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N-WIDTHS FOR SINGULARLY PERTURBED PROBLEMS

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Abstract. Kolmogorov N-widths are an approximation theory concept that, for a given problem, yields information about the optimal rate of convergence attainable by any numerical method applied to that problem. We survey sharp bounds recently obtained for the N-widths of certain singularly perturbed convection-diffusion and reaction-diffusion boundary value problems.

Keywords: N-width, singularly perturbed, differential equation, boundary value problem, convection-diffusion, reaction-diffusion

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1. Introduction

Singularly perturbed differential equations arise in the modelling of various physical processes. For example, the Navier-Stokes equations of computational fluid dynamics are singularly perturbed at high Reynolds number. Such equations typically exhibit solutions with layers, which cause severe computational difficulties for standard numerical methods. Consequently many papers have been devoted to the construction and analysis of accurate numerical methods for singularly perturbed problems. Nevertheless, relatively little attention has been paid to a basic question that underlies such methods: *given a specified amount of smoothness of the problem data in a singularly perturbed problem, what is the optimal rate of convergence that can be attained by a numerical method?* We shall survey the main results of four recent papers [5], [6], [7], [9] that throw light on the answer to this fundamental question.

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These papers show that for certain classes of singularly perturbed problems, when one is given a certain amount of smoothness of the data, the rate of convergence that can in general be attained by any numerical method is less than that attainable for classical non-singularly perturbed problems. Moreover, one can precisely quantify the amount of deterioration in the rates of convergence as a function of the singular perturbation parameter.

When computing solutions to classical boundary-value problems, one uses the approximation-theoretic concept of N-widths to quantify optimal attainable rates of convergence. For example, N-widths tell us that for Poisson's equation $-\Delta u = f$ on Ω, where the boundary $\partial\Omega$ is smooth, if f lies only in $L_2[0,1]$ and not in any higher-order Sobolev space, then in general *any* numerical method for computing u can be at best second-order convergent in the L_2 norm. Lorentz [8] and Pinkus [11] discuss the computation and application of N-widths, and Aubin [2] describes their use in a finite element context. Results similar to that just described are well known in the context of elliptic differential equations with moderate coefficients, but to obtain similar results for singularly perturbed problems is more difficult.

The basic definition is as follows. Given a set $S \subset L_2(\Omega)$, where Ω is some domain, then the N-width of S in the L_2 norm is defined to be

$$d_N(S, L_2(\Omega)) = \inf_{X_N} \sup_{u\in S} \inf_{v\in X_N} \|u - v\|_{L_2(\Omega)},$$

for $N = 0, 1, \ldots$, where the outer infimum is taken over all subspaces X_N of dimension N that lie in $L_2(\Omega)$. Thus d_N measures how well a "worst" point $u \in S$ can be approximated using N-dimensional subspaces. The quantity $d_N(S, L_2(\Omega))$ might be infinite; it is finite for all N if and only if S is bounded in $L_2(\Omega)$, and $d_N(S, L_2(\Omega)) \to 0$ as $N \to \infty$ if and only if S is compact in $L_2(\Omega)$. If $d_N(S, L_2(\Omega)) = O(N^{-\alpha})$ as $N \to \infty$, where α is the largest positive constant for which this equation holds, then any numerical method designed to approximate points in S can in general attain at best $O(N^{-\alpha})$ convergence in the $L_2(\Omega)$ norm when N degrees of freedom are used.

2. Methodology and notation

The singularly perturbed elliptic boundary value problems studied in this paper have constant coefficients and a small singular perturbation parameter ε multiplies their second-order derivatives; that is, $0 < \varepsilon << 1$. When the equation has no first-order derivative term, we say it is of *reaction-diffusion* type; when a first-order term is present, we say the problem is of *convection-diffusion* type. This terminology is drawn from the physical models where such equations are derived.

The notation C (sometimes subscripted) denotes a generic positive constant that is independent of ε and of the dimension of any approximating subspace. Note that C may take different values in different places.

Let m be a non-negative integer. Write $H^m(\Omega)$ for the usual Sobolev space of functions whose derivatives up to the mth order are square-integrable over a domain Ω, where Ω may lie in R^1 or R^2. In particular $H^0(\Omega) \equiv L_2(\Omega)$. We use $\|\cdot\|_{m,\Omega}$ to denote the standard $H^m(\Omega)$ norm. Standard interpolation theory [3] can then be used to define the spaces $H^s(\Omega)$ for $s \geqslant 0$. Let $B^s(\Omega)$ denote the unit ball in $H^s(\Omega)$.

When $\Omega \subset R^2$, set $\Gamma = \partial\Omega$; the spaces $H^m(\Gamma)$ are defined analogously to $H^m(\Omega)$ and the trace spaces $H^s(\Gamma)$ are defined as in [1].

The classical theory of N-widths provides a useful result [8, p. 140]:

Theorem 2.1. *Let l_2 be the usual Hilbert space of square summable sequences $\{a_k\}_0^\infty$, with norm $\left(\sum\limits_0^\infty a_k^2\right)^{1/2}$. Let $0 < \delta_k \leqslant \infty$ be a monotone decreasing sequence. Set $D = \left\{\{a_k\}\colon \sum\limits_0^\infty \delta_k^{-2} a_k^2 \leqslant 1\right\}$. Then $d_N(D, l_2) = \delta_N$, for $N = 0, 1, \ldots$.*

Given a differential equation $Lu = f$ on some domain Ω with $u = 0$ on its boundary $\partial\Omega$, define the solution operator E by $u = Ef$. Now, given $f \in T$ for some set T, we want to compute the N-width $d_N(S, L_2(\Omega))$, where $S = ET$.

The methodology of [5], [6], [7] is the following:

(i) upper bound for $d_N(S, L_2(\Omega))$: carefully construct an N-dimensional subspace S_N of $L_2(\Omega)$ with good approximation properties, then compute

$$\sup_{u \in S} \inf_{x_N \in S_N} \|u - x_N\|_{0,\Omega}.$$

(ii) lower bound for $d_N(S, L_2(\Omega))$: choose $\widehat{T} \subset T$ in such a way that $d_N(\widehat{S}, L_2(\Omega))$ (where $\widehat{S} = E\widehat{T}$) can be computed using Theorem 2.1, via Fourier series expansions and Parseval's equation.

Of course one aims to obtain upper and lower bounds that have the same asymptotic behaviour as functions of N and ε, so that the precise asymptotic behaviour of d_N is determined.

The somewhat different approach used by Melenk [9] will be described later.

3. Reaction-diffusion and convection-diffusion problems in one dimension

This Section is based on [5]. Let Ω be the one-dimensional domain $(0, 1)$.

We begin by considering a reaction-diffusion problem with Dirichlet boundary conditions. Let $f \in H^k(\Omega)$, where k is a non-negative integer. Let $u = Ef$ be the solution of the two-point boundary value problem

$$-\varepsilon u'' + u = f \quad \text{on } (0,1), \quad u(0) = u(1) = 0. \tag{3.1}$$

Using Fourier sine series expansions one obtains

Theorem 3.1 [5, Theorem 2.3]. *Let k be a non-negative integer. There are positive constants $C_1(k)$ and $C_2(k)$ such that*

$$\frac{C_1(k)}{N^k(1+\varepsilon N^2)} \leqslant d_N(E(B^k(\Omega)),\ L_2(\Omega)) \leqslant \frac{C_2(k)}{N^k(1+\varepsilon N^2)}, \quad \textit{for} \ \ N = 0, 1, \ldots$$

Consider now the relationship of Theorem 3.1 to the finite element method. In the case $\varepsilon \approx 1$, the N-width $d_N(E(B^k), L_2(\Omega))$ is of order $N^{-(k+2)}$ as $N \to \infty$. The usual finite element error analysis shows that this approximation order is attained by the finite element approximation using a uniform mesh, and numerical experiments confirm this result. In comparison with this classical result, the singularly perturbed nature of reaction-diffusion problems will *in practice* cost a factor N^2 in convergence rates for data of given smoothness; this happens because typically $N^2 << \varepsilon^{-1}$, so the term εN^2 in Theorem 3.1 will not contribute significantly to the rate of convergence.

For example, in order to obtain second-order convergence in practice for small ε, one must in general have $f \in H^2(\Omega)$. If $f \in H^2(\Omega)$ and we solve (3.1) using a Galerkin finite element method with piecewise linear functions on a Shishkin mesh, then it is easy to modify the analysis of [13] to prove that

$$\|u - u_N\|_{0,\Omega} \leqslant CN^{-2} \ln^2 N \|f\|_{2,\Omega}, \tag{3.2}$$

where N is the dimension of the trial space and u_N is the computed solution. We see from Theorem 3.1 that in the case $k = 2$, this method is almost optimal (up to the $\ln^2 N$ factor) with respect to the given data. From a practical point of view the logarithmic factor in (3.2) is not important in assessing the accuracy of the Shishkin mesh method.

We now consider problems where a first-order derivative appears in the differential equation. Consider $f \in H^k(\Omega)$, where k is a non-negative integer. Now let $u = Ef$ denote the solution of the two-point boundary value problem

$$Lu := -\varepsilon u'' + u' = f \quad \text{on } (0,1), \quad u(0) = u(1) = 0. \tag{3.3}$$

Unlike the reaction-diffusion case, the presence of the convection term means that one must use full Fourier expansions to prove

Theorem 3.2 [5, Theorem 3.2]**.** *Let k be a non-negative integer. There are positive constants $C_1(k)$ and $C_2(k)$ such that*

$$\frac{C_1(k)}{N^{k+1}(1+\varepsilon N)} \leqslant d_N(E(B^k(\Omega)),\ L_2(\Omega)) \leqslant \frac{C_2(k)}{N^{k+1}(1+\varepsilon N)}, \quad \textit{for } N = 0, 1, \ldots.$$

Despite the extensive literature dealing with (3.3), we know of no finite element method error bound for this problem that attains (or comes close to) the upper bound of Theorem 3.2.

In the case $\varepsilon \approx 1$, the N-width $d_N(E(B^k(\Omega)), L_2(\Omega))$ is of order $N^{-(k+2)}$ as $N \to \infty$. In comparison with this classical result, the singularly perturbed nature of convection-diffusion problems will *in practice* cost a factor N in convergence rates for data of given smoothness; this happens because typically $N << \varepsilon^{-1}$, so the term εN in Theorem 3.2 does not contribute significantly to the rate of convergence.

We see that for convection-diffusion problems, the approximability of the solution does not deteriorate as badly as in the reaction-diffusion case; compared with the case $\varepsilon \approx 1$, one power of N is lost here, while in Theorem 3.1 the loss was $O(N^2)$. Thus, to obtain a given order of convergence, in practice more smoothness of the data will be needed in the reaction-diffusion case. (This should not be interpreted as saying that convection-diffusion problems are easier to solve numerically than reaction-diffusion ones, since other considerations such as stability of numerical methods matter also.)

4. Reaction-diffusion problems in two dimensions

Consider now the analyses of [6] and [9]. Let Ω be an open bounded domain in R^2 with smooth boundary Γ. In this Section we consider the boundary value problem

$$Lu := -\varepsilon \Delta u + u = f \text{ in } \Omega, \quad \text{where } u = 0 \text{ on } \Gamma, \tag{4.1}$$

and the parameter ε lies in $(0,1]$. This problem is well-posed; if $f \in H^s(\Omega)$ for any $s \geqslant 0$, then the solution u lies in $H^{s+2}(\Omega)$ (see, e.g., [10]). We write $E\colon\ f \mapsto u$ for the solution operator of (4.1).

Our aim is to find upper and lower bounds for the N-width $d_N(E(B^s(\Omega)), L_2(\Omega))$ when $s \geqslant 0$ but (for technical reasons) $s - 1/2$ is not an even integer. We have the following theorem:

Theorem 4.1 [6, Theorem 2.1]**.** *Let $s \geqslant 0$ with $s - 1/2$ not an even integer. There are positive constants $C_1(s)$ and $C_2(s)$ such that*

$$\frac{C_1(s)}{N^{s/2}(1+\varepsilon N)} \leqslant d_N(E(B^s(\Omega)),\ L_2(\Omega)) \leqslant \frac{C_2(s)}{N^{s/2}(1+\varepsilon N)}, \qquad \text{for } N = 0, 1, \ldots.$$

This result says essentially that there is an N-dimensional subspace $X_N \subset L_2(\Omega)$ with the following property: for each $f \in H^s(\Omega)$ there is a $u_N \in X_N$ such that

$$\|u - u_N\|_{0,\Omega} \leqslant \frac{C}{N^{s/2}(1+\varepsilon N)}\|f\|_{s,\Omega}, \tag{4.2}$$

and furthermore, no other N-dimensional subspace of $L_2(\Omega)$ can achieve a similar result with a smaller factor multiplying $\|f\|_{s,\Omega}$. If we rewrite this result in terms of a mesh-width h (as is customary in numerical analysis), then on a quasiuniform mesh we have $N = O(h^{-2})$ so (4.2) becomes

$$\|u - u_N\|_{0,\Omega} \leqslant \frac{Ch^s}{1+\varepsilon h^{-2}}\|f\|_{s,\Omega}, \tag{4.3}$$

and (as just stated) the order of convergence is best possible. When one rewrites the result of Theorem 3.1 in terms of a quasiuniform mesh, then the one-dimensional meshwidth $h = O(N^{-1})$, and it is easy to see that one also obtains (4.3) with s an integer. Thus when expressed in terms of meshwidths, *our results for reaction-diffusion problems in one and two dimensions coincide.*

In the case $\varepsilon = 1$, (4.3) is $\|u - u_N\|_{0,\Omega} \leqslant Ch^{s+2}\|f\|_{s,\Omega}$, which is well-known to be best possible. But when $\varepsilon \approx 0$, (4.3) becomes in effect $\|u - u_N\|_{0,\Omega} \leqslant Ch^s\|f\|_{s,\Omega}$. Heuristically, the loss here of two orders of convergence happens because no extra smoothness is engendered by the term $-\varepsilon\Delta u$ in (4.1); from a numerical point of view, u is only as smooth as f is.

In [6], Theorem 4.1 is proved using the methodology of Section 2. To do this we first decompose the solution u of (4.1) into a smooth component and a component that contains all boundary layers, but this splitting is not the same as the decompositions one finds in standard asymptotic analyses of (4.1). The decomposition is rather intricate and certain exceptional values of s are excluded in Theorem 4.1 because they lead to some technical difficulties regarding traces. Our approximating subspace has $O(N)$ degrees of freedom, of which $O(\sqrt{N})$ degrees of freedom are used to approximate the boundary layer.

Melenk [9] also considers (4.1), but in a d-dimensional domain where $d \geqslant 2$, and the differential operator is permitted to have variable coefficients. The analysis of [9] is very different from that of [6]: Melenk shows that the desired N-width can be expressed in terms of the asymptotic behaviour (as $N \to \infty$) of the Nth eigenvalue of a certain boundary value problem (see [8] for similar arguments), then invokes a classical result regarding the asymptotic behaviour of these eigenvalues. The argument is very elegant, simpler than that of [6], and treats a larger class of norms. In particular it needs no decomposition of u and consequently, unlike Theorem 4.1, no values of s are excluded in the final result. It does not seem possible however to obtain sharp results by applying arguments of this nature to nonsymmetric operators of convection-diffusion type, so we shall not discuss [9] further here.

5. Convection-diffusion problems in two dimensions

In this Section we examine the N-width in L_2 of the set of solutions of two elliptic singularly perturbed convection-diffusion problems posed on the unit square. These N-widths are discussed in [7].

Let $\Omega = (0,1) \times (0,1)$ be the unit square, with boundary Γ. Let Γ_E denote the intersection of Γ with the line $x = 1$. Our two boundary value problems are

$$Lu := -\varepsilon \Delta u + u_x + u = f \text{ in } \Omega,\ u = 0 \text{ on } \Gamma \tag{5.1}$$

and

$$Lv = f \text{ in } \Omega,\ v = 0 \text{ on } \Gamma \setminus \Gamma_E,\ v_x = 0 \text{ on } \Gamma_E, \tag{5.2}$$

where $f \in L_2(\Omega)$.

Consider the bilinear form $\Phi(v,w) = \int_\Omega (\varepsilon \nabla v \cdot \nabla w + v_x w + vw)\,\mathrm{d}x\,\mathrm{d}y$. For problem (5.1), the bilinear form is taken on $H_0^1(\Omega)$, and for problem (5.2) the bilinear form is taken on the space of functions in $H^1(\Omega)$ that vanish on $\Gamma \setminus \Gamma_E$. In each case the bilinear form is easily seen to be bounded and coercive, so it follows from the Lax-Milgram lemma that, for each $f \in L_2(\Omega)$, the problem (5.1) has a solution $u \in H_0^1(\Omega)$ and (5.2) has a solution $v \in H^1(\Omega)$ which vanishes on $\Gamma \setminus \Gamma_E$. The domain Ω is a polygon so the solution of (5.1) or (5.2) has corner singularities at the vertices of Ω. These singularities are however not very severe, and the solution is in $H^2(\Omega)$ (see, e.g., [4, Theorem 3.2.1.2]).

The problems (5.1) and (5.2) differ in that while the solution to each problem has in general an exponential outflow boundary layer on Γ_E, the boundary layer for (5.2) is weaker than that for (5.1). See, e.g., [12]. Parabolic boundary and interior layers may appear in the solutions to both problems.

Let $A_1\colon f \mapsto u$ denote the solution operator of (5.1) and let $A_2\colon f \mapsto v$ denote the solution operator of (5.2). The solution operators A_l (for $l = 1, 2$) are well-defined, bounded maps from $L_2(\Omega)$ to $H^1(\Omega)$.

The main result from [7] is as follows.

Theorem 5.1 [7, Theorem 1.1]**.** *Let N be a positive integer. There are positive constants C_1 and C_2, which are independent of ε and N, such that for $l = 1, 2$, the N-widths satisfy*

$$\frac{C_1}{\varepsilon N} \leqslant d_N(A_l(B^0(\Omega)),\ L_2(\Omega)) \leqslant \frac{C_2}{\varepsilon N} \text{ if } \varepsilon^2 N \geqslant 1, \tag{5.3}$$

$$\frac{C_1}{1+\varepsilon^{1/3}N^{2/3}} \leqslant d_N(A_l(B^0(\Omega)),\ L_2(\Omega)) \leqslant \frac{C_2}{1+\varepsilon^{1/3}N^{2/3}} \text{ if } \varepsilon^2 N \leqslant 1. \tag{5.4}$$

This theorem should be interpreted in terms of approximations. The upper bound follows from the following approximation results. For each integer $N > 1$ there is a subspace $U_N \subset L_2(\Omega)$ of dimension N such that for any $f \in L_2(\Omega)$ the solution $A_l f$ of (5.1) or (5.2) can be approximated by a $u_N \in U_N$ with approximation error

$$\|A_l f - u_N\|_{0,\Omega} \leqslant \frac{C}{\varepsilon N}\|f\|_{0,\Omega}. \tag{5.5}$$

For each integer $N > 1$ that satisfies $\varepsilon^2 N \leqslant 1$ there is a subspace $U_N \subset L_2(\Omega)$ of dimension N such that for any $f \in L_2(\Omega)$ the solution $A_l f$ of (5.1) or (5.2) can be approximated by a $u_N \in U_N$ with approximation error

$$\|A_l f - u_N\|_{0,\Omega} \leqslant \frac{C}{1+\varepsilon^{1/3}N^{2/3}}\|f\|_{0,\Omega}, \text{ if } \varepsilon^2 N \leqslant 1. \tag{5.6}$$

Notice that although (5.5) holds true for all N, the bound in (5.6) is sharper in the parameter range $\varepsilon^2 N \leqslant 1$.

In our proof of the upper bound for the N-width, the construction of the subspace U_N can be modified to make it a subset of $H^1(\Omega)$ and to satisfy the essential boundary conditions in (5.1) or (5.2).

Theorem 5.1 shows that while the solution of (5.1) has a stronger boundary layer than the solution of (5.2), this does not affect the approximability of the solution in our L_2 setting. This latter qualification is important; it might be expected that stronger norms are needed to discern the effect of the boundary layer. On the other hand, a decomposition of u into its boundary layer and smooth components is needed for the proof of Theorem 5.1 in the case of problem (5.1).

On a quasiuniform mesh of diameter h we have $N = O(h^{-2})$, so (5.3) becomes

$$d_N(A_l(B^0(\Omega)), L_2(\Omega)) = O\left(\frac{h^2}{\varepsilon}\right) = O\left(\frac{h^2}{h+\varepsilon}\right)$$

since $\varepsilon^2 N \geqslant 1$ is equivalent to $h \leqslant C\varepsilon$. That is, when the right-hand side f lies only in L_2 and ε is not small relative to the mesh diameter, the N-width for convection-diffusion problems in two variables agrees with the N-width for convection-diffusion two-point boundary value problems (Theorem 3.2).

The curious formula (5.4) for $d_N(A_l(B^0(\Omega)), L_2(\Omega))$ in the case of small ε should also be noted. This formula comes from the area enclosed by a level curve of the symbol of the operator L, as can be seen by a perusal of the proof of the lower bound in [7]. To construct an approximating subspace with an error that achieves this lower bound we have found it necessary to use a subspace that is not a tensor product of functions in x and functions in y. In our subspace, functions whose y variations are in a certain frequency range have degrees of freedom in the x variable that depend on this frequency range.

As suggested by the above discussion, the proofs in [7] rely on very specific constructions. We do not at present know how to generalize the results beyond what is stated here. In particular, we do not see how to apply the general methods of [9] to our problem. Furthermore, in the previous Sections we considered $f \in H^k(\Omega)$, but in [7] we are able to push through the analysis only when $f \in L_2(\Omega)$.

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