Omega, \partial\varphi_1/\partial n < 0$ on $\partial\Omega$, where n denotes the exterior unit normal to $\partial\Omega$. One also has $\varphi_1 \in C^{1,\nu}(\overline{\Omega})$ with some $\nu \in (0,1)$ (see e.g. [8, Lemma 2.1, p. 115]). Moreover, λ_1 can be characterized as the best (the greatest) constant $C > 0$ in the Poincaré inequality

$$\int_\Omega |\nabla u|^p \geqslant C \int_\Omega |u|^p \tag{1.4}$$

for all $u \in W_0^{1,p}(\Omega)$, where the identity

$$\int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p = 0$$

holds exactly for the multiples of the first eigenfunction φ_1.

In our further considerations we will use the standard spaces $W_0^{1,p}(\Omega)$, $L^p(\Omega)$, $C(\overline{\Omega})$ and $C^1(\overline{\Omega})$ (or $C_0^1(\overline{\Omega})$, respectively), with the corresponding norms $\|u\| = (\int_\Omega |\nabla u|^p)^{\frac{1}{p}}$, $\|u\|_{L^p} = (\int_\Omega |u|^p)^{1/p}$, $\|u\|_C = \max\limits_{x\in\Omega} |u(x)|$, $\|u\|_{C^1} = \|u\|_C + \max\limits_{x\in\Omega} |\nabla u(x)|$, respectively (here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$ or $\mathbb{R}^N$). The subscript 0 indicates that the traces (or values) of functions equal zero on $\partial\Omega$. Moreover, for element h we use the (L^2-orthogonal) decomposition

$$h(x) = \tilde{h}(x) + \overline{h}\varphi_1(x),$$

and also the L^2-nonorthogonal decomposition

$$h(x) = \tilde{h}(x) + \hat{h},$$

where $\bar{h}, \hat{h} \in \mathbb{R}$ and

$$\int_\Omega \tilde{h}(x)\varphi_1(x)\,\mathrm{d}x = 0.$$

The particular subspace formed by $\tilde{h}(x)$ will be denoted by $\widetilde{C}(\overline{\Omega})$.

By $B_C(\tilde{f}, \varrho)$ we denote the open ball in the space $C(\overline{\Omega})$ with the center $\tilde{f}$ and radius ϱ.

We introduce the energy functional associated with (1.1):

$$E_f(u) \colon = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega fu, \ \ u \in W_0^{1,p}(\Omega).$$

This functional is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and its *critical points* correspond one-to-one to *solutions* of (1.1).

Our main results concern the geometry of E_f and the structure of the set of its critical points on the one hand and the solvability properties of (1.1) on the other. They are formulated in theorems below.

Theorem 1.1 ([5]). *Let $1 < p < 2$ and $0 \neq \tilde{f} \in \widetilde{C}(\overline{\Omega})$. Then there exists $\varrho = \varrho(\tilde{f}) > 0$ such that for any $f \in B_C(\tilde{f}, \varrho)$ the functional E_f is unbounded from below and has at least one critical point (which is the saddle point). Moreover, for $f \in B_C(\tilde{f}, \varrho) \setminus \widetilde{C}(\overline{\Omega})$ the functional E_f has at least two distinct critical points.*

Theorem 1.2 ([5]). *Let $p > 2$ and $0 \neq \tilde{f} \in \widetilde{C}(\overline{\Omega})$. Then the functional $E_{\tilde{f}}$ is bounded from below and has at least one critical point (which is the global minimizer). Moreover, there exists $\varrho = \varrho(\tilde{f}) > 0$ such that for $f \in B_C(\tilde{f}, \varrho) \setminus \widetilde{C}(\overline{\Omega})$ the functional E_f has at least two distinct critical points.*

Theorem 1.3 ([5]). *Let $p > 1, p \neq 2, \tilde{f} \in \widetilde{C}(\overline{\Omega})$. Then the problem (1.1) has at least one solution if $f = \tilde{f}$. For $0 \neq \tilde{f} \in \widetilde{C}(\overline{\Omega})$ there exists $\varrho = \varrho(\tilde{f}) > 0$ such that (1.1) has at least one solution for any $f \in B_C(\tilde{f}, \varrho)$. Moreover, there exist real numbers $F_- < 0 < F_+$ (see Fig. 1) such that the problem (1.1) with $f = \tilde{f} + \hat{f}$ has*

(i) *no solution for $\hat{f} \notin [F_-, F_+]$;*

(ii) *at least two distinct solutions for $\hat{f} \in (F_-, 0) \cup (0, F_+)$;*

(iii) *at least one solution for $\hat{f} \in \{F_-, 0, F_+\}$.*

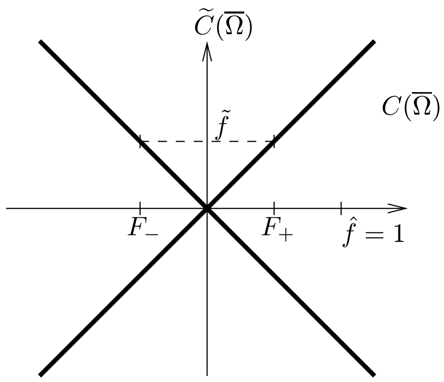

Fig. 1. "Slice" of $C(\overline{\Omega})$ containing all constants and one fixed $\tilde{f} \in \widetilde{C}(\overline{\Omega})$.

2. Remarks

Remark 2.1. Note that a standard bootstrap regularity argument implies that any solution from Theorems 1.1–1.3 belongs to $L^\infty(\Omega)$ (cf. Drábek, Kufner, Nicolosi [9]). It follows then from the regularity results of Tolksdorf [19] (see also Di Benedetto [4] and Liebermann [14]) that it belongs to $C^{1,\nu}(\overline{\Omega})$ with some $\nu \in (0,1)$. In particular, our solution is an element of $C^1_0(\overline{\Omega})$.

Remark 2.2. In particular, it follows from our results that the set of $f \in C(\overline{\Omega})$ for which (1.1) with $p \neq 2$ has at least one solution has a nonempty interior in $C(\overline{\Omega})$.

Remark 2.3. Note that Theorem 1.3 provides a necessary and sufficient condition for solvability of the problem (1.1). This condition is in fact of Landesman-Lazer type (see [13], cf. also [10]). Indeed, given $\tilde{f} \in \widetilde{C}(\overline{\Omega}), \tilde{f} \neq 0$, the problem (1.1) with the right hand side $f(x) = \tilde{f}(x) + \hat{f}$ has a solution if and only if

$$F_-(\tilde{f}) \leqslant \frac{1}{\|\varphi_1\|_{L^1}} \int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x \leqslant F_+(\tilde{f}).$$

However, it should be pointed out that this condition differs from the original condition of Landesman and Lazer due to the fact that F_- and F_+ depend on the component $\tilde{f}$ of the right hand side f and not on the perturbation term (which is actually not present in our problem (1.1)). By homogeneity we have that for any $t > 0$,

$$F_\pm(t\tilde{f}) = tF_\pm(\tilde{f}).$$

Our proofs can be found in paper [5] and rely on the combination of the variational approach and the method of lower and upper solutions. We also use essentially the results obtained by Drábek and Holubová [7], Takáč [17] and Fleckinger-Pellé and Takáč [12]. In fact, Theorem 1.1 was proved already in [7], however, here a different approach is used. During the preparation of this manuscript the author received a preprint of Takáč [18], where a result similar to our Theorem 1.3 is proved. However, the approach used in [18] is very different from ours.

Our objective in this paper is to *avoid* complicated *technical assumptions*. For this reason we restrict to rather special domains Ω and right hand sides f. On the other hand, we believe that in our approach the main ideas appear more clearly and that a possible generalization of Ω or f will bring new insight neither into the geometry of E_f nor to the solvability of (1.1).

It should be mentioned that our approach covers also the case $N = 1$, and completes thus the previous results in this direction proved by Del Pino, Drábek and Manásevich [3], Drábek, Girg and Manásevich [6], Manásevich and Takáč [16], Binding, Drábek and Huang [2], Drábek and Takáč [11]. In fact, the first relevant result which led to better understanding of the problem appeared in [3].

Note also that our Theorems 1.1, 1.2 and 1.3 express not only the difference between the linear case $p = 2$ and the nonlinear case $p \neq 2$ but also the striking difference between the case $1 < p < 2$ and the case $p > 2$. The main goal of this paper is actually to emphasize this fact.

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