THE LEAST EIGENVALUES OF NONHOMOGENEOUS DEGENERATED QUASILINEAR EIGENVALUE PROBLEMS

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Summary. We prove the existence of the least positive eigenvalue with a corresponding nonnegative eigenfunction of the quasilinear eigenvalue problem

$$\begin{aligned} -\operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) &= \lambda b(x,u)|u|^{p-2}u \quad \text{in } \Omega,\\ u &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain, $p>1$ is a real number and $a(x,u)$, $b(x,u)$ satisfy appropriate growth conditions. Moreover, the coefficient $a(x,u)$ contains a degeneration or a singularity. We work in a suitable weighted Sobolev space and prove the boundedness of the eigenfunction in $L^{\infty}(\Omega)$. The main tool is the investigation of the associated homogeneous eigenvalue problem and an application of the Schauder fixed point theorem.

Keywords: weighted Sobolev space, degenerated quasilinear partial differential equations, weak solutions, eigenvalue problems, Schauder fixed point theorem, boundedness of the solution

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1. Introduction

The aim of this paper is to prove the existence of the least positive eigenvalue λ and the corresponding nonnegative eigenfunction u of the *nonhomogeneous degenerated quasilinear eigenvalue problem*

$$\begin{aligned} -\operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) &= \lambda b(x,u)|u|^{p-2}u \text{ in } \Omega,\\ u = 0 &\qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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where Ω is a bounded domain, $p > 1$ is a real number and $a(x,s), b(x,s)\colon \Omega\times\mathbb{R}\to\mathbb{R}$ are real functions satisfying appropriate growth conditions (see Section 4). Moreover, the function $a(x,s)$ may contain a *degeneration* or a *singularity*. We work in a suitable *weighted Sobolev space* $W_0^{1,p}(w,\Omega)$ with the weight function $w > 0$ a.e. in Ω (see Section 2) and prove that *for a given* $R > 0$ *there exists the least* $\lambda > 0$ *and a corresponding* $u \in W_0^{1,p}(w,\Omega)\cap L^\infty(\Omega)$ *such that* $u \geqslant 0$ *a.e. in* Ω, $\|u\|_{L^p(\Omega)} = R$ *and the equation in* (1.1) *is fulfilled in the weak sense* (see Theorem 4.10). In fact, a more general result (dealing with more general growth conditions imposed on $b(x,s)$) is proved in Theorem 4.5.

This paper generalizes the result of Boccardo [5] and Drábek, Kučera [6] (where *nondegenerated* uniformly elliptic quasilinear operators were considered) and completes the papers on eigenvalues of p-Laplacian published by Anane [2], Barles [3], Bhattacharya [4], García Azorero, Peral Alonso [9], Otani, Teshima [14] and others (where *nondegenerated* and *homogeneous* operators were considered). Let us note that neither global results for nonlinear eigenvalue problems, nor Ljusternik-Schnirelmann theory can be used, since the operator in (1.1) is not (in general) a potential operator.

The *paper is organized* as follows. In Section 2, which has a *preliminary character*, we define appropriate weighted Sobolev spaces and prove some useful imbeddings. We prove also a version of Friedrichs inequality in the weighted Sobolev space. Moreover, an auxiliary assertion due to Stampacchia is proved and we present some consequences of Clarkson's inequality. In Section 3 we study the *homogeneous eigenvalue problem* associated with (1.1) (i.e. we consider the problem (1.1) with $a(x,u) := a(x)$ and $b(x,u) := b(x)$). We prove the existence of the least positive eigenvalue and the corresponding nonnegative eigenfunction of this problem. We show that the eigenfunction belongs to $L^\infty(\Omega)$. We also prove the simplicity of the least eigenvalue and study some useful properties of the homogeneous operator associated with the principal part. The *main result* we prove in Section 4. The tools are an a apriori estimate in $L^\infty(\Omega)$, the results for the homogeneous eigenvalue problem (namely the continuous dependence of the least eigenvalue and the corresponding nonnegative eigenfunction of the homogeneous problem with respect to $a(x)$, $b(x)$) and the Schauder fixed point theorem. Finally, Section 5 contains *examples* which illustrate our general result.

2.1. Weighted Sobolev space. Let us suppose that Ω is an open bounded subset of the n-dimensional Euclidean space $\mathbb{R}^n$, $p > 1$ is an arbitrary real number and w is a *weight function* (i.e. positive and measurable) in Ω. Assume that

$$w \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \frac{1}{w} \in L^{\frac{1}{p-1}}_{\mathrm{loc}}(\Omega). \tag{2.1}$$

Let us define the *weigted Sobolev space* $W^{1,p}(w,\Omega)$ as the set of all real valued functions u defined in Ω for which

$$\|u\|_{1,p,w} = \left(\int_\Omega |u|^p \,\mathrm{d}x + \int_\Omega w|\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty. \tag{2.2}$$

It follows from (2.1) that $W^{1,p}(w,\Omega)$ is a *reflexive Banach space* and that $W^{1,p}_0(w,\Omega)$ is well defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(w,\Omega)$ with respect to the norm $\|\cdot\|_{1,p,w}$ (see e.g. Kufner, Sändig [11]).

Let $s \geqslant \frac{1}{p-1}$ be a real number. A simple application of the Hölder inequality yields that the *continuous imbedding*

$$W^{1,p}(w,\Omega) \hookrightarrow W^{1,p_1}(\Omega) \tag{2.3}$$

holds provided

$$\frac{1}{w} \in L^s(\Omega) \text{ and } p_1 = \frac{ps}{s+1}.$$

2.2. Compact imbeddings. It follows from (2.3) and from the Sobolev imbedding theorem (see e.g. Adams [1], Kufner, John, Fučík [10]) that for $s+1 \leqslant ps < n(s+1)$ we have

$$W^{1,p}_0(w,\Omega) \hookrightarrow W^{1,p_1}_0(\Omega) \hookrightarrow L^q(\Omega), \tag{2.4}$$

where $1 \leqslant q = \frac{np_1}{n-p_1} = \frac{nps}{n(s+1)-ps}$, and for $ps \geqslant n(s+1)$ the imbedding (2.4) holds with arbitrary $1 \leqslant q < \infty$.

Moreover, the *compact imbedding*

$$W^{1,p}_0(w,\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$$

holds provided $1 \leqslant r < q$.

An easy calculation yields that $s > \frac{n}{p}$ implies $q > p$. In particular, we have

$$W^{1,p}_0(w,\Omega) \hookrightarrow\hookrightarrow L^{p+\eta}(\Omega) \tag{2.5}$$

for $0 \leqslant \eta < q - p$ provided

$$\frac{1}{w} \in L^s(\Omega) \text{ and } s \in \left(\frac{n}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right). \tag{2.6}$$

2.3. Friedrichs inequality in weighted Sobolev spaces. In what follows we will always assume that (2.6) is fulfilled. Let $u \in C_0^\infty(\Omega)$. Then due to $q > p$ and the imbedding $W_0^{1,p_1}(\Omega) \hookrightarrow L^q(\Omega)$ we have

$$\left(\int_\Omega |u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leqslant c_1 \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant c_2 \left(\int_\Omega [|u|^{p_1} + |\nabla u|^{p_1}] \,\mathrm{d}x\right)^{\frac{1}{p_1}}. \tag{2.7}$$

The Friedrichs inequality in $W_0^{1,p_1}(\Omega)$ yields

$$\left(\int_\Omega [|u|^{p_1} + |\nabla u|^{p_1}] \,\mathrm{d}x\right)^{\frac{1}{p_1}} \leqslant c_3 \left(\int_\Omega |\nabla u|^{p_1} \,\mathrm{d}x\right)^{\frac{1}{p_1}}. \tag{2.8}$$

Using the Hölder inequality we obtain

$$\begin{aligned}
\left(\int_\Omega |\nabla u|^{p_1} \,\mathrm{d}x\right)^{\frac{1}{p_1}} &= \left(\int_\Omega |\nabla u|^{p_1} w^{\frac{p_1}{p}} \frac{1}{w^{\frac{p_1}{p}}} \,\mathrm{d}x\right)^{\frac{1}{p_1}} \\
&\leqslant \left(\int_\Omega w |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\Omega \frac{1}{w^{\frac{p_1}{p} \frac{p}{p-p_1}}} \,\mathrm{d}x\right)^{\frac{p-p_1}{p} \cdot \frac{1}{p_1}} \\
&\leqslant \left(\int_\Omega \frac{1}{w^s} \,\mathrm{d}x\right)^{\frac{1}{ps}} \left(\int_\Omega w |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}
\end{aligned} \tag{2.9}$$

(see Subsection 2.1 for the relation between s, p and p_1). It follows from (2.7)–(2.9) that

$$\int_\Omega |u|^p \,\mathrm{d}x \leqslant c_4 \int_\Omega w |\nabla u|^p \,\mathrm{d}x$$

with a constant $c_4 > 0$ independent of $u \in C_0^\infty(\Omega)$. Hence the norm

$$\|u\|_w = \left(\int_\Omega w |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

on the space $W_0^{1,p}(w, \Omega)$ is equivalent to the norm $\|\cdot\|_{1,p,w}$ defined by (2.2).

2.4. Equivalent norms. Let us assume that $\tilde{w}$ is a weight function defined in Ω and satisfying inequalities

$$c_5 w(x) \leqslant \tilde{w}(x) \leqslant c_6 w(x) \tag{2.10}$$

for a.e. $x \in \Omega$ with some constants $c_6 \geqslant c_5 > 0$. Then obviously

$$W_0^{1,p}(\tilde{w}, \Omega) = W_0^{1,p}(w, \Omega)$$

and the norms $\|\cdot\|_{\tilde{w}}$ and $\|\cdot\|_w$ are *equivalent* on $W_0^{1,p}(w, \Omega)$. It follows from Clarkson's inequality (see Kufner, John, Fučík [10]) that $W_0^{1,p}(w, \Omega)$ is a *uniformly convex Banach space* with respect to the norm $\|\cdot\|_{\tilde{w}}$ for any $\tilde{w}$ satisfying (2.10).

2.5. Lemma. *(cf. Murthy, Stampacchia [13]). Let $\zeta = \zeta(t)$ be a nonnegative, nonincreasing function on a half line $t \geqslant k_0 \geqslant 0$ such that*

$$\zeta(h) \leqslant c_7 (h-k)^{-\sigma} (\zeta(k))^{\delta} \tag{2.11}$$

for $h > k \geqslant k_0$. Then $\sigma > 0$, $\delta > 1$ imply

$$\zeta(k_0 + d) = 0,$$

where $d = c_7^{\frac{1}{\sigma}} (\zeta(k_0))^{\frac{\delta-1}{\sigma}} \cdot 2^{\frac{\delta}{\delta-1}}$.

P r o o f. Let us define a sequence (k_n) by

$$k_n = k_{n-1} + \frac{d}{2^n}, \; n = 1, 2, \ldots. \tag{2.12}$$

Substituting (2.12) into (2.11) we get by induction

$$\zeta(k_n) \leqslant \frac{\zeta(k_0)}{2^{n \frac{\sigma}{\delta-1}}} \to 0$$

for $n \to \infty$. Since $\lim_{n\to\infty} k_n = k_0 + d$ and ζ is nonincreasing, we obtain $\zeta(k_0 + d) = 0$. □

2.6. Lemma. *Let $p \geqslant 2$. Then*

$$|\boldsymbol{t}_2|^p - |\boldsymbol{t}_1|^p \geqslant p|\boldsymbol{t}_1|^{p-2}\boldsymbol{t}_1(\boldsymbol{t}_2 - \boldsymbol{t}_1) + \frac{|\boldsymbol{t}_2 - \boldsymbol{t}_1|^p}{2^{p-1}-1} \tag{2.13}$$

for all points $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ in $\mathbb{R}^n$.

Let $1 < p < 2$. Then

$$|\boldsymbol{t}_2|^p - |\boldsymbol{t}_1|^p \geqslant p|\boldsymbol{t}_1|^{p-2}\boldsymbol{t}_1(\boldsymbol{t}_2 - \boldsymbol{t}_1) + \frac{3p(p-1)}{16} \frac{|\boldsymbol{t}_2 - \boldsymbol{t}_1|^2}{(|\boldsymbol{t}_1| + |\boldsymbol{t}_2|)^{2-p}} \tag{2.14}$$

for all points $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ in $\mathbb{R}^n$.

Proof of this lemma is based on Clarkson's inequality and can be found in Lindqvist [12].

2.7. Remark. It follows from (2.13) and (2.14) that the inequality

$$|\boldsymbol{t}_2|^p - |\boldsymbol{t}_1|^p > p|\boldsymbol{t}_1|^{p-2}\boldsymbol{t}_1(\boldsymbol{t}_2 - \boldsymbol{t}_1) \tag{2.15}$$

holds for any $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathbb{R}^n$, $\boldsymbol{t}_1 \neq \boldsymbol{t}_2$ and for any $p > 1$. Note that inequality (2.15) is just a restating of the strict convexity of the mapping $\boldsymbol{t} \mapsto |\boldsymbol{t}|^p$ and can be proved independently of (2.13) and (2.14).

3. Homogeneous eigenvalue problem

3.1. Weak formulation. Let us suppose that w is the weight function satisfying (2.1) and (2.6). Let $a(x)$, $b(x)$ be measurable functions satisfying

$$\frac{w(x)}{c_8} \leqslant a(x) \leqslant c_8 w(x), \tag{3.1}$$

$$0 \leqslant b(x) \tag{3.2}$$

for a.e. $x \in \Omega$ with some constant $c_8 > 1$, and $b(x) \in L^{\frac{q^*}{q^*-p}}(\Omega)$ for $p < q^* < q$, $b(x) \in L^\infty(\Omega)$ for $q^* = p$ (see Subsection 2.2 for q). Moreover, let

$$\operatorname{meas}\{x \in \Omega;\, b(x) > 0\} > 0. \tag{3.3}$$

Further we will assume that $p < q^* < q$. The proofs in the forthcoming subsections can be performed in the same way also in the case $q^* = p$.

Let us consider *homogeneous eigenvalue problem*

$$\begin{aligned} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) &= \lambda b(x)|u|^{p-2}u \text{ in } \Omega,\\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.4}$$

We will say that $\lambda \in \mathbb{R}$ is an *eigenvalue* and $u \in W_0^{1,p}(w,\Omega)$, $u \neq 0$, is the corresponding *eigenfunction* of the eigenvalue problem (3.4) if

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla \varphi \,\mathrm{d}x = \lambda \int_\Omega b(x)|u|^{p-2}u\varphi \,\mathrm{d}x \tag{3.5}$$

holds for any $\varphi \in W_0^{1,p}(w,\Omega)$.

3.2. Lemma. *There exists the least (the first) eigenvalue $\lambda_1 > 0$ and at least one corresponding eigenfunction $u_1 \geqslant 0$ a.e. in $\Omega(u_1 \not\equiv 0)$ of the eigenvalue problem (3.4).*

Proof. Set

$$\lambda_1 = \inf\left\{ \int_\Omega a(x)|\nabla v|^p \,\mathrm{d}x\,;\ \int_\Omega b(x)|v|^p \,\mathrm{d}x = 1 \right\}.$$

Obviously $\lambda_1 \geqslant 0$. Let (v_n) be the minimizing sequence for λ_1, i.e.

$$\int_\Omega b(x)|v_n|^p \,\mathrm{d}x = 1 \text{ and } \int_\Omega a(x)|\nabla v_n|^p \,\mathrm{d}x = \lambda_1 + \delta_n, \tag{3.6}$$

with $\delta_n \to 0_+$ for $n \to \infty$. It follows from (3.6) that $\|v_n\|_a \leqslant c_9$, with $c_9 > 0$ independent of n. The reflexivity of $W_0^{1,p}(w,\Omega)$ (see Subsection 2.4) yields the weak convergence $v_n \rightharpoonup u_1$ in $W_0^{1,p}(w,\Omega)$ for some u_1 (at least for some subsequence of (v_n)). The compact imbedding $W_0^{1,p}(w,\Omega) \hookrightarrow\hookrightarrow L^{q^*}(\Omega)$ implies the strong convergence $v_n \to u_1$ in $L^{q^*}(\Omega)$. It follows from (3.2),(3.6), from the Minkowski and the Hölder inequality that

$$\begin{aligned}
1 &= \lim_{n\to\infty} \left(\int_\Omega b(x)|v_n|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\
&\leqslant \lim_{n\to\infty} \left(\int_\Omega b(x)|v_n - u_1|^p \,\mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_\Omega b(x)|u_1|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\
&\leqslant \lim_{n\to\infty} \left(\int_\Omega (b(x))^{\frac{q^*}{q^*-p}} \,\mathrm{d}x \right)^{\frac{q^*-p}{pq^*}} \cdot \left(\int_\Omega |v_n - u_1|^{q^*} \,\mathrm{d}x \right)^{\frac{1}{q^*}} + \left(\int_\Omega b(x)|u_1|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \\
&= \left(\int_\Omega b(x)|u_1|^p \,\mathrm{d}x \right)^{\frac{1}{p}},
\end{aligned}$$

and analogously

$$\begin{aligned}
\left(\int_\Omega b(x)|u_1|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \leqslant \lim_{n\to\infty} \left(\int_\Omega (b(x))^{\frac{q^*}{q^*-p}} \,\mathrm{d}x \right)^{\frac{q^*-p}{pq^*}} \cdot \left(\int_\Omega |u_1 - v_n|^{q^*} \,\mathrm{d}x \right)^{\frac{1}{q^*}} \\
+ \lim_{n\to\infty} \left(\int_\Omega b(x)|v_n|^p \,\mathrm{d}x \right)^{\frac{1}{p}} = 1.
\end{aligned}$$

Hence

$$\int_\Omega b(x)|u_1|^p \,\mathrm{d}x = 1.$$

In particular, $u_1 \not\equiv 0$. The property of the weakly convergent sequence (v_n) in $W_0^{1,p}(w,\Omega)$ yields

$$\begin{aligned}
\lambda_1 &\leqslant \int_\Omega a(x)|\nabla u_1|^p \,\mathrm{d}x = \|u_1\|_a^p \leqslant \liminf_{n\to\infty} \|v_n\|_a^p \\
&= \liminf_{n\to\infty} \int_\Omega a(x)|\nabla v_n|^p \,\mathrm{d}x = \liminf_{n\to\infty} (\lambda_1 + \delta_n) = \lambda_1,
\end{aligned}$$

i.e.

$$\lambda_1 = \int_\Omega a(x)|\nabla u_1|^p \,\mathrm{d}x. \tag{3.7}$$

It follows from (3.7) that $\lambda_1 > 0$ and it is easy to see that λ_1 is the least eigenvalue of (3.4) with the corresponding eigenfunction u_1. Moreover, if u is an eigenfunction corresponding to λ_1 then $|u|$ is also an eigenfunction corresponding to λ_1. Hence we can suppose that $u_1 \geqslant 0$ a.e. in Ω. □

3.3. Remark. It follows from the proof of Lemma 3.2 that $v_n \rightharpoonup u_1$ in $W_0^{1,p}(w,\Omega)$ and $\|v_n\|_a \to \|u_1\|_a$. The uniform convexity of $W_0^{1,p}(w,\Omega)$ (see Subsection 2.4) then implies the *strong convergence* $v_n \to u_1$ in $W_0^{1,p}(w,\Omega)$.

3.4. Lemma. *Let $u \in W_0^{1,p}(w,\Omega)$, $u \geqslant 0$ a.e. in Ω, be the eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the eigenvalue problem* (3.4). *Then $u \in L^r(\Omega)$ for any $1 \leqslant r < \infty$.*

Proof. The assertion of lemma is fulfilled automatically if $ps \geqslant n(s+1)$ (see Subsection 2.2). Let us suppose that $ps < n(s+1)$. For $M > 0$ define

$$v_M(x) = \inf\{u(x), M\} \in W_0^{1,p}(w,\Omega) \cap L^\infty(\Omega).$$

Let us choose $\varphi = v_M^{\kappa p+1} (\kappa \geqslant 0)$ in

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla \varphi \,\mathrm{d}x = \lambda_1 \int_\Omega b(x)|u|^{p-2}u\varphi \,\mathrm{d}x. \tag{3.8}$$

Obviously $\varphi \in W_0^{1,p}(w,\Omega) \cap L^\infty(\Omega)$. It follows from (3.8) that

$$(\kappa p+1)\int_\Omega a(x)v_M^{\kappa p}|\nabla v_M|^p \,\mathrm{d}x = \lambda_1 \int_\Omega b(x)u^{p-1}v_M^{\kappa p+1} \,\mathrm{d}x. \tag{3.9}$$

Due to (3.1) and the imbedding $W_0^{1,p}(w,\Omega) \hookrightarrow L^q(\Omega)$ we have

$$\begin{aligned} &(\kappa p+1)\int_\Omega a(x)v_M^{\kappa p}|\nabla v_M|^p \,\mathrm{d}x \\ &\qquad \geqslant \frac{\kappa p+1}{c_8}\int_\Omega w(x)v_M^{\kappa p}|\nabla v_M|^p \,\mathrm{d}x \\ &\qquad = \frac{\kappa p+1}{c_8(\kappa+1)^p}\int_\Omega w(x)|\nabla(v_M^{\kappa+1})|^p \,\mathrm{d}x \geqslant c_9\left(\int_\Omega (v_M^{\kappa+1})^q \,\mathrm{d}x\right)^{\frac{p}{q}}. \end{aligned} \tag{3.10}$$

Hence it follows from (3.2), (3.8), (3.9), (3.10) and the Hölder inequality that

$$
\begin{aligned}
\left(\int_\Omega v_M^{(\kappa+1)q}\,\mathrm{d}x\right)^{\frac{p}{q}} &\leqslant c_{10}\int_\Omega b(x)u^{p-1}v_M^{\kappa p+1}\,\mathrm{d}x \\
&\leqslant c_{10}\left(\int_\Omega b(x)^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\cdot\left(\int_\Omega u^{(\kappa+1)q^*}\,\mathrm{d}x\right)^{\frac{p}{q^*}}.
\end{aligned}
\tag{3.11}
$$

Since $u \in L^r(\Omega)$ for any $1 \leqslant r \leqslant q$ (see Subsection 2.2), we can choose κ in (3.11) in the following way:

$$(\kappa+1)q^* = q. \tag{3.12}$$

Then substituting (3.12) into (3.11) we obtain

$$\left(\int_\Omega v_M^{(\kappa+1)q}\,\mathrm{d}x\right)^{\frac{p}{q}} \leqslant c_{11}\left(\int_\Omega u^q\,\mathrm{d}x\right)^{\frac{p}{q^*}} \leqslant c_{12}, \tag{3.13}$$

i.e. $v_M \in L^{q'}(\Omega), q' = (\kappa+1)q$, for any $M > 0$. We have $u(x) = \lim\limits_{M\to\infty} v_M(x), x \in \Omega$. Then the Fatou lemma and (3.13) yield

$$\left(\int_\Omega u^{q'}\,\mathrm{d}x\right)^{\frac{p}{q}} \leqslant \liminf_{M\to\infty}\left(\int_\Omega v_M^{q'}\,\mathrm{d}x\right)^{\frac{p}{q}} \leqslant c_{12},$$

i.e. $u \in L^{q'}(\Omega)$, where

$$q' = \frac{q}{q^*}q.$$

Repeating the same argument we can choose κ in (3.11) as $(\kappa+1)q^* = q'$ and get $u \in L^{q''}(\Omega), q'' = q(\frac{q}{q^*})^2$, etc. Since $q > q*$, the bootstrap argument implies the assertion of lemma. □

3.5. Lemma. *Let $u \in W_0^{1,p}(w,\Omega), u \geqslant 0$ a.e. in Ω be the eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the eigenvalue problem* (3.4). *Then $u \in L^\infty(\Omega)$.*

Proof. Let $k \geqslant 0$ be a real number. Set

$$\varphi(x) = \sup\,\{u(x), k\} - k$$

in (3.8). We obtain

$$\int_{\Omega(u>k)} a(x)|\nabla\varphi|^p\,\mathrm{d}x = \lambda_1\int_{\Omega(u>k)} b(x)(\varphi+k)^{p-1}\varphi\,\mathrm{d}x,$$

i.e.

$$\int_{\Omega(u>k)} w(x)|\nabla\varphi|^p\,\mathrm{d}x \leqslant \lambda_1 c_8 \int_{\Omega(u>k)} b(x)(\varphi+k)^{p-1}\varphi\,\mathrm{d}x. \tag{3.14}$$

Let us choose

$$r > \max\left\{\frac{(p-1)qq^*}{p(q-q^*)}, q\right\}.$$

Due to the homogeneity of (3.8) and Lemma 3.4 we can assume without loss of generality that

$$\|u\|_{L^r(\Omega)} = \tilde{R} > 0.$$

The imbedding $W_0^{1,p}(w,\Omega) \hookrightarrow L^q(\Omega)$ implies

$$\int_{\Omega(u>k)} w(x)|\nabla\varphi|^p\,\mathrm{d}x \geqslant c_{13}\|\varphi\|^p_{L^q(\Omega)}. \tag{3.15}$$

Since $r > q$, the Hölder inequality yields

$$\begin{aligned}
&\int_{\Omega(u>k)} b(x)(\varphi+k)^{p-1}\varphi\,\mathrm{d}x\\
&\quad\leqslant \left(\int_{\Omega(u>k)} b(x)^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}} \left(\int_{\Omega(u>k)} (\varphi+k)^{\frac{p-1}{p}q^*}\varphi^{\frac{q^*}{p}}\,\mathrm{d}x\right)^{\frac{p}{q^*}}\\
&\quad\leqslant c_{14}\left(\int_{\Omega(u>k)} (\varphi+k)^{q^*}\,\mathrm{d}x\right)^{\frac{p-1}{q^*}} \left(\int_{\Omega(u>k)} \varphi^{q^*}\,\mathrm{d}x\right)^{\frac{1}{q^*}}\\
&\quad\leqslant c_{14}\left(\int_{\Omega} u^r\,\mathrm{d}x\right)^{\frac{p-1}{r}} \big(\operatorname{meas}\Omega(u>k)\big)^{\frac{p-1}{q^*}(1-\frac{q^*}{r})}\\
&\qquad\quad\times\left(\int_{\Omega} \varphi^q\,\mathrm{d}x\right)^{\frac{1}{q}} \big(\operatorname{meas}\Omega(u>k)\big)^{\frac{1}{q^*}(1-\frac{q^*}{q})}.
\end{aligned} \tag{3.16}$$

It follows from (3.14)–(3.16) that

$$\|\varphi\|^{p-1}_{L^q(\Omega)} \leqslant c_{15}(\tilde{R})\big(\operatorname{meas}\Omega(u>k)\big)^{\frac{p-1}{q^*}(1-\frac{q^*}{r})+\frac{1}{q^*}(1-\frac{q^*}{q})}. \tag{3.17}$$

On the other hand, for $h > k$ we obtain

$$\begin{aligned}
\|\varphi\|^{p-1}_{L^q(\Omega)} &= \left(\int_{\Omega(u>k)} |u-k|^q\,\mathrm{d}x\right)^{\frac{p-1}{q}}\\
&\geqslant \left(\int_{\Omega(u>h)} |u-k|^q\,\mathrm{d}x\right)^{\frac{p-1}{q}} \geqslant (h-k)^{p-1}\big(\operatorname{meas}\Omega(u>h)\big)^{\frac{p-1}{q}}.
\end{aligned} \tag{3.18}$$

Set $\zeta(t) = \operatorname{meas}\Omega(u > t)$. Then $\zeta(t)$ is a nonnegative and nonincreasing function and it follows from (3.17), (3.18) that

$$\zeta(h)^{\frac{p-1}{q}} \leqslant \frac{c_{15}(\tilde{R})}{(h-k)^{p-1}} \big(\zeta(k)\big)^{\frac{p-1}{q^*}(1-\frac{q^*}{r})+\frac{1}{q^*}(1-\frac{q^*}{q})},$$

i.e.

$$\zeta(h) \leqslant \tilde{c}_{15}(\tilde{R})(h-k)^{-q}\big(\zeta(k)\big)^{\delta},$$

where

$$\sigma = q, \quad \delta = \frac{q}{p-1}\Big[\frac{p-1}{q^*}\Big(1-\frac{q^*}{r}\Big) + \frac{1}{q^*}\Big(1-\frac{q^*}{q}\Big)\Big].$$

Due to the choice of r we have $\delta > 1$. It follows from Lemma 2.5 that there exists $d = d(r, q, \tilde{R}, \operatorname{meas}\Omega) > 0$ such that $\zeta(d) = 0$. Hence $u(x) \leqslant d$ for a.e. $x \in \Omega$. □

3.6. Proposition. *There exists precisely one nonnegative eigenfunction* u_1, $\|u_1\|_{L^{q^*}(\Omega)} = 1$, *corresponding to the first eigenvalue* $\lambda_1 > 0$ *of the eigenvalue problem* (3.4).

Proof. Due to the variational characterization of λ_1 the function $u \in W_0^{1,p}(w,\Omega)$ is an eigenfunction corresponding to λ_1 if and only if

$$\int_\Omega a(x)|\nabla u|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|u|^p \,\mathrm{d}x = 0$$

$$= \inf_{v \in W_0^{1,p}(w,\Omega)} \Big\{ \int_\Omega a(x)|\nabla v|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|v|^p \,\mathrm{d}x \Big\}.$$

This imlies that if $u_1, u_2 \in W_0^{1,p}(w,\Omega)$ are two eigenfunctions corresponding to λ_1 then also

$$v_1(x) = \max_{x\in\Omega}\{u_1(x), u_2(x)\}, \quad v_2(x) = \min_{x\in\Omega}\{u_1(x), u_2(x)\}$$

are eigenfunctions corresponding to λ_1 provided that $v_2 \not\equiv 0$. Indeed, we have $v_1, v_2 \in W_0^{1,p}(w,\Omega)$ and

$$\int_\Omega a(x)|\nabla v_1|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|v_1|^p \,\mathrm{d}x + \int_\Omega a(x)|\nabla v_2|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|v_2|^p \,\mathrm{d}x$$

$$= \int_\Omega a(x)|\nabla u_1|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|u_1|^p \,\mathrm{d}x + \int_\Omega a(x)|\nabla u_2|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|u_2|^p \,\mathrm{d}x.$$

Hence

$$\int_\Omega a(x)|\nabla v_1|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|v_1|^p \,\mathrm{d}x = \int_\Omega a(x)|\nabla v_2|^p \,\mathrm{d}x - \lambda_1 \int_\Omega b(x)|v_2|^p \,\mathrm{d}x = 0.$$

Let $u_1 \geqslant 0$ and $u_2 \geqslant 0$ be two eigenfunctions corresponding to λ_1 such that $u_1 \not\equiv u_2$, $\min\limits_{x\in\Omega}\{u_1(x), u_2(x)\} \not\equiv 0$ and

$$\|u_1\|_{L^{q^*}(\Omega)} = \|u_2\|_{L^{q^*}(\Omega)} = 1.$$

Denote $v_3(x) = k_1 v_2(x) = k_1 \min\limits_{x\in\Omega}\{u_1(x), u_2(x)\}$, where $k_1 > 0$ is chosen in such a way that

$$\|v_3\|_{L^{q^*}(\Omega)} = 1.$$

Then $v_3 \in W_0^{1,p}(w,\Omega)$ is again an eigenfunction corresponding to λ_1 such that $v_3 \not\equiv u_1$. Moreover,

$$\{x \in \Omega;\, u_1(x) = 0\} \subseteq \{x \in \Omega;\, v_3(x) = 0\}.$$

Set $v_5(x) = k_2 v_4(x) = k_2 \max\limits_{x\in\Omega}\{u_1(x), v_3(x)\}$, where $k_2 > 0$ is chosen such that

$$\|v_5\|_{L^{q^*}(\Omega)} = 1.$$

Then $v_5 \in W_0^{1,p}(w,\Omega)$ is an eigenfunction corresponding to λ_1 such that $v_5 \not\equiv u_1$ and

$$\{x \in \Omega;\, v_5(x) = 0\} = \{x \in \Omega;\, u_1(x) = 0\}.$$

Let, now, $u_1 \geqslant 0$ and $u_2 \geqslant 0$ be two eigenfunctions corresponding to λ_1 such that $u_1 \not\equiv u_2$, $\|u_1\|_{L^{q^*}(\Omega)} = \|u_2\|_{L^{q^*}(\Omega)} = 1$ and

$$\min_{x\in\Omega}\{u_1(x), u_2(x)\} \equiv 0.$$

Denote $\tilde{u}_1 = k_3 \max\{u_1(x), u_2(x)\}$, where $0 < k_3 < 1$ is chosen such that

$$\|\tilde{u}_1\|_{L^{q^*}(\Omega)} = 1,$$

and $\tilde{u}_2 = k_4 \max\{u_1(x), \tilde{u}_1(x)\}$, where $0 < k_4 < 1$ is such that

$$\|\tilde{u}_2\|_{L^{q^*}(\Omega)} = 1.$$

Then $\tilde{u}_1$ and $\tilde{u}_2$ are eigenfunctions corresponding to λ_1 such that $\tilde{u}_1 \not\equiv \tilde{u}_2$ and

$$\{x \in \Omega;\, \tilde{u}_1 = 0\} = \{x \in \Omega;\, \tilde{u}_2 = 0\}.$$

We will prove the assertion of proposition via contradiction. Due to the argument presented above we assume that $u \geqslant 0$ and $v \geqslant 0$ are eigenfunctions corresponding to λ_1 such that

$$\|u\|_{L^{q^*}(\Omega)} = \|v\|_{L^{q^*}(\Omega)} = 1, \quad u \not\equiv v, \tag{3.19}$$

and vanishing in Ω on the same set (almost everywhere in the sense of the Lebesgue measure). Then

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,\mathrm{d}x = \lambda_1\int_\Omega b(x)|u|^{p-2}u\varphi\,\mathrm{d}x \tag{3.20}$$

for any $\varphi \in W_0^{1,p}(w,\Omega)$, and

$$\int_\Omega a(x)|\nabla v|^{p-2}\nabla v\nabla\psi\,\mathrm{d}x = \lambda_1\int_\Omega b(x)|v|^{p-2}v\psi\,\mathrm{d}x \tag{3.21}$$

for any $\psi \in W_0^{1,p}(w,\Omega)$. For $\varepsilon > 0$ set

$$u_\varepsilon = u+\varepsilon \text{ and } v_\varepsilon = v+\varepsilon.$$

Substitute

$$\varphi = \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}}$$

into (3.20) and

$$\psi = \frac{v_\varepsilon^p - u_\varepsilon^p}{v_\varepsilon^{p-1}}$$

into (3.21). Since $\frac{u_\varepsilon}{v_\varepsilon}, \frac{v_\varepsilon}{u_\varepsilon} \in L^\infty(\Omega)$ and

$$\nabla\varphi = \left[1+(p-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^p\right]\nabla u - p\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}\nabla v,$$

$$\nabla\psi = \left[1+(p-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^p\right]\nabla v - p\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p-1}\nabla u,$$

we have $\varphi, \psi \in W_0^{1,p}(w,\Omega)$. Adding (3.20) and (3.21) (with φ and ψ chosen above) we obtain

$$\int_\Omega a(x)\left\{\left[1+(p-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^p\right]|\nabla u|^p + \left[1+(p-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^p\right]|\nabla v|^p\right\}\mathrm{d}x$$

$$-\int_\Omega a(x)\left\{p\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}|\nabla u|^{p-2}\nabla u\nabla v + p\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p-1}|\nabla v|^{p-2}\nabla v\nabla u\right\}\mathrm{d}x$$

$$= \lambda_1\int_\Omega b(x)\left[\left(\frac{u}{u_\varepsilon}\right)^{p-1} - \left(\frac{v}{v_\varepsilon}\right)^{p-1}\right](u_\varepsilon^p - v_\varepsilon^p)\,\mathrm{d}x.$$

Since $|\nabla \log u_\varepsilon| = \frac{|\nabla u|}{u_\varepsilon}$, the last equality is equivalent to

$$\begin{aligned}
&\int_\Omega a(x)(u_\varepsilon^p - v_\varepsilon^p)[|\nabla \log u_\varepsilon|^p - |\nabla \log v_\varepsilon|^p]\,\mathrm{d}x \\
&\quad - \int_\Omega a(x) p v_\varepsilon^p |\nabla \log u_\varepsilon|^{p-2} \nabla \log u_\varepsilon (\nabla \log v_\varepsilon - \nabla \log u_\varepsilon)\,\mathrm{d}x \\
&\quad - \int_\Omega a(x) p u_\varepsilon^p |\nabla \log v_\varepsilon|^{p-2} \nabla \log v_\varepsilon (\nabla \log u_\varepsilon - \nabla \log v_\varepsilon)\,\mathrm{d}x \\
&\quad = \lambda_1 \int_\Omega b(x) \Big[\Big(\frac{u}{u_\varepsilon}\Big)^{p-1} - \Big(\frac{v}{v_\varepsilon}\Big)^{p-1}\Big](u_\varepsilon^p - v_\varepsilon^p)\,\mathrm{d}x.
\end{aligned} \tag{3.22}$$

Let $p \geqslant 2$. We use (2.13) in order to estimate the left hand side of (3.22) (we first set $\boldsymbol{t}_1 = \nabla \log u_\varepsilon$, $\boldsymbol{t}_2 = \nabla \log v_\varepsilon$ and then $\boldsymbol{t}_1 = \nabla \log v_\varepsilon$, $\boldsymbol{t}_2 = \nabla \log u_\varepsilon$). We obtain

$$\begin{aligned}
&\lambda_1 \int_\Omega b(x) \Big[\Big(\frac{u}{u_\varepsilon}\Big)^{p-1} - \Big(\frac{v}{v_\varepsilon}\Big)^{p-1}\Big](u_\varepsilon^p - v_\varepsilon^p)\,\mathrm{d}x \\
&\quad \geqslant \frac{1}{2^{p-1}-1} \int_\Omega a(x) |\nabla \log u_\varepsilon - \nabla \log v_\varepsilon|^p (u_\varepsilon^p + v_\varepsilon^p)\,\mathrm{d}x \\
&\quad = \frac{1}{2^{p-1}-1} \int_\Omega a(x) \Big(\frac{1}{v_\varepsilon^p} + \frac{1}{u_\varepsilon^p}\Big) |v_\varepsilon \nabla u - u_\varepsilon \nabla v|^p\,\mathrm{d}x \geqslant 0.
\end{aligned} \tag{3.23}$$

Let $1 < p < 2$. We use (2.14) in order to estimate the left hand side of (3.22) (similarly as above) obtaining

$$\begin{aligned}
&\lambda_1 \int_\Omega b(x) \Big[\Big(\frac{u}{u_\varepsilon}\Big)^{p-1} - \Big(\frac{v}{v_\varepsilon}\Big)^{p-1}\Big](u_\varepsilon^p - v_\varepsilon^p)\,\mathrm{d}x \\
&\quad \geqslant \frac{3p(p-1)}{16} \int_\Omega a(x) \Big(\frac{1}{u_\varepsilon^p} + \frac{1}{v_\varepsilon^p}\Big) \frac{|v_\varepsilon \nabla u - u_\varepsilon \nabla v|^2}{(v_\varepsilon |\nabla u| + u_\varepsilon |\nabla v|)^{2-p}}\,\mathrm{d}x \geqslant 0.
\end{aligned} \tag{3.24}$$

We have $u, v \in L^\infty(\Omega)$ (see Lemma 3.5) and

$$\frac{u}{u_\varepsilon} \to 1, \quad \frac{v}{v_\varepsilon} \to 1 \quad (\varepsilon \to 0_+) \tag{3.25}$$

a.e. in Ω where $u > 0$ and $v > 0$, respectively;

$$\frac{u}{u_\varepsilon} = 0, \quad \frac{v}{v_\varepsilon} = 0 \text{ (for any } \varepsilon > 0) \tag{3.26}$$

elsewhere (since u and v vanish on the same set in Ω). Hence it follows from (3.25), (3.26) and the Lebesgue theorem that for any p, $1 < p < \infty$,

$$\lambda_1 \int_\Omega b(x) \Big[\Big(\frac{u}{u_\varepsilon}\Big)^{p-1} - \Big(\frac{v}{v_\varepsilon}\Big)^{p-1}\Big](u_\varepsilon^p - v_\varepsilon^p)\,\mathrm{d}x \to 0 \quad (\varepsilon \to 0_+).$$

This together with (3.23), (3.24) and the Fatou lemma implies

$$|v \nabla u - u \nabla v| = 0 \text{ a.e. in } \Omega$$

for any $1 < p < \infty$. Hence there exists a constant $k > 0$ such that $u = kv$ a.e. in Ω. But (3.19) yields $k = 1$, i.e. $u = v$ a.e. in Ω, which is a contradiction. $\square$

The proof of Proposition 3.6 follows the lines of the proof of Lemma 3.1 in Lindqvist [12] for the nondegenerate case ($a(x) \equiv 1$ in Ω).

3.7. Lemma. *Let $J\colon W_0^{1,p}(w,\Omega) \longrightarrow [W_0^{1,p}(w,\Omega)]^*$ be an operator defined by*

$$\langle J(u), \varphi \rangle = \int_\Omega a(x) |\nabla u|^{p-2} \nabla u \nabla \varphi \, \mathrm{d}x$$

for any $u, \varphi \in W_0^{1,p}(w,\Omega)$ (here $\langle \cdot, \cdot \rangle$ denotes the duality between $[W_0^{1,p}(w,\Omega)]^*$ and $W_0^{1,p}(w,\Omega)$). Then J is surjective and $J^{-1}\colon [W_0^{1,p}(w,\Omega)]^* \longrightarrow W_0^{1,p}(w,\Omega)$ is bounded and continuous.

P r o o f . The operator J is bounded, strictly monotone, continuous and coercive. Then it follows from the Browder theorem (see e.g. Fučík, Kufner [8]) that J is surjective. It follows from the Hölder inequality that

$$\langle J(v) - J(u), v - u \rangle \geqslant (\|v\|_a^{p-1} - \|u\|_a^{p-1})(\|v\|_a - \|u\|_a) \tag{3.27}$$

for any $u, v \in W_0^{1,p}(w,\Omega)$. The boundedness of J^{-1} follows immediately from (3.27). Let us suppose to the contrary that J^{-1} is not continuous. Then there exists a sequence (f_n) such that $f_n \to f$ in $[W_0^{1,p}(w,\Omega)]^*$ and $\|J^{-1}(f_n) - J^{-1}(f)\|_a \geqslant \delta$ for some $\delta > 0$. Denote $u_n = J^{-1}(f_n), u = J^{-1}(f)$. It follows from (3.27) that

$$\|f_n\|_* \cdot \|u_n\|_a \geqslant \langle f_n, u_n \rangle = \langle J(u_n), u_n \rangle \geqslant \|u_n\|_a^p,$$

i.e.

$$\|u_n\|_a^{p-1} \leqslant \|f_n\|_*$$

($\| \cdot \|_*$ denotes the norm in the dual space $[W_0^{1,p}(w,\Omega)]^*$). Then (u_n) is bounded in $W_0^{1,p}(w,\Omega)$ and we can assume that there exists $\tilde{u} \in W_0^{1,p}(w,\Omega)$ such that $u_n \rightharpoonup \tilde{u}$ in $W_0^{1,p}(w,\Omega)$. Hence we have

$$\begin{aligned} &\langle J(u_n) - J(\tilde{u}), u_n - \tilde{u} \rangle = \\ &= \langle J(u_n) - J(u), u_n - \tilde{u} \rangle + \langle J(u) - J(\tilde{u}), u_n - \tilde{u} \rangle \longrightarrow 0 \end{aligned} \tag{3.28}$$

since $J(u_n) \to J(u)$ in $[W_0^{1,p}(w,\Omega)]^*$. It follows from (3.27) (where we set $v = u_n, u = \tilde{u}$) and (3.28) that $\|u_n\|_a \to \|\tilde{u}\|_a$. The uniform convexity of $W_0^{1,p}(w,\Omega)$ equipped with the norm $\| \cdot \|_a$ (see Subsection 2.4) implies $u_n \to \tilde{u}$ in $W_0^{1,p}(w,\Omega)$. This convergence together with the convergence $J(u_n) \to J(u)$ in $[W_0^{1,p}(w,\Omega)]^*$ implies $\tilde{u} = u$ which is a contradiction. The continuity of J^{-1} is proved. $\square$

4.1. Weak formulation. In this section we will consider the *nonhomogeneous eigenvalue problem*

$$\begin{aligned} -\operatorname{div}(a(x,u)|\nabla u|^{p-2}\nabla u) &= \lambda b(x,u)|u|^{p-2}u \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{4.1}$$

Let $g\colon [0,\infty) \to [1,\infty)$ be a nondecreasing function, $\alpha(x) \in L^{\frac{q^*}{q^*-p}}(\Omega)$ for $q > q^* > p, \alpha(x) \in L^\infty(\Omega)$ for $q^* = p$ (for q, q^* see Subsection 3.1), $\beta > 0$ a constant. We assume that $a(x,s), b(x,s)$ are Carathéodory functions (i.e. continuous in s for a.e. $x \in \Omega$ and measurable in x for all $s \in \mathbb{R}$) and

$$\frac{w(x)}{c_8} \leqslant a(x,s) \leqslant c_8 g(|s|)w(x), \tag{4.2}$$

$$0 \leqslant b(x,s) \leqslant \alpha(x) + \beta|s|^{q^*-p} \tag{4.3}$$

hold for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$.

Moreover, assume that

$$\operatorname{meas}\{x \in \Omega;\, b(x,v(x)) > 0\} > 0 \tag{4.4}$$

for any $v \in L^{q^*}(\Omega), v \not\equiv 0$. (Note that the condition (4.4) is fulfilled e.g. if $b(x,s) > 0$ for a.e. $x \in \Omega$ and for all $s \neq 0$.)

We will say that $\lambda \in \mathbb{R}$ is an *eigenvalue* and $u \in W_0^{1,p}(w,\Omega), u \not\equiv 0$, is the corresponding *eigenfunction* of the eigenvalue problem (4.1) if

$$\int_\Omega a(x,u(x))|\nabla u|^{p-2}\nabla u \nabla \varphi \,\mathrm{d}x = \lambda \int_\Omega b(x,u(x))|u|^{p-2}u\varphi \,\mathrm{d}x \tag{4.5}$$

holds for any $\varphi \in W_0^{1,p}(w,\Omega)$.

4.2. Proposition (apriori estimate). *Let $u \in L^\infty(\Omega), \|u\|_{L^{q^*}(\Omega)} = R > 0$, $u \geqslant 0$ be any eigenfunction of* (4.1) *corresponding to the eigenvalue λ. Then there exists $d(R) > 0$ (independent of g) such that $\|u\|_{L^\infty(\Omega)} \leqslant d(R)$.*

Proof. Choose $\varphi = u^{\kappa p+1}$ in (4.5) with $\kappa \geqslant 0$. We obtain

$$(\kappa p+1)\int_\Omega a(x,u(x))u^{\kappa p}|\nabla u|^p \,\mathrm{d}x = \lambda \int_\Omega b(x,u(x))u^{(\kappa+1)p}\,\mathrm{d}x, \text{ i.e.}$$

$$(4.6)\qquad \frac{\kappa p+1}{(\kappa+1)^p}\int_\Omega a(x,u(x))|\nabla(u^{\kappa+1})|^p\,\mathrm{d}x=\lambda\int_\Omega b(x,u(x))u^{(\kappa+1)p}\,\mathrm{d}x.$$

It follows from (4.2) and the imbedding $W_0^{1,p}(w,\Omega)\hookrightarrow L^q(\Omega)$ that

$$(4.7)\qquad \begin{aligned}\int_\Omega a(x,u(x))|\nabla(u^{\kappa+1})|^p\,\mathrm{d}x &\geqslant \frac{1}{c_8}\int_\Omega w(x)|\nabla(u^{\kappa+1})|^p\,\mathrm{d}x\\ &\geqslant c_{16}\Big(\int_\Omega u^{(\kappa+1)q}\,\mathrm{d}x\Big)^{\frac{p}{q}}\end{aligned}$$

with $c_{16}>0$ independent of κ, R and g.

Applying the Hölder inequality, (4.3) and the Minkowski inequality we obtain

$$(4.8)\qquad \begin{aligned}&\int_\Omega b(x,u(x))u^{(\kappa+1)p}\,\mathrm{d}x\\ &\leqslant\left(\int_\Omega\Big(b(x,u(x))\Big)^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\left(\int_\Omega u^{(\kappa+1)q^*}\,\mathrm{d}x\right)^{\frac{p}{q^*}}\\ &\leqslant\left[\left(\int_\Omega \alpha(x)^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}+\beta\left(\int_\Omega u^{q^*}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\right]\left(\int_\Omega u^{(\kappa+1)q^*}\,\mathrm{d}x\right)^{\frac{p}{q^*}}.\end{aligned}$$

It follows from (4.6), (4.7) and (4.8) that

$$(4.9)\qquad \begin{aligned}&\int_\Omega u^{(\kappa+1)q}\,\mathrm{d}x\\ &\leqslant c_{17}\frac{(\kappa+1)^q}{(\kappa p+1)^{\frac{q}{p}}}\Big[\|\alpha\|_{L^{\frac{q^*}{q^*-p}}(\Omega)}+\beta R^{q^*-p}\Big]^{\frac{q}{p}}\cdot\left(\int_\Omega u^{(\kappa+1)q^*}\,\mathrm{d}x\right)^{\frac{q}{q^*}},\end{aligned}$$

with $c_{17}>0$ independent of κ, R and g. Let j be a nonnegative integer. Substitute $\kappa=\frac{q^j-(q^*)^j}{(q^*)^j}$ into (4.9):

$$(4.10)\qquad \begin{aligned}\int_\Omega u^{\frac{q^{j+1}}{(q^*)^j}}\,\mathrm{d}x\leqslant c_{17}&\frac{\left[\frac{q^j}{(q^*)^j}\right]^q}{\left[\frac{q^j-(q^*)^j}{(q^*)^j}p+1\right]^{\frac{q}{p}}}\\ &\times\Big[\|\alpha\|_{L^{\frac{q^*}{q^*-p}}(\Omega)}+\beta R^{q^*-p}\Big]^{\frac{q}{p}}\left(\int_\Omega u^{\frac{q^j}{(q^*)^{j-1}}}\,\mathrm{d}x\right)^{\frac{q}{q^*}}.\end{aligned}$$

Since

$$\lim_{j\to\infty}\frac{q^{j+1}}{(q^*)^j}=\infty,$$

there exists the least j_0 such that

$$r=\frac{q^{j_0+1}}{(q^*)^{j_0}}>\max\Big\{\frac{(p-1)qq^*}{p(q-q^*)},q\Big\}.$$

It follows from (4.9), (4.10) (setting $j = j_0, j_0 - 1, \cdots, 1$) that

$$\left(\int_\Omega u^r \, \mathrm{d}x\right)^{\frac{1}{r}} \leqslant \tilde{R}(R),$$

where $\tilde{R} > 0$ is independent of g.

Now we set $a(x) := a(x, u(x))$ and $b(x) := b(x, u(x))$ in the proof of Lemma 3.5. Following the lines of this proof we obtain

$$\|u\|_{L^\infty(\Omega)} \leqslant d(R),$$

where $d = d(R)$ is independent of g. This completes the proof of Proposition 4.2. □

4.3. Truncation in the principal part. Let $R > 0$ and $d = d(R) > 0$ be as above. We define

$$\tilde{a}(x, s) = \begin{cases} a(x, s) & \text{for } x \in \Omega, |s| \leqslant d(R), \\ a(x, d(R)) & \text{for } x \in \Omega, s > d(R), \\ a(x, -d(R)) & \text{for } x \in \Omega, s < -d(R). \end{cases} \tag{4.11}$$

Let us consider the nonhomogeneous eigenvalue problem

$$\begin{gathered} -\operatorname{div}(\tilde{a}(x, u)|\nabla u|^{p-2} \nabla u) = \lambda b(x, u)|u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{gathered} \tag{4.12}$$

Then it follows from Proposition 4.2 that $u \in W_0^{1,p}(w, \Omega)$, $\|u\|_{L^{q^*}(\Omega)} = R$, $u \geqslant 0$ is an eigenfunction of (4.12) *if and only if* it is an eigenfunction of (4.1).

4.4. Application of the fixed point theorem. For a given $v \in L^{q^*}(\Omega)$ set $a_v(x) = \tilde{a}(x, v(x)), b_v(x) = b(x, v(x))$. It follows from (4.2), (4.3), (4.4) and (4.11) that $a_v(x)$ and $b_v(x)$ fulfil (3.1), (3.2), (3.3) for any fixed $v \in L^{q^*}(\Omega)$. Let us consider the homogeneous eigenvalue problem

$$\begin{gathered} -\operatorname{div}(a_v(x)|\nabla u|^{p-2} \nabla u) = \lambda b_v(x)|u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{gathered} \tag{4.13}$$

for any fixed $v \in L^{q^*}(\Omega)$. Due to the results of Section 3 there exists the *least* eigenvalue $\lambda_v > 0$ of (4.13) and *precisely one* corresponding eigenfunction u_v such that $u_v \geqslant 0$ a.e. in $\Omega, u_v \in L^\infty(\Omega)$ and $\|u_v\|_{L^{q^*}(\Omega)} = R$. Hence we can define the *operator*

$$S\colon L^{q^*}(\Omega) \to L^{q^*}(\Omega)$$

which associates with $v \in L^{q^*}(\Omega)$ the first nonnegative eigenfunction u_v of (4.13) such that $\|u_v\|_{L^{q^*}(\Omega)} = R$.

Let us assume for a moment that S is a *compact operator*. Since it maps the ball $B_R = \{u \in L^{q^*}(\Omega), \|u\|_{L^{q^*}(\Omega)} \leqslant R\}$ into itself it follows from the Schauder fixed point theorem (see e.g. Fučík, Kufner [8]) that S has a *fixed point* $u \in B_R$. Hence there exists $\lambda_u > 0$ such that

$$\begin{aligned} -\operatorname{div}(a_u(x)|\nabla u|^{p-2}\nabla u) &= \lambda_u b_u(x)|u|^{p-2}u \text{ in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

and it follows from the considerations in Subsection 4.3 that $\lambda_u > 0$ is the least eigenvalue of (4.1) and $u \in L^\infty(\Omega), u \geqslant 0$ a.e. in Ω, is the corresponding eigenfunction satisfying $\|u\|_{L^{q^*}(\Omega)} = R$.

The *main result of this paper* follows from the considerations presented above.

4.5. Theorem. *Let the assumptions from Subsection* 4.1 *be fulfilled. Then for a given real number $R > 0$ there exists the least eigenvalue $\lambda > 0$ and the corresponding eigenfunction $u \in W_0^{1,p}(w,\Omega) \cap L^\infty(\Omega)$ of the nonhomogeneous eigenvalue problem* (4.1) *such that $u \geqslant 0$ a.e. in Ω and $\|u\|_{L^{q^*}(\Omega)} = R$.*

In the forthcoming subsections it remains to prove the compactness of the operator S in order to justify our assumption in Subsection 4.4.

4.6. The Nemytskii operators. Let us define the Nemytskii operators

$$G_1\colon u \mapsto |u|^{p-2}u, \quad G_2\colon u \mapsto |u|^p, \quad G_3\colon u \mapsto b(x, u(x)).$$

Then G_i is a bounded and continuous operator from $L^{q^*}(\Omega)$ into $L^{\frac{q^*}{p-1}}(\Omega)$ for $i = 1$, from $L^{q^*}(\Omega)$ into $L^{\frac{q^*}{p}}(\Omega)$ for $i = 2$, and from $L^{q^*}(\Omega)$ into $L^{\frac{q^*}{q^*-p}}(\Omega)$ for $i = 3$ (see e.g. Vajnberg [15], Fučík, Kufner [8]). The Nemytskii operator

$$G_4\colon (u, z_1, \ldots, z_n) \mapsto \tilde{a}(x, u(x))(z_1^2(x) + \ldots + z_n^2(x))^{\frac{p-1}{2}}$$

is bounded and continuous from $L^{q^*}(\Omega) \times L^p(w,\Omega) \times \ldots \times L^p(w,\Omega)$ into $L^{\frac{p}{p-1}}(w^{-\frac{1}{p-1}}, \Omega)$ (see e.g. Drábek, Kufner, Nicolosi [7], Kufner, Sändig [11]).

4.7. Lemma. *Let $z, z_n \in W_0^{1,p}(w,\Omega)$ and*

$$\int_\Omega a_v(x)|\nabla z|^{p-2}\nabla z \nabla\varphi\,\mathrm{d}x = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x,$$

$$\int_\Omega a_{v_n}(x)|\nabla z_n|^{p-2}\nabla z_n \nabla\psi\,\mathrm{d}x = \int_\Omega f_n(x)\psi(x)\,\mathrm{d}x$$

for any $\varphi, \psi \in W_0^{1,p}(w,\Omega)$ and let $v_n \to v$ in $L^{q^*}(\Omega)$, $f_n \to f$ in $[W_0^{1,p}(w,\Omega)]^*$. Then $z_n \to z$ in $W_0^{1,p}(w,\Omega)$.

Proof. Define operators $J, J_n\colon W_0^{1,p}(w,\Omega) \to [W_0^{1,p}(w,\Omega)]^*$ by

$$\langle J(u),\varphi\rangle = \int_\Omega a_v(x)|\nabla u|^{p-2}\nabla u\nabla\varphi\,\mathrm{d}x,$$

$$\langle J_n(u),\psi\rangle = \int_\Omega a_{v_n}(x)|\nabla u|^{p-2}\nabla u\nabla\psi\,\mathrm{d}x$$

for any $\varphi, \psi, u \in W_0^{1,p}(w,\Omega)$. Hence $J(z) = f$ and $J_n(z_n) = f_n$.

Let $n \in \mathbb{N}$ be fixed. Consider the equation

$$J_n(u) = h.$$

It follows that

$$\int_\Omega a_{v_n}(x)|\nabla u|^p\,\mathrm{d}x = \int_\Omega h(x)u(x)\,\mathrm{d}x,$$

$$\|u\|_w^p \leqslant c_{18}\|h\|_*\|u\|_w,$$

$$\|J_n^{-1}(h)\|_w \leqslant c_{18}\|h\|_*^{\frac{1}{p-1}} \tag{4.14}$$

for any $h \in [W_0^{1,p}(w,\Omega)]^*$, where $c_{18} > 0$ is independent of n and h. Analogously

$$\|J^{-1}(h)\|_w \leqslant c_{18}\|h\|_*^{\frac{1}{p-1}} \tag{4.15}$$

(cf. Lemma 3.7). Applying Lemma 3.7 for $a(x) := a_v(x)$ we obtain continuity of J^{-1} (with J defined in this subsection).

Assume that (u_n) is a sequence satisfying $u_n \to z$ in $W_0^{1,p}(w,\Omega)$. It follows from the continuity of the Nemytskii operator G_4 that

$$\begin{aligned}\|J_n(u_n) - J(u_n)\|_* &= \sup_{\|\varphi\|_w\leqslant 1}|\langle J_n(u_n) - J(u_n),\varphi\rangle|\\ &= \sup_{\|\varphi\|_w\leqslant 1}\left|\int_\Omega (a_{v_n}(x) - a_v(x))|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,\mathrm{d}x\right|\\ &\leqslant \sup_{\|\varphi\|_w\leqslant 1}\left|\int_\Omega \left[a_{v_n}(x)|\nabla u_n|^{p-2}\nabla u_n - a_v(x)|\nabla z|^{p-2}\nabla z\right]\nabla\varphi\,\mathrm{d}x\right|\\ &\quad + \sup_{\|\varphi\|_w\leqslant 1}\left|\int_\Omega \left[a_v(x)|\nabla z|^{p-2}\nabla z - a_v(x)|\nabla u_n|^{p-2}\nabla u_n\right]\nabla\varphi\,\mathrm{d}x\right|\end{aligned} \tag{4.16}$$

$$\leqslant \sup_{\|\varphi\|_w\leqslant 1}\Big(\int_\Omega w(x)^{-\frac{1}{p-1}}\big|a_{v_n}(x)|\nabla u_n|^{p-2}\nabla u_n - a_v(x)|\nabla z|^{p-2}\nabla z\big|^{\frac{p}{p-1}}\,\mathrm{d}x\Big)^{\frac{p-1}{p}}$$

$$\times\Big(\int_\Omega w(x)|\nabla\varphi|^p)\,\mathrm{d}x\Big)^{\frac{1}{p}}$$

$$+\sup_{\|\varphi\|_w\leqslant 1}\Big(\int_\Omega w(x)^{-\frac{p-1}{p}}\big|a_v(x)|\nabla z|^{p-2}\nabla z - a_v(x)|\nabla u_n|^{p-2}\nabla u_n\big|^{\frac{p}{p-1}}\,\mathrm{d}x\Big)^{\frac{p-1}{p}}$$

$$\times\Big(\int_\Omega w(x)|\nabla\varphi|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\to 0$$

for $n\to\infty$.

Set $u_n = J^{-1}(f_n)$. Then the assumptions of lemma and the continuity of J^{-1} imply

$$u_n\to z \text{ in } W_0^{1,p}(w,\Omega). \tag{4.17}$$

The relations (4.14)–(4.17) and the continuity of J^{-1} now yield

$$\begin{aligned}\|z_n - z\|_w &\leqslant \|J_n^{-1}(f_n) - J^{-1}(f_n)\|_w + \|J^{-1}(f_n) - J^{-1}(f)\|_w\\ &\leqslant \|J_n^{-1}(J_n - J)J^{-1}(f_n)\|_w + \|J^{-1}(f_n) - J^{-1}(f)\|_w\\ &\leqslant c_{18}\|J_n(u_n) - J(u_n)\|_*^{\frac{1}{p-1}} + \|J^{-1}(f_n) - J^{-1}(f)\|_w\to 0\end{aligned}$$

for $n\to\infty$, which completes the proof. □

4.8. Proposition. *The operator $S\colon L^{q^*}(\Omega)\to L^{q^*}(\Omega)$ defined in Subsection* 4.4 *is compact.*

Proof. We prove that S is a continuous operator from $L^{q^*}(\Omega)$ into $W_0^{1,p}(w,\Omega)$. The assertion then follows from the compact imbedding $W_0^{1,p}(w,\Omega)\hookrightarrow\hookrightarrow L^{q^*}(\Omega)$ (see Subsection 2.2). Let $u_{v_n} = S(v_n), u_v = S(v)$. Suppose to the contrary that $v_n\to v$ in $L^{q^*}(\Omega)$ and

$$\|u_{v_n} - u_v\|_w\geqslant\delta \tag{4.18}$$

for some $\delta > 0$. We have

$$\int_\Omega a_v(x)|\nabla u_v|^{p-2}\nabla u_v\nabla\varphi\,\mathrm{d}x = \lambda_v\int_\Omega b_v(x)|u_v|^{p-2}u_v\varphi\,\mathrm{d}x, \tag{4.19}$$

$$\int_\Omega a_{v_n}(x)|\nabla u_{v_n}|^{p-2}\nabla u_{v_n}\nabla\psi\,\mathrm{d}x = \lambda_{v_n}\int_\Omega b_{v_n}(x)|u_{v_n}|^{p-2}u_{v_n}\psi\,\mathrm{d}x \tag{4.20}$$

for any $\varphi, \psi \in W_0^{1,p}(w,\Omega)$. It follows from Lemma 3.7 that for any $v_n \in L^{q^*}(\Omega)$ there exists $z_n \in W_0^{1,p}(w,\Omega)$ such that

$$\int_\Omega a_{v_n}(x)|\nabla z_n|^{p-2}\nabla z_n \nabla \varphi \,\mathrm{d}x = \lambda_v \int_\Omega b_v(x)|u_v|^{p-2}u_v\varphi \,\mathrm{d}x \tag{4.21}$$

for any $\varphi \in W_0^{1,p}(w,\Omega)$. Lemma 4.7 yields $z_n \to u_v$ in $W_0^{1,p}(w,\Omega)$ (and hence also in $L^{q^*}(\Omega)$). Applying the Hölder inequality, (4.3) and the Minkowski inequality, we obtain

$$\begin{aligned}
&\left|\int_\Omega b(x,v(x))|u_v|^{p-2}u_v(z_n-u_v)\,\mathrm{d}x\right| \\
&\leqslant \left(\int_\Omega (b(x,v(x)))^{\frac{q^*}{q^*-1}}|u_v|^{\frac{q^*(p-1)}{q^*-1}}\,\mathrm{d}x\right)^{\frac{q^*-1}{q^*}}\left(\int_\Omega |z_n-u_v|^{q^*}\,\mathrm{d}x\right)^{\frac{1}{q^*}} \\
&\leqslant \left(\int_\Omega (b(x,v(x)))^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}} \\
&\qquad\times\left(\int_\Omega |u_v|^{q^*}\,\mathrm{d}x\right)^{\frac{p-1}{q^*}}\left(\int_\Omega |z_n-u_v|^{q^*}\,\mathrm{d}x\right)^{\frac{1}{q^*}} \\
&\leqslant \left[\left(\int_\Omega \alpha(x)^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}} + \beta\left(\int_\Omega |v(x)|^{q^*}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\right] \\
&\qquad\times\left(\int_\Omega |u_v|^{q^*}\,\mathrm{d}x\right)^{\frac{p-1}{q^*}}\left(\int_\Omega |z_n-u_v|^{q^*}\,\mathrm{d}x\right)^{\frac{1}{q^*}} \to 0
\end{aligned} \tag{4.22}$$

for $n \to \infty$. Applying the Hölder inequality, (4.3), the Minkowski inequality and the continuity of the Nemytskii operators G_2, G_3 we obtain

$$\begin{aligned}
&\left|\int_\Omega \Big[b(x,v_n(x))|z_n|^p - b(x,v(x))|u_v|^p\Big]\,\mathrm{d}x\right| \\
&\leqslant \left|\int_\Omega b(x,v_n(x))\big[|z_n|^p-|u_v|^p\big]\,\mathrm{d}x\right| \\
&\quad+\left|\int_\Omega \big[b(x,v_n(x))-b(x,v(x))\big]|u_v|^p\,\mathrm{d}x\right| \\
&\leqslant \left[\left(\int_\Omega \alpha(x)^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}} + \beta\left(\int_\Omega |v_n(x)|^{q^*}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\right] \\
&\qquad\times\left(\int_\Omega \big||z_n|^p-|u_v|^p\big|^{\frac{q^*}{p}}\,\mathrm{d}x\right)^{\frac{p}{q^*}} \\
&\quad+\left(\int_\Omega \big|b(x,v_n(x))-b(x,v(x))\big|^{\frac{q^*}{q^*-p}}\,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\left(\int_\Omega |u_v|^{q^*}\,\mathrm{d}x\right)^{\frac{p}{q^*}} \to 0
\end{aligned} \tag{4.23}$$

for $n \to \infty$. It follows from the variational characterization of λ_{v_n}, (4.19)–(4.23) that

$$\lambda_{v_n} \leqslant \frac{\int_\Omega a_{v_n}(x)|\nabla z_n|^p \,\mathrm{d}x}{\int_\Omega b_{v_n}(x)|z_n|^p \,\mathrm{d}x} = \frac{\lambda_v \int_\Omega b_v(x)|u_v|^{p-2}u_v z_n \,\mathrm{d}x}{\int_\Omega b_{v_n}(x)|z_n|^p \,\mathrm{d}x} \to \lambda_v \frac{\int_\Omega b_v(x)|u_v|^p \,\mathrm{d}x}{\int_\Omega b_v(x)|u_v|^p \,\mathrm{d}x} = \lambda_v.$$

Hence

$$\limsup \lambda_{v_n} \leqslant \lambda_v. \tag{4.24}$$

Applying the Hölder inequality, the Minkowski inequality and the assumptions (4.2), (4.3) we obtain from (4.20) (with $\psi = u_{v_n}$):

$$\begin{aligned}\frac{1}{c_8}\|u_{v_n}\|_w^p &\leqslant \int_\Omega a_{v_n}(x)|\nabla u_{v_n}|^p \,\mathrm{d}x = \lambda_{v_n}\int_\Omega b_{v_n}(x)|u_{v_n}|^p \,\mathrm{d}x \\ &\leqslant \lambda_{v_n}\left[\left(\int_\Omega |\alpha(x)|^{\frac{q^*}{q^*-p}} \,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}} + \beta\left(\int_\Omega |v_n(x)|^{q^*} \,\mathrm{d}x\right)^{\frac{q^*-p}{q^*}}\right] \\ &\qquad \times \left(\int_\Omega |u_{v_n}|^{q^*} \,\mathrm{d}x\right)^{\frac{p}{q^*}}.\end{aligned} \tag{4.25}$$

It follows from the assumption $\|u_{v_n}\|_{L^{q^*}(\Omega)} = R$, from $v_n \to v$ in $L^{q^*}(\Omega)$ and from(4.25) that

$$\|u_{v_n}\|_w \leqslant \text{ const} \tag{4.26}$$

for any $n \in \mathbb{N}$. Due to (4.26) we have

$$u_{v_n} \rightharpoonup u \text{ in } W_0^{1,p}(w,\Omega) \tag{4.27}$$

(at least for some subsequence) for some $u \in W_0^{1,p}(w,\Omega)$ and hence $u_n \to u$ in $L^{q^*}(\Omega)$.

The Hölder inequality, the Minkowski inequality, (4.3) and the continuity of the Nemytskii operators G_1 and G_3 imply

$$\begin{aligned}&\left|\int_\Omega [b(x,v_n(x))|u_{v_n}|^{p-2}u_{v_n} - b(x,v(x))|u|^{p-2}u]\varphi \,\mathrm{d}x\right| \\ &\leqslant \left|\int_\Omega \big[b(x,v_n(x)) - b(x,v(x))\big]|u_{v_n}|^{p-2}u_{v_n}\varphi \,\mathrm{d}x\right| \\ &\quad + \left|\int_\Omega b(x,v(x))\big[|u_{v_n}|^{p-2}u_{v_n} - |u|^{p-2}u\big]\varphi \,\mathrm{d}x\right| \leqslant\end{aligned} \tag{4.28}$$

$$\leqslant \left(\int_\Omega |b(x, v_n(x)) - b(x, v(x))|^{\frac{q^*}{q^*-p}} \, \mathrm{d}x \right)^{\frac{q^*-p}{q^*}} \left(\int_\Omega |u_{v_n}|^{q^*} \, \mathrm{d}x \right)^{\frac{p-1}{q^*}}$$

$$\times \left(\int_\Omega |\varphi|^{q^*} \, \mathrm{d}x \right)^{\frac{1}{q^*}}$$

$$+ \left[\left(\int_\Omega |\alpha(x)|^{\frac{q^*}{q^*-p}} \, \mathrm{d}x \right)^{\frac{q^*-p}{q^*}} + \beta \left(\int_\Omega |v(x)|^{\frac{q^*}{q^*-p}} \, \mathrm{d}x \right)^{\frac{q^*-p}{q^*}} \right]$$

$$\times \left(\int_\Omega \left| |u_{v_n}|^{p-2} u_{v_n} - |u|^{p-2} u \right|^{\frac{q^*}{p-1}} \, \mathrm{d}x \right)^{\frac{p-1}{q^*}} \left(\int_\Omega |\varphi|^{q^*} \, \mathrm{d}x \right)^{\frac{1}{q^*}} \to 0$$

for any $\varphi \in W_0^{1,p}(w, \Omega)$. Passing to suitable subsequences we can assume that

$$\lambda_{v_n} \to \lambda \in [0, \lambda_v] \tag{4.29}$$

(see (4.24)).

Let $\overline{u} \in W_0^{1,p}(w, \Omega)$ be the unique solution of

$$\int_\Omega a_v(x) |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi \, \mathrm{d}x = \lambda \int_\Omega b_v(x) |u|^{p-2} u \varphi \, \mathrm{d}x \tag{4.30}$$

for any $\varphi \in W_0^{1,p}(w, \Omega)$ (Lemma 3.7 guarantees the existence of $\overline{u}$). It follows from (4.28)–(4.30) and from Lemma 4.7 that

$$u_{v_n} \to \overline{u} \text{ in } W_0^{1,p}(w, \Omega). \tag{4.31}$$

Now, (4.27), (4.31) imply $u = \overline{u}$ and $u_{v_n} \to u$ in $W_0^{1,p}(w, \Omega)$. Hence we have

$$\lambda_v \geqslant \lambda = \frac{\int_\Omega a_v(x) |\nabla u|^p \, \mathrm{d}x}{\int_\Omega b_v(x) |u|^p \, \mathrm{d}x} \geqslant \inf_{\substack{\tilde{u} \neq 0 \\ \tilde{u} \in W_0^{1,p}(w,\Omega)}} \frac{\int_\Omega a_v(x) |\nabla \tilde{u}|^p \, \mathrm{d}x}{\int_\Omega b_v(x) |\tilde{u}|^p \, \mathrm{d}x}$$

$$= \frac{\int_\Omega a_v(x) |\nabla u_v|^p \, \mathrm{d}x}{\int_\Omega b_v(x) |u_v|^p \, \mathrm{d}x} = \lambda_v.$$

This implies that $\lambda = \lambda_v$ and $u = u_v$ (see the uniqueness of $u_v \geqslant 0$, $\|u_v\|_{L^{q^*}(\Omega)} = R$ in Section 3).

In particular, this means that

$$u_{v_n} \to u_v \text{ in } W_0^{1,p}(w, \Omega),$$

which contradicts (4.18). This completes the proof of Proposition 4.8. $\square$

4.9. Remark. The proofs in Section 4 can be performed in the same way working with $L^\infty(\Omega)$ instead of $L^{\frac{q^*}{q^*-p}}(\Omega)$ in the case $q^* = p$. Hence we obtain the following *special version* of Theorem 4.5.

4.10. Theorem. *Let* (4.2)–(4.4) *be fulfilled with* $\alpha(x) \in L^\infty(\Omega)$ *and* $q^* = p$. *Then for a given real number* $R > 0$ *there exists the least eigenvalue* $\lambda > 0$ *and the corresponding eigenfunction* $u \in W_0^{1,p}(w,\Omega) \cup L^\infty(\Omega)$ *of* (4.1) *such that* $u \geqslant 0$ *a.e. in* Ω *and* $\|u\|_{L^p(\Omega)} = R$.

4.11. Remark. Since the eigenvalue problem (4.13) is *homogeneous*, we can define the operator $\tilde{S}\colon L^{q^*}(\Omega) \longrightarrow L^{q^*}(\Omega)$ which associates with $v \in L^{q^*}(\Omega)$ the first nonpositive eigenfunction $-u_v$ of (4.13) such that $\|-u_v\|_{L^{q^*}(\Omega)} = R$. It is clear from the above considerations that $\tilde{S}$ has the *same properties* as S defined in Subsection 4.4. Hence repeating the same arguments as in Subsections 4.2–4.4, 4.6–4.8 we prove the following *dual version* of Theorem 4.5.

4.12. Theorem. *Let the assumptions of Theorem* 4.5 *be fulfilled. Then for a given real number* $R > 0$ *there exists the least eigenvalue* $\tilde{\lambda} > 0$ *and the corresponding eigenfunction* $\tilde{u} \in W_0^{1,p}(w,\Omega) \cap L^\infty(\Omega)$ *of the nonhomogeneous eigenvalue problem* (4.1) *such that* $\tilde{u} \leqslant 0$ *a.e. in* Ω *and* $\|\tilde{u}\|_{L^{q^*}(\Omega)} = R$.

4.13. Remark. Let λ and $\tilde{\lambda}$ be the least eigenvalues guaranteed by Theorem 4.5 and 4.12, respectively, for a given fixed $R > 0$. Then $\lambda \neq \tilde{\lambda}$ may hold due to the fact that the eigenvalue problem (4.1) is not homogeneous in general.

5. Examples

5.1. Example. Let Ω be a bounded domain in $\mathbb{R}^n$, $p > 1$, $w(x)$ be positive and measurable in Ω satisfying $w(x) \in L^1_{\mathrm{loc}}(\Omega)$, $\frac{1}{w(x)} \in L^s(\Omega)$ for $s > \max\{\frac{n}{p}, \frac{1}{p-1}\}$. Consider the eigenvalue problem

$$-\operatorname{div}(w(x)e^{u^2}|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{5.1}$$

In this case we have

$$a(x,s) = w(x)e^{s^2}, b(x,s) \equiv 1$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$.

It follows from Theorem 4.10 that *for any given real number* $R > 0$ *there exists the least eigenvalue* $\lambda > 0$ *and the corresponding eigenfunction* $u \in W_0^{1,p}(w,\Omega) \cap L^\infty(\Omega)$ *of* (5.1) *such that* $u \geqslant 0$ *a.e. in* Ω *and* $\|u\|_{L^p(\Omega)} = R$.

5.2. Example. Let us consider for Ω the plane domain $\Omega = (-1,1) \times (-1,1)$ (i.e. $\Omega \subset \mathbb{R}^2$). For $x = (x_1, x_2) \in \Omega$ set

$$w(x) = \begin{cases} 1, & x_1 \leqslant 0, \\ x_2^{\nu}(1-x_1)^{\gamma}, & x_1 > 0,\ x_2 > 0, \\ |x_2|^{\mu}(1-x_1)^{\gamma}, & x_1 > 0,\ x_2 < 0 \end{cases}$$

with ν, μ, γ real numbers. Consider the eigenvalue problem

$$\begin{aligned} -\operatorname{div}(w(x)(1+u^4)|\nabla u)|^2 \nabla u) &= \lambda u^9 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{5.2}$$

In this case we have $p = 4$,

$$a(x,s) = w(x)(1+s^4), b(x,s) = s^6$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$. Thus the principal part of the differential operator has a *degeneration* (or *singularity*) which is concentrated on a part Γ_1 of the boundary $\partial\Omega$,

$$\Gamma_1 = \{x = (x_1, x_2);\, x_1 = 1, x_2 \in (-1,1)\},$$

as well as on a segment Γ_2 in the interior of Ω,

$$\Gamma_2 = \{x = (x_1, x_2);\, x_1 \in (0,1), x_2 = 0\}.$$

Condition (2.1) indicates that we have to choose ν and μ from the interval $(-1,3)$ with no condition on γ. Let us assume that

$$\nu, \mu \in \Big(-1, \frac{4}{3}\Big), \quad \gamma \in \Big(-\infty, \frac{4}{3}\Big). \tag{5.3}$$

It follows from (5.3) that $\frac{1}{w(x)} \in L^{\frac{3}{4}}(\Omega)$ and $q = 12$ (see Subsection 2.2). Hence the growth condition (4.3) is fulfilled e.g. with $q^* = 10$. Applying Theorem 4.5 we have the following assertion.

Let us assume (5.3). *Then for a given real number $R > 0$ there exists the least eigenvalue $\lambda > 0$ and the corresponding eigenfunction $u \in W_0^{1,4}(w,\Omega) \cap L^{\infty}(\Omega)$ of* (5.2) *such that $u \geqslant 0$ a.e. in Ω and $\|u\|_{L^{10}(\Omega)} = R$.*

Note that for ν, μ and γ *positive* we have a *degeneration* of the same extent at Γ_1 and Γ_2. On the other hand, the *singularity* can occur in a limited extent at Γ_2 (for ν or μ *negative*, but bigger than -1), but big enough at Γ_1 (for any $\gamma < 0$).

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