SOME RESULTS ABOUT THE HENSTOCK-KURZWEIL FOURIER TRANSFORM

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Dedicated to Prof. Vladimir A. Borovikov on the first anniversary of his death

Abstract. We consider the Fourier transform in the space of Henstock-Kurzweil integrable functions. We prove that the classical results related to the Riemann-Lebesgue lemma, existence and continuity are true in appropriate subspaces.

Keywords: Fourier transform, Henstock-Kurzweil integral, bounded variation functions

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1. Introduction

Given a function $f\colon \mathbb{R}\to\mathbb{R}$, its Fourier transform at $s\in\mathbb{R}$ is defined by $\hat{f}(s) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xs} f(x)\,\mathrm{d}x$. Here the integral is the Henstock-Kurzweil integral, which is equivalent to the Denjoy and Perron integrals.

The study of the Fourier transform in the space of the Henstock-Kurzweil integrable functions has been recently developed by E. Talvila [3]. He has shown some theorems on existence and continuity for the Fourier transform in certain subspaces. In general, neither existence nor continuity nor the Riemann-Lebesgue lemma are valid in the space of the Henstock-Kurzweil integrable functions.

These facts motivate us to look at a subspace of the Henstock-Kurzweil integrable functions that is not contained in the space of Lebesgue integrable functions and on which these classical properties are valid.

Notation 1.1. Let I be a finite or infinite closed interval. We work on the following subspaces:

- $\mathcal{HK}(I) = \{f;\ f$ is Henstock-Kurzweil integrable on $I\}$.
- $\mathcal{HK}_{\mathrm{loc}}(\mathbb{R}) = \{f;\ f \in \mathcal{HK}(I)$ for each finite closed interval $I\}$.
- $\mathcal{BV}(I) = \{f;\ f$ is of bounded variation on $I\}$.
  If $f \in \mathcal{BV}(I)$, $V_I f$ is the total variation of f on I.
- $\mathcal{BV}([\pm\infty]) = \{f;\ f \in \mathcal{BV}([a,\infty]) \cap \mathcal{BV}([-\infty,b])$ for some $a, b \in \mathbb{R}\}$.
- $\mathcal{BV}_0([\pm\infty]) = \{f \in \mathcal{BV}([\pm\infty]);\ \lim\limits_{|x|\to\infty} f(x) = 0\}$.
- $L(I) = \{f;\ f$ is Lebesgue integrable on $I\}$.

Main results 1.2. Our main results are the following:

(i) $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \subseteq \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$ and $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \nsubseteq L(\mathbb{R})$.

(ii) An existence theorem for $\hat{f}$ on $\mathbb{R}$ when f is in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$.

(iii) Continuity of $\hat{f}$ on $\mathbb{R} \setminus \{0\}$ for functions $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$.

(iv) A Riemann-Lebesgue lemma in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$.

In the following sections we prove these results.

2. The $\mathcal{HK}(I) \cap \mathcal{BV}(I)$ subspace

If I is a compact interval, we know that

$$\mathcal{BV}(I) \subset L(I) \subset \mathcal{HK}(I),$$

and consequently $\mathcal{HK}(I) \cap \mathcal{BV}(I) \subset L(I)$.

Now, if I is unbounded, the first two observations which we have are

$$\mathcal{BV}(I) \nsubseteq L(I) \tag{2.1}$$

and

$$L(I) \nsubseteq \mathcal{HK}(I) \cap \mathcal{BV}(I). \tag{2.2}$$

Really, it is easy to demonstrate that the function $f(x) = 1/x$ defined in $[1,\infty]$ is of bounded variation with

$$V_{[1,\infty]} f = 1$$

and

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \infty.$$

This implies that (2.1) is true.

To verify (2.2), we consider the function $f\colon [0,\infty] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x}\sin(1/x) & \text{if } x \in (0,1],\\ 0 & \text{if } x = 0,\ x \in (1,\infty] \end{cases}$$

which is in $L([0,\infty]) \setminus \mathcal{BV}([0,\infty])$.

Next, we will prove that $\mathcal{HK}(I) \cap \mathcal{BV}(I) \nsubseteq L(I)$.

Proposition 2.1. *Let $\varphi\colon [a,\infty] \to \mathbb{R}$ be a non-negative function which is decreasing to zero when $x \to \infty$. If $\varphi \notin \mathcal{HK}([a,\infty])$, then the functions $\varphi(t)\sin t$ and $\varphi(t)\cos t$ are in $\mathcal{HK}([a,\infty]) \setminus L([a,\infty])$.*

Proof. We will demonstrate that $\varphi(t)\sin t \notin L([a,\infty])$. The proof that $\varphi(t)\cos t \notin L([a,\infty])$ can be done in a similar way.

Suppose that n_0 is the first natural number for which $a < (1+4n_0)\pi/4$. For $t \in [a,\infty]$ we have

$$|\sin t| \geqslant \frac{1}{\sqrt{2}} \quad \text{if and only if} \quad t \in \bigcup_{k=n_0}^{\infty} [(1+4k)\pi/4, (3+4k)\pi/4].$$

Let $n \in \mathbb{N}$ with $n \geqslant n_0$. Since $(3+4n)\pi/4 < (1+n)\pi$, we have

$$\begin{aligned} \int_a^{(1+n)\pi} \varphi(t)|\sin t|\,\mathrm{d}t &\geqslant \frac{1}{\sqrt{2}} \sum_{k=n_0}^{n} \int_{(1+4k)\pi/4}^{(3+4k)\pi/4} \varphi(t)\,\mathrm{d}t \\ &\geqslant \frac{1}{\sqrt{2}} \sum_{k=n_0}^{n} \int_{(1+4k)\pi/4}^{(3+4k)\pi/4} \varphi((3+4k)\pi/4)\,\mathrm{d}t \\ &= \frac{\pi}{2\sqrt{2}} \sum_{k=n_0}^{n} \varphi((3+4k)\pi/4) \\ &\geqslant \frac{\pi}{2\sqrt{2}} \sum_{k=n_0}^{n} \varphi((1+k)\pi). \end{aligned} \tag{2.3}$$

On the other hand,

$$\begin{aligned} \int_a^{(1+n)\pi} \varphi(t)\,\mathrm{d}t &= \int_a^{n_0\pi} \varphi(t)\,\mathrm{d}t + \int_{n_0\pi}^{(1+n)\pi} \varphi(t)\,\mathrm{d}t \\ &= \int_a^{n_0\pi} \varphi(t)\,\mathrm{d}t + \sum_{k=n_0}^{n} \int_{k\pi}^{(1+k)\pi} \varphi(t)\,\mathrm{d}t \\ &\leqslant \int_a^{n_0\pi} \varphi(t)\,\mathrm{d}t + \pi \sum_{k=n_0}^{n} \varphi(k\pi). \end{aligned} \tag{2.4}$$

Since $\varphi \notin \mathcal{HK}([a,\infty])$, we have $\int_a^\infty \varphi(t)\,\mathrm{d}t = \infty$ and (2.4) implies

$$\sum_{k=n_0}^{\infty} \varphi(k\pi) = \infty. \tag{2.5}$$

Using (2.5) and letting $n \to \infty$ in (2.3), we conclude that $\varphi(t)\sin t \notin L([a,\infty])$. For any $x \in [a,\infty)$,

$$\left|\int_a^x \sin t\,\mathrm{d}t\right| \leqslant 2 \quad \text{and} \quad \left|\int_a^x \cos t\,\mathrm{d}t\right| \leqslant 2.$$

Hence according to [1, Theorem 16.10] (Chartier-Dirichlet) we have that $\varphi(t)\sin t$ and $\varphi(t)\cos t$ are in $\mathcal{HK}[a,\infty]$. $\square$

Example 2.2. For any $a > 0$,

$$\frac{\sin t}{t} \in \mathcal{HK}([a,\infty]) \setminus L([a,\infty]).$$

Proposition 2.3. *Let $1 > \alpha > 0$. The function $f_\alpha \colon [\pi^{1/\alpha},\infty] \to \mathbb{R}$ defined as*

$$f_\alpha(t) = \frac{\sin(t^\alpha)}{t}$$

satisfies

(a) $f_\alpha \in \mathcal{HK}[\pi^{1/\alpha},\infty] \setminus L([\pi^{1/\alpha},\infty])$,

(b) $f_\alpha \in \mathcal{BV}([\pi^{1/\alpha},\infty])$.

Proof. (a) This is a consequence of [3, Lemma 23].

(b) Let $x \in (\pi^{1/\alpha},\infty)$. We know that $f'_\alpha \in \mathcal{HK}([\pi^{1/\alpha},x])$. Now since

$$f'_\alpha(t) = \frac{\alpha\cos(t^\alpha)}{t^{2-\alpha}} - \frac{\sin(t^\alpha)}{t^2},$$

we have that

$$|f'_\alpha(t)| \leqslant \frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2}. \tag{2.6}$$

The function $g(t) = \alpha/t^{2-\alpha} + 1/t^2$ satisfies $g \in \mathcal{HK}([\pi^{1/\alpha},x])$, hence by (2.6) and [1, Theorem 7.7] we conclude that $f'_\alpha \in L([\pi^{1/\alpha},x])$ and

$$\begin{aligned}\int_{\pi^{1/\alpha}}^x |f'_\alpha| &\leqslant \int_{\pi^{1/\alpha}}^x \Big(\frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2}\Big)\,\mathrm{d}t \\ &= \Big(\frac{1}{\alpha-1}\Big)[x^{\alpha-1} - \pi^{(\alpha-1)/\alpha}] - \frac{1}{x} + \frac{1}{\pi^{1/\alpha}}.\end{aligned}$$

Consequently, by [1, Theorem 7.5],

$$V_{[\pi^{1/\alpha},x]} f_\alpha \leqslant \Big(\frac{1}{\alpha-1}\Big)[x^{\alpha-1} - \pi^{(\alpha-1)/\alpha}] - \frac{1}{x} + \frac{1}{\pi^{1/\alpha}}.$$

Therefore, as $1-\alpha > 0$, we have that

$$V_{[\pi^{1/\alpha},\infty]} f_\alpha \leqslant \frac{1}{(1-\alpha)\pi^{(1-\alpha)/\alpha}} + \frac{1}{\pi^{1/\alpha}}.$$

Thus, $f_\alpha \in \mathcal{BV}([\pi^{1/\alpha},\infty])$. $\square$

Similarly, we can prove that for $1 > \alpha > 0$, the function $g_\alpha\colon [-\infty, -\pi^{1/\alpha}] \to \mathbb{R}$ defined as

$$g_\alpha(t) = \frac{\sin(-t)^\alpha}{-t}$$

belongs to $\mathcal{HK}([-\infty,-\pi^{1/\alpha}]) \cap \mathcal{BV}([-\infty,-\pi^{1/\alpha}]) \setminus L([-\infty,-\pi^{1/\alpha}])$.

Let $h \in \mathcal{BV}([-\pi^{1/\alpha},\pi^{1/\alpha}])$. For $1 > \alpha > 0$, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} h(x) & \text{if } x \in (-\pi^{1/\alpha}, \pi^{1/\alpha}), \\ \dfrac{\sin|t|^\alpha}{|t|} & \text{if } x \in (-\infty, -\pi^{1/\alpha}] \cup [\pi^{1/\alpha}, \infty) \end{cases}$$

is in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \setminus L(\mathbb{R})$. With this example and Proposition 2.3 we have the following theorem.

Theorem 2.4. *There exists a function f in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \setminus L(\mathbb{R})$.*

Now, since $\mathcal{BV}(\mathbb{R}) \subset \mathcal{BV}([\pm\infty])$, we have immediately the next corollary.

Corollary 2.5. $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty]) \not\subseteq L(\mathbb{R})$.

We observe that $\mathcal{BV}(\mathbb{R}) \subset \mathcal{BV}([\pm\infty])$ properly, because instead of the function h in $\mathcal{BV}([-\pi^{1/\alpha},\pi^{1/\alpha}])$ we can take a function in $\mathcal{HK}([-\pi^{1/\alpha},\pi^{1/\alpha}]) \setminus \mathcal{BV}([-\pi^{1/\alpha},\pi^{1/\alpha}])$.

3. An existence theorem for $\hat{f}(s)$ in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$

A part from Proposition 2.1(b) in [3] by E. Talvila tells us that, if $f \in \mathcal{HK}_{\rm loc}(\mathbb{R}) \cap \mathcal{BV}_0([\pm\infty])$, then $\hat{f}(s)$ exists for all $s \in \mathbb{R}$. If $s \neq 0$, then the result is true. However, under these conditions, it is not necessarily true for $\hat{f}(0)$. For example, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in (-1,1), \\ 1/x & \text{if } x \in (-\infty,-1] \cup [1,\infty) \end{cases}$$

is in $\mathcal{HK}_{\rm loc}(\mathbb{R}) \cap \mathcal{BV}_0([\pm\infty])$ but $\hat{f}(0)$ does not exist.

In order to have the existence of $\hat{f}(0)$, we need that $f \in \mathcal{HK}(\mathbb{R})$.

We will demonstrate that the Fourier transform exists in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$ for every $s \in \mathbb{R}$.

Theorem 3.1. *If $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$, then $\hat{f}(s)$ exists for all $s \in \mathbb{R}$.*

Proof. The result is true for $s = 0$ because $f \in \mathcal{HK}(\mathbb{R})$. Now let $s \neq 0$; since $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty]) \subset \mathcal{HK}_{\rm loc}(\mathbb{R}) \cap \mathcal{BV}_0([\pm\infty])$, by [3, Proposition 2.1 (b)] it follows that $\hat{f}(s)$ exists. $\square$

4. Continuity of $\hat{f}$

We know that the continuity of the Lebesgue-Fourier transform on $\mathbb{R}$ is a consequence of the dominated convergence theorem and that the Lebesgue integral is absolute. Now to prove the continuity of the Henstock-Kurzweil Fourier transform we can not use the same arguments, because the Henstock-Kurzweil integral is not absolute. Two results about this are given in the following theorems. The first of them is an immediate consequence of [3, Theorem 5].

Theorem 4.1. *Let f be a function with support in a compact interval such that $f \in \mathcal{HK}(\mathbb{R})$. Then $\hat{f}$ is continuous on $\mathbb{R}$.*

Theorem 4.2. *If $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$, then $\hat{f}$ is continuous on $\mathbb{R} \setminus \{0\}$.*

Proof. Let $t_0 \in \mathbb{R}\setminus\{0\}$ and consider $a < 0$ and $b > 0$ such that $f \in \mathcal{BV}(-\infty,a] \cap \mathcal{BV}[b,\infty)$. If we show that $\widehat{f\chi_{(-\infty,a]}}$, $\widehat{f\chi_{[a,b]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous at t_0, then $\hat{f}$ is continuous at t_0, because

$$\hat{f}(t) = \widehat{f\chi_{(-\infty,a]}}(t) + \widehat{f\chi_{[a,b]}}(t) + \widehat{f\chi_{[b,\infty)}}(t) \text{ for all } t \in \mathbb{R}.$$

By Theorem 4.1, $\widehat{f\chi_{[a,b]}}$ is continuous at t_0. To prove that $\widehat{f\chi_{(-\infty,a]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous at t_0 we will use [3, Proposition 6(a)]. The conditions f is Henstock-Kurzweil integrable on $\mathbb{R}$ and f is of bounded variation on $(-\infty,a]\cup[b,\infty)$ imply that $\lim\limits_{|x|\to\infty} f(x)=0$. Now since $t_0\neq 0$, there exist $K>0$ and $\delta>0$ such that if $|t-t_0|<\delta$, then $1/|t|<K$. Thus for all $|t-t_0|<\delta$,

$$\left|\int_u^v \mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}x\right| \leqslant \frac{2}{|t|} < 2K \quad \text{for all } [u,v]\subseteq\mathbb{R}.$$

Therefore, by [3, Proposition 6(a)], $\widehat{f\chi_{(-\infty,a]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous at t_0. □

5. The Riemann-Lebesgue lemma

First we give a corollary proved by Talvila in [2].

Corollary 5.1. *If $|\int_a^x g_n|\leqslant M$ for all $n\geqslant 1$ and all $x\in[a,b)$, if each f_n is of bounded variation, if $\lim\limits_{x\to b^-} f_n(x)=0$ uniformly in n, if $f_n\to 0$ on $[a,b]$ and if $V(f_n)\to 0$, then $\int_a^b g_nf_n\to 0$.*

Theorem 5.2. *If $f\in\mathcal{HK}(\mathbb{R})\cap\mathcal{BV}(\mathbb{R})$, then $\lim\limits_{|t|\to\infty}\hat f(t)=0$.*

P r o o f. First we will prove that for every sequence $\{t_n\}_{n\in\mathbb{N}}\subseteq[0,\infty)$ such that $n\leqslant t_n$ for all $n\in\mathbb{N}$ it is true that $\lim\limits_{n\to\infty}\hat f(t_n)=0$.

Let $\{t_n\}_{n\in\mathbb{N}}\subseteq[0,\infty)$ be a sequence such that $n\leqslant t_n$ for all $n\in\mathbb{N}$. For every $n\in\mathbb{N}$, define $f_n(x)=n^{-1}f(x)$, $g_n(x)=n\mathrm{e}^{-\mathrm{i}xt_n}$ on $[0,\infty)$ and $f_n(\infty)=0$, $g_n(\infty)=0$.

For all $n\in\mathbb{N}$ and all $s\in[0,\infty)$,

$$\left|\int_0^s g_n(x)\,\mathrm{d}x\right| = \left|n\int_0^s \mathrm{e}^{-\mathrm{i}xt_n}\,\mathrm{d}x\right| \leqslant \frac{2n}{t_n}\leqslant 2.$$

Since $f\in\mathcal{BV}([0,\infty])\cap\mathcal{HK}([0,\infty])$, we have that each f_n is in $\mathcal{BV}([0,\infty])\cap\mathcal{HK}([0,\infty])$ and

$$\lim_{n\to\infty} V_{[0,\infty]}f_n = \lim_{n\to\infty}\frac{1}{n}V_{[0,\infty]}f = 0.$$

We observe too that $\lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty} n^{-1}f(x)=0$ for all $x\in[0,\infty]$.

Thus according to Corollary 5.1,

$$\lim_{n\to\infty}\int_0^\infty f(x)\mathrm{e}^{-\mathrm{i}xt_n}\,\mathrm{d}x = \lim_{n\to\infty}\int_0^\infty f_n(x)g_n(x)\,\mathrm{d}x = 0.$$

Using Corollary 5.1 for intervals of the type $(a, b]$ we can prove too that

$$\lim_{n\to\infty} \int_{-\infty}^{0} f(x)\mathrm{e}^{-\mathrm{i}xt_n}\,\mathrm{d}x = 0.$$

Thus $\lim\limits_{n\to\infty} \hat{f}(t_n) = 0$.

We now prove that $\lim\limits_{t\to\infty} \hat{f}(t) = 0$. Suppose that it is not true, then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $t_n > n$ such that $|\hat{f}(t_n)| \geqslant \varepsilon$. The sequence $\{t_n\}_{n\in\mathbb{N}}$ satisfies $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, \infty)$ and $n \leqslant t_n$ for all $n \in \mathbb{N}$, hence by the first part of this proof we have $\lim\limits_{n\to\infty} \hat{f}(t_n) = 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $|\hat{f}(t_n)| < \varepsilon$ for all $n \geqslant n_0$. If we take $n_1 > n_0$ then $\varepsilon \leqslant |\hat{f}(t_{n_1})| < \varepsilon$, which is a contradiction.

The proof of $\lim\limits_{t\to-\infty} \hat{f}(t) = 0$ is analogous. $\square$

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