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PROBLEMS INVOLVING p-LAPLACIAN TYPE EQUATIONS AND MEASURES

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Abstract. In this paper I discuss two questions on p-Laplacian type operators: I characterize sets that are removable for Hölder continuous solutions and then discuss the problem of existence and uniqueness of solutions to $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu$ with zero boundary values; here μ is a Radon measure. The joining link between the problems is the use of equations involving measures.

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1. Removable sets

Throughout this paper let Ω be an open set in $\mathbb{R}^n$ and $1 < p < \infty$ a fixed number. Continuous solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = 0 \tag{1.1}$$

are called $\mathcal{A}$-*harmonic* in Ω; here $\mathcal{A}\colon\ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to verify Leray-Lions type conditions, that is, for some constants $0 < \lambda \leqslant \Lambda < \infty$:

$$\begin{aligned}&\text{the function } x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n\text{, and}\\ &\text{the function } \xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n;\end{aligned} \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

$$\mathcal{A}(x,\xi)\cdot\xi \geqslant \lambda|\xi|^p, \tag{1.3}$$

$$|\mathcal{A}(x,\xi)| \leqslant \Lambda|\xi|^{p-1}, \tag{1.4}$$

$$(\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta))\cdot(\xi-\zeta) > 0 \tag{1.5}$$

whenever $\xi \neq \zeta$. A prime example of the operators is the p-Laplacian

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Here we understand the divergence in the sense of distributions, i.e.

$$-\operatorname{div}\mathcal{A}(\cdot,\nabla u)(\varphi) = \int \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x, \quad \varphi \in C_0^\infty(\Omega).$$

Definition. We say that a closed set $E \subset \Omega$ is *removable* for $\mathcal{A}$-harmonic functions in $\mathcal{F}$, if every $u \in \mathcal{F}$ that is $\mathcal{A}$-harmonic in $\Omega \setminus E$ is $\mathcal{A}$-harmonic in the whole of Ω.

Examples. The following results are well known and can be found e.g. in [11] or [17].

A set E is removable for $\mathcal{A}$-harmonic functions in $W^{1,p}_{\mathrm{loc}}(\Omega)$ if and only if $\operatorname{cap}_p(E) = 0$; here the p-capacity of the set E is defined as

$$\operatorname*{cap}_p(E) = \inf_\varphi \int_{\mathbb{R}^n} |\nabla\varphi|^p + |\varphi|^p\,\mathrm{d}x,$$

where the infimum is taken over all φ that are $\geqslant 1$ on an open neighborhood of E. Observe that $\operatorname{cap}_p E = 0$ roughly means that the Hausdorff dimension of E does not exceed $n - p$.

Similarly, E is removable for $\mathcal{A}$-harmonic functions in $L^\infty(\Omega)$ if and only if $\operatorname{cap}_p(E) = 0$.

Further, E is removable for $\mathcal{A}$-harmonic functions in $L^s(\Omega)$ if and only if $\operatorname{cap}_q(E) = 0$, where

$$p < q = \frac{ps}{s-p} \leqslant n.$$

Next, I consider the case where $\mathcal{F} = C^{0,\alpha}(\Omega)$, $0 < \alpha \leqslant 1$. The following theorem was proved in [14]:

1.6. Theorem. *A closed set E is removable for $\mathcal{A}$-harmonic functions in $C^{0,\alpha}(\Omega)$ if and only if E is of $n - p + \alpha(p-1)$ Hausdorff measure zero. For the only if part we assume that $0 < \alpha < \varkappa$, where $\varkappa$ is the best local Hölder continuity exponent for the $\mathcal{A}$-harmonic functions.*

For the p-Laplacian we have $\varkappa = 1$. In the case when $\alpha = \varkappa$ the necessity part does not hold. Then the problem is a way more difficult. For instance, in the case of the classical Laplacian the question which sets are removable for Lipschitz continuous p-harmonic functions was treated by David and Mattila [7] in the case $n = p = 2$:

a compact set E of finite 1-Hausdorff measure is removable for Lipschitz continuous harmonic functions if and only if E is purely unrectifiable. The other cases have remained open.

Carleson [5] proved Theorem 1.6 for the Laplacian ($p = 2$). As to the quasilinear case, Heinonen and Kilpeläinen [10, 4.5] proved the sufficiency part for $\alpha = 1$. Trudinger and Wang [19] had a version of sufficiency under the assumption that u has an $\mathcal{A}$-superharmonic extension to Ω, which assumption can be dispensed with for small α.

Our method of proof combines some ideas from [12], [15], and [19]. We use solutions of equations

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = \mu,$$

where μ is a nonnegative Radon measure from $W^{-1,p'}_{\mathrm{loc}}(\Omega)$.

Sketch of the proof of Theorem 1.6[1]. Suppose first that

$$\mathcal{H}^{n-p+\alpha(p-1)}(E) = 0$$

and let $u \in C^{0,\alpha}(\Omega)$ be $\mathcal{A}$-harmonic in $\Omega\setminus E$. Let v be the smallest $\mathcal{A}$-superharmonic function not smaller than u, i.e. v is the pointwise infimum of all functions $\tilde v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$-\operatorname{div}\mathcal{A}(x,\nabla\tilde v) \geqslant 0$$

and $\tilde v \geqslant u$ in Ω. Then v is $\mathcal{A}$-superharmonic [11] and there is a nonnegative Radon measure μ such that

$$\mu = -\operatorname{div}\mathcal{A}(x,\nabla v).$$

Claim: $\mu(B(x,r)) \leqslant cr^{n-p+\alpha(p-1)}$ if $B(x,8r)\subset\Omega$.

We consider two separate cases:

Case (i): $u(x) = v(x)$. Then one can show by using the weak Harnack inequality that

$$\operatorname{osc}(v,B(x,r)) \leqslant c\operatorname{osc}(u,B(x,2r)) \leqslant c\,r^{\alpha},$$

whence for a usual cut-off function $\eta\in C_0^\infty(B(x,2r))$

$$\begin{aligned}\mu(B(x,r)) &\leqslant \int_{B(x,2r)} \eta^p\,\mathrm{d}\mu = \int_{B(x,2r)} \mathcal{A}(y,\nabla v)\cdot\nabla\eta^p\,\mathrm{d}y\\ &\leqslant cr^{(n-p)/p}\Big(\int_{B(x,2r)} |\nabla v|^p\eta^p\,\mathrm{d}y\Big)^{(p-1)/p}\\ &\leqslant c\,r^{n-p}\operatorname{osc}(v,B(x,2r))^{p-1}\\ &\leqslant c\,r^{n-p+\alpha(p-1)},\end{aligned}$$

which shows the claim in the case (i).

[1] A detailed proof can be found in [14].

Case (ii): $u(x) < v(x)$. Now either $B(x,r) \cap \{u = v\} = \emptyset$, whence $\mu = 0$ in $B(x,r)$, or there is a point $y \in B(x,r)$ such that $u(y) = v(y)$. Next we have by case (i) that

$$\mu(B(x,r)) \leqslant \mu(B(y,2r)) \leqslant c\, r^{n-p+\alpha(p-1)},$$

as desired.

Using the above estimate we can easily conclude the proof for the "if" part: Let $K \subset E$ be compact and $\varepsilon > 0$. Choose balls $B(x_j, r_j)$, $x_j \in K$ so that

$$\sum_j r_j^{n-p+\alpha(p-1)} < \varepsilon.$$

Then by the above claim

$$\mu(K) \leqslant \sum_j \mu(B(x_j, r_j)) \leqslant \sum_j r_j^{n-p+\alpha(p-1)} < \varepsilon,$$

whence $\mu(E) = 0$. It follows that v is $\mathcal{A}$-harmonic in Ω.

Next, we make the same construction from below: let w be such that $-w$ is the smallest $\mathcal{A}$-superharmonic function not smaller than $-u$. Arguing as above, we find that w is also $\mathcal{A}$-harmonic. Because w and v coincide on the boundary of Ω with u, the uniqueness yields that $v = w$. It follows that $u = v$ is $\mathcal{A}$-harmonic in Ω.

To prove the "only if" part we need the following regularity theorem which is of independent interest.

1.7. Theorem. *Suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)$ is a nonnegative Radon measure. Then $u \in C^{0,\alpha}(\Omega)$ if and only if there is a constant $M > 0$ such that*

$$\mu(B(x,r)) \leqslant M r^{n-p+\alpha(p-1)}$$

whenever $B(x,3r) \subset \Omega$. For the if part we assume that $0 < \alpha < \varkappa$, where $\varkappa$ is the best local Hölder continuity exponent for the $\mathcal{A}$-harmonic functions.

If $\mathcal{H}^{n-p+\alpha(p-1)}(K) > 0$ for some compact $K \subset E$, then by Frostman's lemma ([1], 5.1.12, [5]) there is a nonnegative Radon measure μ on K with $\mu(K) > 0$ and

$$\mu(B(x,r) \leqslant r^{n-p+\alpha(p-1)}.$$

Any solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

is $\mathcal{A}$-harmonic in $\Omega \setminus E$ [16, 3.19] and $u \in C^{0,\alpha}(\Omega)$ by Theorem 1.7, but u fails to have an $\mathcal{A}$-harmonic extension to E, since $\mu(E) > 0$.

2. Uniqueness

A question that is under intensive research is the unique solvability of the Dirichlet problem

$$\text{(P)} \qquad \begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \mu & \text{on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Choosing $\mu = \delta$, the Dirac measure, one easily sees that $|\nabla u|^{p-1}$ cannot in general be in $L^{n/(n-1)}_{\mathrm{loc}}$. Thus the best regularity one can hope for is not $W^{1,p}(\Omega)$, but $W_0^{1,n(p-1)/(n-1)}(\Omega)$. We reformulate the problem:

$$\text{(P}'\text{)} \qquad \begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = \mu & \text{on } \Omega, \\ u \in \bigcap\limits_{q < \frac{n(p-1)}{n-1}} W_0^{1,q}(\Omega). \end{cases}$$

The *existence* of solutions to problem (P′) is well known, cf. [3]: If μ is in the dual of $W^{1,p}(\Omega)$, this is the classical Leray-Lions result. If not, then approximate μ by smooth nonnegative functions μ_j with uniformly bounded masses such that

$$\mu_j \to \mu \quad \text{weakly in the sense of measures.}$$

Then solve the problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_j) = \mu_j & \text{on } \Omega, \\ u_j = 0 \in W_0^{1,p}(\Omega) \end{cases}$$

and prove the estimate

$$\|u_j\|_{1,q} \leqslant c \quad \text{for all } q < \frac{n(p-1)}{n-1}.$$

Then infer that there is

$$u \in \bigcap_{q < \frac{n(p-1)}{n-1}} W_0^{1,q}(\Omega)$$

such that $u_j \to u$ and $\nabla u_j \to \nabla u$ a.e. so that $-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$.

W h a t a b o u t t h e u n i q u e n e s s ?

E x a m p l e (Serrin [18]). Given $q < 2 = p$, there is a linear operator

$$Lu = -\operatorname{div} a(x, \nabla u)$$

and $u \in W_0^{1,q}(\Omega)$ such that $u \not\equiv 0$, but $Lu = 0$. This example shows that the uniqueness of solutions to (P′) fails in general (at least if $n \geqslant 3$) and one needs an extra condition to get both the existence and uniqueness.

One would like to prove uniqueness for (P′) by using the difference of two solutions as the test function. This is, however, not legitimate. There are different attempts to treat uniqueness by changing the concept of solution to one that allows testing by (truncations of) solutions: For example,
– *entropy solution* [2], i.e.

$$\int \mathcal{A}(x, \nabla u) \cdot \nabla T_k(u - \varphi) \, \mathrm{d}x \leqslant \int T_k(u - \varphi) \, \mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\Omega)$ and $k > 0$; here $T_k(t) = \min(\max(t, -k), k)$ is the truncation at level k;
– *renormalized solution* [6], i.e.

$$\int \mathcal{A}(x, \nabla u) \cdot \nabla (h(u)\varphi) \, \mathrm{d}x = \int h(u)\varphi \, \mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\Omega)$ and $h \in W^{1,\infty}(\mathbb{R})$ with $h(u)\varphi \in W^{1,p}(\Omega)$.

The following result has been proved by several authors, see [4], [13], [6].

2.1. Theorem. *If μ is absolutely continuous with respect to the p-capacity, then there is a unique entropy/renormalized solution of* (P′) *with*

$$T_k u \in W^{1,p}(\Omega) \text{ for all } k > 0.$$

Trudinger and Wang [20] have recently proved the following very interesting result which does not employ any artificial concept of solutions:

2.2. Theorem. *If Ω is Lipschitz and μ is absolutely continuous with respect the p-capacity, then there is a unique solution of* (P′) *with*

$$T_k u \in W^{1,p}(\Omega) \text{ for all } k > 0.$$

In the borderline case $p = n$ there are a couple of good uniqueness results; usually they apply for operators satisfying a strong monotonicity assumption. Therefore we formulate them only for the n-Laplacian: Suppose that $p = n$, Ω is smooth and

$$\mathcal{A}(x,\xi) = |\xi|^{n-2}\xi.$$

Then [9] there is a unique solution u of (P′) in $W^{1,n)}(\Omega)$, i.e.

$$u \in \bigcap_{q<n} W_0^{1,q}(\Omega) \text{ and } \sup_{\varepsilon>0} \varepsilon \int |\nabla u|^{n-\varepsilon} < \infty;$$

[8] there is a unique solution u of (P′) such that

$$u \in \bigcap_{q<n} W_0^{1,q}(\Omega) \text{ and } \nabla u \in \text{weak}\, L^n(\Omega),$$

i.e.

$$\sup_{t>0} t^n |\{\nabla u > t\}| < \infty.$$

Finally, Zhong proved in his thesis [21] that for $p = n$ and Ω "smooth" there is a unique solution u of (P′).

There are other partial results concerning uniqueness, but for $p < n$ the problem seems not to be well understood yet.

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