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Eduard Feireisl

Antonín Novotný

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Eduard Feireisl *   Antonín Novotný †

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

IMATH, Université du Sud Toulon-Var
BP 20139, 839 57 La Garde, France

Abstract

We consider the full Navier-Stokes-Fourier system in the singular regime of small Mach and large Reynolds and Péclet numbers, with ill prepared initial data on an unbounded domain$\Omega \subset R^3$ with a compact boundary. We perform the singular limit in the framework of weak solutions and identify the Euler-Boussinesq system as the target problem.

Contents

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1 Introduction

The present paper is an extension of our previous results concerning the inviscid incompressible limit of the Navier-Stokes-Fourier system [7]. In contrast with [7], where the problem is considered on the whole space R^3 without any driving force imposed, we consider a more realistic situation when the fluid is subject to a gravitational force due to the physical objects placed *outside* the fluid domain. Accordingly, we shall assume that the fluid occupies an unbounded *exterior* domain $\Omega \subset R^3$ with smooth (compact) boundary. Such a situation is interesting from the point of view of possible applications in various meteorological models as the singular limit in the low Mach, Froude, and large Reynolds and Péclet numbers leads to a target system driven by the buoyancy force proportional to temperature deviations. In particular, we provide a rigorous justification of the so-called Euler-Boussinesq approximation. Our approach is based on the recently discovered relative entropy inequality [6] and the related concept of *dissipative solution* for the Navier-Stokes-Fourier system. In comparison with [7], the present problem features some additional mathematical difficulties related to the geometry of the underlying spatial domain and the presence of a driving force. In particular, we have to handle perturbations of weakly stratified equilibrium states, whereas those are simply constant in [7].

We consider the motion of a compressible, viscous and heat conducting fluid, with the density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$, and the absolute temperature $\vartheta = \vartheta(t,x)$ governed by the scaled *Navier-Stokes-Fourier system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) + \frac{1}{\varepsilon}\varrho\nabla_x F, \tag{1.2}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \varepsilon^\beta \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\right) = \frac{1}{\vartheta}\left(\varepsilon^{2+a}\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \varepsilon^b \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right), \tag{1.3}$$

where $p = p(\varrho,\vartheta)$ is the pressure, $s = s(\varrho,\vartheta)$ the specific entropy, the symbol $\mathbb{S}(\vartheta,\nabla_x\mathbf{u})$ denotes the viscous stress satisfying *Newton's law*

$$\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) = \mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{1.4}$$

and $\mathbf{q} = \mathbf{q}(\vartheta,\nabla_x\vartheta)$ is the heat flux determined by *Fourier's law*

$$\mathbf{q}(\vartheta,\nabla_x\vartheta) = -\kappa(\vartheta)\nabla_x\vartheta, \tag{1.5}$$

where the quantities μ, η, κ are temperature dependent transport coefficients.

The fluid occupies an exterior domain $\Omega \subset R^3$, with impermeable, thermally insulating and frictionless boundary, specifically,

$$\mathbf{u}\cdot\mathbf{n} = [\mathbb{S}(\vartheta,\nabla_x\mathbf{u})\cdot\mathbf{n}]_{\tan}|_{\partial\Omega} = 0,\ \nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{1.6}$$

In addition, we consider the far field boundary conditions

$$\varrho \to \overline{\varrho},\ \vartheta \to \overline{\vartheta},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty, \tag{1.7}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants.

The scaling in (1.1 - 1.3), expressed by means of a single (small) parameter ε, corresponds to:

Mach number .... ε,
Froude number .... $\varepsilon^{1/2}$,
Reynolds number .... ε^{-a},
Péclet number .... ε^{-b}.

In accordance with the previous discussion, we consider the driving force induced by a potential

$$F(x) = \int_{R^3} \frac{1}{x-y} m(y)\mathrm{d}y,\ m \geq 0,\ \mathrm{supp}[m] \subset R^3 \setminus \Omega, \tag{1.8}$$

meaning the fluid is driven by the gravitational force of objects lying outside the fluid domain.

Finally, the initial data are taken in the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho}_\varepsilon + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{1.9}$$

where $(\overline{\varrho}_\varepsilon, \overline{\vartheta})$ is the equilibrium solution associated with the far field values of $\overline{\varrho}$, $\overline{\vartheta}$, namely

$$\nabla_x p(\overline{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon \overline{\varrho}_\varepsilon \nabla_x F, \quad \overline{\varrho}_\varepsilon \to \overline{\varrho} \quad \text{as } |x| \to \infty. \tag{1.10}$$

The limit (target) problem can be formally identified as the incompressible *Euler-Boussinesq system*:

$$\mathrm{div}_x \mathbf{v} = 0, \tag{1.11}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = -a(\overline{\varrho}, \overline{\vartheta}) \theta \nabla_x F, \tag{1.12}$$

$$c_p(\overline{\varrho}, \overline{\vartheta}) \left(\partial_t \theta + \mathbf{v} \cdot \nabla_x \theta \right) - \overline{\vartheta} a(\overline{\varrho}, \overline{\vartheta}) \mathbf{v} \cdot \nabla_x F = 0, \tag{1.13}$$

where we have denoted

thermal expansion coefficient .... $a(\overline{\varrho}, \overline{\vartheta})$,
specific heat at constant pressure .... $c_p(\overline{\varrho}, \overline{\vartheta})$,

cf. [5, Chapter 5] and [7]. Here, the function $\mathbf{v}$ is the limit velocity, while θ is associated with the asymptotic temperature (entropy) deviations

$$\theta \approx \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}.$$

The exact statement of our results including the initial data for the target system (1.11 - 1.13) will be specified in Theorem 3.1 below.

We address the problem in the framework of weak solutions for the Navier-Stokes-Fourier system (1.1 - 1.3), developed in [5], and later extended to problems on unbounded domains in [11]. The main advantage of this approach is the convergence towards the target system on any time interval $[0,T]$, on which the Euler-Boussinsesq system (1.11), (1.12) possesses a regular solution. We refer to Masmoudi [16] for related results on the compressible barotropic Navier-Stokes system in the whole space R^3, see also the survey [17]. An alternative approach to singular limits, proposed in the seminal paper by Klainerman and Majda [13], uses the strong solutions for both the primitive and the target system that may exist, however, only on a possible very short time interval. Using the same framework, Alazard [1], [2], [3] addresses several singular limits of the compressible Euler and/or Navier-Stokes-Fourier system, in the absence of external forcing. The present setting, where the action of the gravitation gives rise to the buoyancy force proportional to $-\theta\nabla_x F$, represents a stronger coupling between the equations, typical for certain models used in meteorology and physics of the atmosphere, see Klein [14], [15], Zeytounian [18].

The necessary preliminary material including various concepts of weak solutions to the Navier-Stokes-Fourier system is collected in Section 2. Section 3 contains the main result on the asymptotic limit for $\varepsilon \to 0$, the proof of which is the main objective of the remaining part for the paper. In Section 4, the relative entropy inequality is used to establish the necessary uniform bounds independent of $\varepsilon \to 0$. The problem of propagation and dispersion of the associated acoustic waves is discussed in Section 5. The proof of convergence towards the limit system is completed in Section 6.

2 Preliminaries, weak solutions to the Navier-Stokes-Fourier system

Motivated by [6], we introduce the *relative entropy functional*

$$\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right)=\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+\frac{1}{\varepsilon^2}\left(H_\Theta(\varrho,\vartheta)-\frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho-r)-H_\Theta(r,\Theta)\right)\right]\ \mathrm{d}x, \tag{2.1}$$

where

$$H_\Theta(\varrho,\vartheta)=\varrho\Big(e(\varrho,\vartheta)-\Theta s(\varrho,\vartheta)\Big) \tag{2.2}$$

is the ballistic free energy. We say that a trio of functions $\{\varrho,\vartheta,\mathbf{u}\}$ represents a *dissipative weak solution* of the Navier-Stokes-Fourier system (1.1 - 1.7) in $(0,T)\times\Omega$ if:

- $\varrho\geq 0$, $\vartheta>0$ a.a. in $(0,T)\times\Omega$,

$$(\varrho-\overline{\varrho}_\varepsilon)\in L^\infty(0,T;L^2+L^{5/3}(\Omega)),\ (\vartheta-\overline{\vartheta})\in L^\infty(0,T;L^2+L^4(\Omega)),$$

$$\nabla_x\vartheta,\ \nabla_x\log(\vartheta)\in L^2(0,T;L^2(\Omega;R^3)),$$

$$\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,$$

  where $[\overline{\varrho}_\varepsilon,\overline{\vartheta}]$ stands for the equilibrium solution introduced in (1.10);

- the equation of continuity (1.1) holds as a family of integral identities

$$\int_\Omega\Big[\varrho(\tau,\cdot)\varphi(\tau,\cdot)-\varrho_{0,\varepsilon}\varphi(0,\cdot)\Big]\ \mathrm{d}x=\int_0^\tau\int_{R^3}\Big(\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.3}$$

  for any $\tau\in[0,T]$ and any test function $\varphi\in C^\infty_c([0,T]\times\overline{\Omega})$;

- the momentum equation (1.2), together with the initial condition (1.9), are satisfied in the sense of distributions,

$$\int_\Omega\Big[\varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot)-\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\varphi(0,\cdot)\Big]\ \mathrm{d}x \tag{2.4}$$

$$=\int_0^\tau\int_\Omega\Big(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi+\frac{1}{\varepsilon^2}p(\varrho,\vartheta)\mathrm{div}_x\varphi-\varepsilon^a\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\varphi+\frac{1}{\varepsilon}\nabla_xF\cdot\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t$$

  for any $\tau\in[0,T]$, and any $\varphi\in C^\infty_c([0,T]\times\overline{\Omega};R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega}=0$;

- the entropy production equation (1.3) is relaxed to the entropy inequality

$$\int_\Omega \Big[\varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \varphi(0,\cdot) - \varrho s(\varrho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot) \Big] \ \mathrm{d}x \tag{2.5}$$

$$+ \int_0^\tau \int_\Omega \frac{1}{\vartheta} \left(\varepsilon^{2+a} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq - \int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta) \partial_t \varphi + \varrho s(\varrho,\vartheta) \mathbf{u} \cdot \nabla_x \varphi + \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t$$

  for a.a. $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$, $\varphi \geq 0$;

- the *relative entropy inequality*

$$\Big[\mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U} \right) \Big]_{t=0}^{\tau} + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^{b-2} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t \tag{2.6}$$

$$\leq \int_0^\tau \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \Big[\Big(p(r,\Theta) - p(\varrho,\vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r,\Theta) \Big] \ \mathrm{d}x\mathrm{d}t$$

$$- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left(\varrho \Big(s(\varrho,\vartheta) - s(r,\Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho,\vartheta) - s(r,\Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta + \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r,\Theta) + \mathbf{U} \cdot \nabla_x p(r,\Theta) \Big) \ \mathrm{d}x \ \mathrm{d}t - \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x F \cdot (\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon) \mathrm{d}x.$$

  holds for a.a. $t \in (0,T)$ and for any trio of continuously differentiable "test" functions defined on $[0,T] \times \overline{\Omega}$,

$$r > 0, \ \Theta > 0, \ r \equiv \overline{\varrho}, \ \Theta \equiv \overline{\vartheta} \text{ outside a compact subset of } \overline{\Omega},$$

$$\mathbf{U} \in C([0,T]; W^{k,2}(\Omega; R^3)), \ \partial_t \mathbf{U} \in \ C([0,T]; W^{k-1,2}(\Omega; R^3)), \ k > \frac{5}{2}, \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Remark 2.1 *Note that the above definition of dissipative weak solutions on* unbounded *domains, proposed in [11], is different from that on bounded domains introduced in [6]. In [6], the relative entropy inequality (2.6) is replaced by the total energy balance, whereas (2.6) is automatically satisfied for any weak solution to the Navier-Stokes-Fourier system. The weak solutions introduced in this paper can be therefore viewed as "very weak dissipative solutions" of the primitive system.*

2.1 Structural restrictions imposed on constitutive relations

We study our singular limit problem under certain physically motivated restrictions imposed on constitutive equations. They are basically the same as required by the existence theory developed in [5, Chapter 3]. Although they might be slightly relaxed if only the convergence towards the target system is studied, we list them in the form presented in [5, Chapter 3], where the interested reader may find more information concerning the physical background as well as possible generalizations.

The pressure $p = p(\varrho, \vartheta)$ is given by the formula

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0; \tag{2.7}$$

the specific internal energy $e = e(\varrho, \vartheta)$ and the specific entropy $s = s(\varrho, \vartheta)$ read

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\vartheta^4 \tag{2.8}$$

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.9}$$

where

$$P \in C^1[0, \infty) \cap C^3(0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.10}$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0, \tag{2.11}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0, \tag{2.12}$$

and

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}, \ \lim_{Z \to \infty} S(Z) = 0. \tag{2.13}$$

The relation (2.12) expresses positivity and uniform boundedness of the specific heat at constant volume.

The transport coefficients μ, η, and κ are effective functions of the temperature,

$$\mu, \ \eta \in C^1[0, \infty) \text{ are globally Lipschitz in, } [0, \infty), \ 0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta), \ \eta(\vartheta) \geq 0, \text{ for all } \vartheta \geq 0, \tag{2.14}$$

$$\kappa \in C^1[0, \infty), \ 0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3) \text{ for all } \vartheta \geq 0. \tag{2.15}$$

2.2 Target system

As noted in the introduction, the expected limit is the Euler-Boussinesq system (1.11- 1.13) endowed with the initial data

$$\theta_0(0, \cdot) = \theta_0, \ \mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{2.16}$$

In agreement with the nowadays standard theory of well-posedness for hyperbolic systems, see e.g. Kato [12], we suppose that the system (1.11- 1.13), endowed with the initial data

$$(\theta_0, \mathbf{v}_0) \in W^{k,2}(\Omega; R^4), \ \|(\theta_0, \mathbf{v}_0)\|_{W^{k,2}(\Omega; R^4)} \leq D, \ \mathrm{div}_x \mathbf{v}_0 = 0, \ \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \ k > \frac{5}{2}, \tag{2.17}$$

possesses a regular solution $(\theta, \mathbf{v})$,

$$(\theta, \mathbf{v}) \in C([0, T_{\max}); W^{k,2}(\Omega; R^4)), \ (\partial_t \mathbf{v}, \ \nabla_x \Pi) \in C([0, T_{\max}); W^{k-1,2}(\Omega; R^6)), \tag{2.18}$$

defined on a maximal time interval $[0, T_{\max})$, $T_{\max} = T_{\max}(D)$.

2.3 Equilibrium state

We finish this preliminary part by recalling the basic properties of the equilibrium solution $(\overline{\varrho}_\varepsilon, \overline{\vartheta})$. Since the potential F is given by (1.8), it is easy to check that

$$\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon F + \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}); \tag{2.19}$$

whence, under the assumptions (2.7), (2.10–2.12),

$$\overline{\varrho}_\varepsilon \in C^3(\Omega),\ \left| \frac{\overline{\varrho}_\varepsilon(x) - \overline{\varrho}}{\varepsilon} \right| \leq cF(x) \quad |\nabla_x \varrho_\varepsilon(x)| \leq \varepsilon |\nabla_x F(x)|,\ x \in \Omega. \tag{2.20}$$

The reader can consult [9] for details.

3 Main result

For a vector field $\mathbf{U} \in L^2(\Omega; R^3)$, we denote by $\mathbf{H}[\mathbf{U}]$ the standard *Helmholtz projection* on the space of solenoidal functions.

We are ready to state the main result of this paper.

Theorem 3.1 *Let the thermodynamic functions p, e, s, and the transport coefficients μ, η, κ satisfy the hypotheses (2.7 - 2.13), (2.14), (2.15). Let the potential force F be given by (1.8). Let the exponents a, b, determining the Reynold and Péclet number scales, satisfy*

$$b > 0,\ 0 < a < \frac{10}{3}. \tag{3.1}$$

Next, let the initial data (1.9) be chosen in such a way that

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(\Omega),\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega), \tag{3.2}$$

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(\Omega; R^3),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3), \tag{3.3}$$

where

$$\varrho^{(1)}_0,\ \vartheta^{(1)}_0 \in W^{1,2} \cap W^{1,\infty}(\Omega),\ \mathbf{H}[\mathbf{u}_0] = \mathbf{v}_0 \in W^{k,2}(\Omega; R^3) \text{ for a certain } k > \frac{5}{2}. \tag{3.4}$$

Suppose that the Euler-Boussinesq system (1.11–1.13), endowed with the initial data

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],\ \theta_0 = \frac{\overline{\vartheta}}{c_p(\overline{\varrho}, \overline{\vartheta})} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \varrho^{(1)}_0 + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta^{(1)}_0 \right), \tag{3.5}$$

admits a regular solution $[\mathbf{v}, \theta]$ in the class (2.18) defined on a maximal time interval $[0, T_{\max})$.

Finally, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ be a dissipative weak solution of the Navier-Stokes-Fourier system (1.1 - 1.7) in $(0,T) \times R^3$, $T < T_{\max}$.

Then

$$\operatorname*{ess\ sup}_{t \in (0,T)} \|\varrho_\varepsilon(t, \cdot) - \overline{\varrho}\|_{L^{5/3}_{\rm loc}(\overline{\Omega})} \leq \varepsilon c, \tag{3.6}$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\ \mathbf{v} \text{ in } L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\overline{\Omega}; R^3)) \text{ and weakly-(*) in } L^\infty(0,T; L^2(\Omega; R^3)), \tag{3.7}$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \theta \text{ in } L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\overline{\Omega})),\ \text{and weakly-(*) in } L^\infty(0,T; L^2(\Omega)). \tag{3.8}$$

Remark 3.1 *Under the hypotheses (2.7 - 2.15), the existence of dissipative weak solutions to the Navier-Stokes-Fourier system in* $(0,T) \times \Omega$ *was shown in [11].*

The rest of the paper is devoted to the proof of Theorem 3.1.

4 Uniform bounds

In this section, we derive uniform bounds on the family of solutions $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon]$ *independent* of the scaling parameter $\varepsilon \to 0$.

4.1 Energy bounds

Taking $r = \overline{\varrho}_\varepsilon$, $\Theta = \overline{\vartheta}$, $\mathbf{U} = 0$ as test functions in the relative entropy inequality (2.6) we obtain

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} (\varrho_\varepsilon - \overline{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\overline{\varrho}_\varepsilon, \overline{\vartheta}) \right) \right] \mathrm{d}x \tag{4.1}$$
$$+ \overline{\vartheta} \int_0^\tau \int_\Omega \frac{1}{\vartheta_\varepsilon} \left(\varepsilon^a \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon - \varepsilon^{b-2} \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon}{\vartheta} \right) \mathrm{d}x \ \mathrm{d}t$$
$$\leq \int_\Omega \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \frac{\partial H_{\overline{\vartheta}}(\overline{\varrho}_\varepsilon, \overline{\vartheta})}{\partial \varrho} (\varrho_{0,\varepsilon} - \overline{\varrho}_\varepsilon) - H_{\overline{\vartheta}}(\overline{\varrho}_\varepsilon, \overline{\vartheta}) \right) \right] \mathrm{d}x$$

for a.a. $\tau \in [0,T]$. Note that such a choice of test functions can be justified by means of a density argument. Thanks to the hypotheses (3.2), (3.3), the integral on the right-hand side of (4.1) remains bounded uniformly for $\varepsilon \to 0$.

In accordance with the structural properties of the thermodynamic functions imposed through (2.7 - 2.13), the ballistic free energy enjoys the following properties: For any compact $K \subset (0,\infty)^2$ and $(r,\Theta) \in K$, there exists a strictly positive constant $c(K)$, depending only on K and the structural properties of P, such that

$$\left(H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r,\Theta) \right) \geq c(K) \left(|\varrho - r|^2 + |\vartheta - \Theta|^2 \right) \ \text{if} \ (\varrho, \vartheta) \in K, \tag{4.2}$$

$$\left(H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r,\Theta) \right) \geq c(K) \Big(1 + \varrho^\gamma + \vartheta^4 \Big) \ \text{if} \ (\varrho, \vartheta) \in (0,\infty)^2 \setminus K. \tag{4.3}$$

Similarly to [5, Chapter 4.7], we introduce a decomposition of a function h:

$$h = [h]_{\mathrm{ess}} + [h]_{\mathrm{res}} \ \text{for a measurable function} \ h,$$

where

$$[h]_{\mathrm{ess}} = h \ 1_{\{\overline{\varrho}/2 < \varrho_\varepsilon < 2\overline{\varrho}; \ \overline{\vartheta}/2 < \vartheta_\varepsilon < 2\overline{\vartheta}\}}, \ [h]_{\mathrm{res}} = h - h_{\mathrm{ess}}.$$

Combining (4.1), (4.2), (4.3), (2.20) with the hypotheses (2.7 - 2.15) we deduce the following estimates:

$$\operatorname*{ess\ sup}_{t \in (0,T)} \| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon(t,\cdot) \|_{L^2(\Omega;R^3)} \leq c, \tag{4.4}$$

$$\operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon}(t,\cdot) \right]_{\mathrm{ess}} \right\|_{L^2(\Omega;R^3)} + \operatorname*{ess\ sup}_{t \in (0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}(t,\cdot) \right]_{\mathrm{ess}} \right\|_{L^2(\Omega;R^3)} \leq c, \tag{4.5}$$

$$\text{ess}\sup_{t\in(0,T)}\int_\Omega\left(\left[\varrho_\varepsilon^{5/3}(t,\cdot)\right]_{\rm res}^{5/3}+[\vartheta_\varepsilon(t,\cdot)]^4_{\rm res}+1_{\rm res}(t,\cdot)\right)\ {\rm d}x\le\varepsilon^2 c, \tag{4.6}$$

and

$$\left\|\varepsilon^{a/2}\mathbf{u}_\varepsilon\right\|_{L^2(0,T;W^{1,2}(\Omega;R^3))}\le c, \tag{4.7}$$

$$\left\|\varepsilon^{(b-2)/2}\left(\vartheta_\varepsilon-\overline{\vartheta}\right)\right\|_{L^2(0,T;W^{1,2}(\Omega;R^3))}+\left\|\varepsilon^{(b-2)/2}\left(\log(\vartheta_\varepsilon)-\log(\overline{\vartheta})\right)\right\|_{L^2(0,T;W^{1,2}(\Omega;R^3))}\le c, \tag{4.8}$$

where the symbol c stands for a generic constant independent of ε. We remark that (4.7) follows from the generalized Korn's inequality $(\int_\Omega\varrho_\varepsilon\mathbf{w}^2{\rm d}x)^{1/2}+\|\nabla_x\mathbf{w}+\nabla_x^t\mathbf{w}-\frac{2}{3}{\rm div}_x\mathbf{w}\mathbb{I}\|_{L^2}\ge c\|\nabla_x\mathbf{w}\|_{L^2}$ for $\mathbf{w}\in W^{1,2}$, combined with the estimates (4.4), (4.6). Similar arguments based on the Sobolev inequality and (4.5), (4.6) yield (4.8).

4.2 Convergence

To begin, we denote

$$\alpha=\frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho},\ \beta=\frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},\ \delta=\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},\ a(\overline{\varrho},\overline{\vartheta})=\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}. \tag{4.9}$$

It follows from (4.5–4.6) and the structural assumptions on the pressure (2.7), (2.10–2.12) that

$$\left[\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\right]_{\rm res}\to 0\text{ in }L^\infty(0,T;L^{5/3}(\Omega)),\ \left[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\right]_{\rm res}\to 0\text{ in }L^\infty(0,T;L^4(\Omega)). \tag{4.10}$$

Next, writing

$$\frac{1}{\varepsilon}\nabla_x p(\varrho_\varepsilon,\vartheta_\varepsilon)-\varrho_\varepsilon\nabla_x F=\frac{1}{\varepsilon}\nabla_x p(\varrho_\varepsilon,\vartheta_\varepsilon)-\overline{\varrho}_\varepsilon\nabla_x F+\varepsilon\frac{\overline{\varrho}_\varepsilon-\varrho_\varepsilon}{\varepsilon}\nabla_x F=\frac{1}{\varepsilon}\nabla_x\left(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\right)+\varepsilon\frac{\overline{\varrho}_\varepsilon-\varrho_\varepsilon}{\varepsilon}\nabla_x F,$$

we deduce from the momentum balance (2.4) that

$$\alpha\left[\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\right]_{\rm ess}+\beta\left[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\right]_{\rm ess}\to 0\text{ weakly-(*) in }L^\infty(0,T;L^2(\Omega)). \tag{4.11}$$

Finally, we use (4.4–4.6) to show that

$$\varrho_\varepsilon\mathbf{u}_\varepsilon\to\overline{\varrho}\overline{\mathbf{u}}\text{ weakly-(*) in }L^\infty(0,T;L^2+L^{5/4}(\Omega;R^3)), \tag{4.12}$$

where, passing to the limit in the continuity equation (2.3), we may infer that

$${\rm div}_x(\overline{\varrho}\overline{\mathbf{u}})=0. \tag{4.13}$$

5 Acoustic and thermal energy transport equations

Similarly to [7], our aim is to use the relative entropy inequality (2.6) to deduce the convergence to the target system. To this end, we take

$$\varrho=\varrho_\varepsilon,\ \vartheta=\vartheta_\varepsilon,\ \mathbf{u}=\mathbf{u}_\varepsilon$$

and choose the test functions $\{r,\Theta,\mathbf{U}\}$ in the following way:

$$r=r_\varepsilon=\overline{\varrho}_\varepsilon+\varepsilon R_\varepsilon,\ \Theta=\Theta_\varepsilon=\overline{\vartheta}+\varepsilon T_\varepsilon,\ \mathbf{U}=\mathbf{U}_\varepsilon=\mathbf{v}+\nabla_x\Phi_\varepsilon; \tag{5.1}$$

where $\mathbf{v}$ is the velocity component of the solution to the incompressible Euler-Boussinesq system (1.11)-(1.13), with the initial condition (3.5), and the functions R_ε, T_ε, and Φ_ε satisfy the *acoustic equation*:

$$\varepsilon\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon) + \omega\Delta\Phi_\varepsilon = 0, \tag{5.2}$$

$$\varepsilon\partial_t\nabla_x\Phi_\varepsilon + \nabla_x(\alpha R_\varepsilon + \beta T_\varepsilon) = 0,\ \nabla_x\Phi_\varepsilon\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{5.3}$$

with the initial values determined by

$$R_\varepsilon(0,\cdot) = R_0,\ T_\varepsilon(0,\cdot) = T_0,\ \Phi_\varepsilon(0,\cdot) = \Phi_0, \tag{5.4}$$

and the constants α, β defined in (4.9),

$$\omega = \overline{\varrho}\left(\alpha + \frac{\beta^2}{\delta}\right).$$

The first equation in (5.2) is nothing other than a linearization of the continuity equation, while the second equation is a linearization of the momentum equation projected onto the space of gradients.

In order to determine R_ε and T_ε in a unique way, we require $\delta R_\varepsilon - \beta T_\varepsilon$, with δ defined in (4.9), to satisfy the *transport equation*

$$\partial_t(\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon\cdot\nabla_x\left(\delta T_\varepsilon - \beta R_\varepsilon - \frac{\beta}{\alpha}F\right) = 0, \tag{5.5}$$

where the initial data are determined by (5.4). Equation (5.5) is obviously related to the limit equation (1.13). Observe that the system of linear equations (5.2–5.5) is well-posed.

5.1 Initial data

In view of the future application of the relative entropy inequality (2.6), the initial data for the test functions must be taken is such a way that

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],\ \Phi_\varepsilon(0,\cdot) = \Phi_{0,\eta},\ \nabla_x\Phi_{0,\eta} \to \mathbf{H}^\perp[\mathbf{u}_0] \text{ in } L^2(\Omega;R^3) \text{ as } \eta\to 0, \tag{5.6}$$

$$R_\varepsilon(0,\cdot) = R_{0,\eta},\ \|R_{0,\eta}\|_{L^\infty(\Omega)} < c(\eta),\ R_{0,\eta}\to\varrho_0^{(1)} \text{ in } L^2(\Omega) \text{ as } \eta\to 0, \tag{5.7}$$

and

$$T_\varepsilon(0,\cdot) = T_{0,\eta},\ \|T_{0,\eta}\|_{L^\infty(\Omega)} < c(\eta),\ T_{0,\eta}\to\vartheta_0^{(1)} \text{ in } L^2(\Omega) \text{ as } \eta\to 0. \tag{5.8}$$

Note that (5.6 - 5.8) imply that

$$\mathcal{E}_\varepsilon\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon},\mathbf{u}_{0,\varepsilon}\Big| r_\varepsilon(0,\cdot),\Theta_\varepsilon(0,\cdot),\mathbf{U}(0,\cdot)\right) \to \chi(\eta) \text{ as } \varepsilon\to 0, \tag{5.9}$$

where

$$\chi(\eta)\to 0 \text{ as } \eta\to 0.$$

Our next goal is to choose suitable approximations for the initial data. Following [8], we consider the Neumann Laplacean Δ_N,

$$\mathcal{D}(\Delta_N) = \Big\{ v\in L^2(\Omega)\ \Big|\ \nabla_x v\in L^2(\Omega;R^3),\ \int_\Omega \nabla_x v\cdot\nabla_x\varphi\ \mathrm{d}x = \int_\Omega g\varphi\ \mathrm{d}x$$
$$\text{for any } \varphi\in C_c^\infty(\overline{\Omega}) \text{ and a certain } g\in L^2(\Omega)\Big\},$$

together with a family of regularizing operators

$$[v]_\eta = G_\eta(\sqrt{-\Delta_N})[\psi_{1/\eta} v], \tag{5.10}$$

with the cut-off functions

$$\psi_\eta(x) = \psi(x/\eta);\ \psi \in C_c^\infty(R),\ 0 \le \psi \le 1,\ \psi(x) = \left\{ \begin{array}{c} 1 \text{ si } |x| \le 1, \\ 0 \text{ si } |x| \ge 2 \end{array} \right\}, \tag{5.11}$$

$$G_\eta \in C_c^\infty(R),\ 0 \le G_\eta \le 1,\ G_\eta(-z) = G_\eta(z),$$

$$G_\eta(z) = 1 \text{ for } z \in \left(-\frac{1}{\eta}, -\eta\right) \cup \left(\eta, \frac{1}{\eta}\right),\ G_\eta(z) = 0 \text{ for } z \in \left(-\infty, -\frac{2}{\eta}\right) \cup \left(-\frac{\eta}{2}, \frac{\eta}{2}\right) \cup \left(\frac{2}{\eta}, \infty\right),$$

where the linear operator $G_\eta(\sqrt{-\Delta_N})$ is defined by means of the standard spectral theory associated to Δ_N.

Accordingly, we consider regularized initial data in the form

$$R_{0,\eta} = [\varrho_0^{(1)}]_\eta,\ T_{0,\eta} = [\vartheta_0^{(1)}]_\eta, \tag{5.12}$$

and

$$\Phi_{0,\eta} = \left[\Delta_N^{-1} \mathrm{div}_x[\mathbf{u}_0]\right]_\eta, \text{ with } \nabla_x \Delta_N^{-1} \mathrm{div}_x[\mathbf{u}_0] \equiv \mathbf{H}^\perp[\mathbf{u}_0]. \tag{5.13}$$

To avoid excessive notation, we omit writing the parameter η in the course of the limit passage $\varepsilon \to 0$.

5.2 Dispersive estimates for the wave equation

The acoustic equation (5.2 - 5.4) has been studied in detail in [8]. In particular, we report the following estimates ([8, estimates (6.6), (6.8)]:

$$\sup_{t \in [0,T]} \left(\|\nabla_x \Phi_{\varepsilon,\eta}\|_{W^{k,2} \cap W^{k,\infty}(\Omega:R^3)} + \|(\alpha R_{\varepsilon,\eta} + \beta T_{\varepsilon,\eta})(t,\cdot)\|_{W^{k,2} \cap W^{k,\infty}(\Omega:R^3)} \right) \tag{5.14}$$

$$\le c(k,\eta) \left(\|\nabla_x \Phi_{0,\eta}\|_{L^2(\Omega;R^3)} + \|\alpha R_{0,\eta} + \beta T_{0,\eta}\|_{L^2(\Omega)} \right),$$

for any $k = 0, 1, \ldots$, $\eta > 0$; and the dispersive estimates

$$\int_0^T \left(\|\nabla_x \Phi_{\varepsilon,\eta}\|_{W^{k,\infty}(\Omega:R^3)} + \|(\alpha R_{\varepsilon,\eta} + \beta T_{\varepsilon,\eta})(t,\cdot)\|_{W^{k,\infty}(\Omega:R^3)} \right) \mathrm{d}t \tag{5.15}$$

$$\le \omega(\varepsilon,\eta,k) \left(\|\nabla_x \Phi_{0,\eta}\|_{L^2(\Omega;R^3)} + \|\alpha R_{0,\eta} + \beta T_{0,\eta}\|_{L^2(\Omega)} \right)$$

where

$$\omega(\varepsilon,\eta,k) \to 0 \text{ as } \varepsilon \to 0 \text{ for any fixed } \eta > 0,\ k \ge 0.$$

The relation (5.15) represents *dispersive* estimates for the wave equation (5.2), (5.3). Note that both (5.14) and (5.15) apply to the regularized initial data, meaning for a fixed $\eta > 0$; they in fact blow up when $\eta \to 0$.

Moreover, as shown in [4, Section 5.3],

$$|x|^s |\partial_x^k [h]_\eta(x)| \le c(s,k,\eta) \|h\|_{L^2(\Omega)} \text{ for all } x \in \Omega,\ s \ge 0, k \ge 0, \tag{5.16}$$

therefore the functions $\Phi_{\varepsilon,\eta}$, $(\alpha R_{\varepsilon,\eta} + \beta T_{\varepsilon,\eta})$ decay fast for $|x| \to \infty$ as long as $\eta > 0$ is fixed.

Remark 5.1 *As a matter of fact, the results of [8] are stated for the domain Ω - a perturbed half-space. However, as pointed out in [8], the same holds for a larger class of domains on which Δ_N, among which the exterior domains in R^3. Alternatively, we may also use the dispersive estimates established by Isozaki [10].*

5.3 L^p estimates for the transport equation

For fixed $\eta > 0$, the initial data for the transport equation (5.5) enjoy the decay properties (5.16). Consequently, in view of (5.14), (5.15), the solutions of the transport equation (5.5) admit the estimates

$$\sup_{t\in[0,T]} \|\delta T_{\varepsilon,\eta} - \beta R_{\varepsilon,\eta}\|_{W^{k,q}(\Omega)} \leq c(\eta, k, F) \left(1 + \|\delta T_{0,\eta} - \beta R_{0,\eta}\|_{L^2(\Omega)}\right), \ k = 0, 1, \ 1 \leq q \leq \infty, \tag{5.17}$$

and the family

$$\{\delta T_{\varepsilon,\eta} - \beta R_{\varepsilon,\eta}\}_{\varepsilon>0} \text{ is precompact in } C([0,T]; W^{k,q}(\Omega)), \ k = 0, 1, \ 1 \leq q \leq \infty. \tag{5.18}$$

Consequently, combining (5.15), (5.17), (5.18) we can let $\varepsilon \to 0$ to obtain

$$T_{\varepsilon,\eta} \to \ T_\eta \ \text{ strongly in } L^\infty_{\rm loc}((0,T]; W^{k,p}(\Omega)), \ \ p > 2, \text{ and weakly}-(*) \text{ in } L^\infty(0,T; W^{k,2}(\Omega)), \ \ k = 0, 1, \text{ as } \varepsilon \to 0, \tag{5.19}$$

$$R_{\varepsilon,\eta} \to \ R_\eta \ \text{ strongly in } L^\infty_{\rm loc}((0,T]; W^{k,p}(\Omega)), \ \ p > 2, \text{ and weakly}-(*) \text{ in } L^\infty(0,T; W^{k,2}(\Omega)), \ k = 0, 1, \text{ as } \varepsilon \to 0, \tag{5.20}$$

where T_η satisfies

$$c_p(\overline{\varrho}, \overline{\vartheta}) \left(\partial_t T_\eta + \mathbf{v} \cdot \nabla_x T_\eta\right) - \overline{\vartheta} a(\overline{\varrho}, \overline{\vartheta}) \mathbf{v} \cdot \nabla_x F = 0, \tag{5.21}$$

with the initial data

$$T_\eta(0, \cdot) = \frac{\overline{\vartheta}}{c_p(\overline{\varrho}, \overline{\vartheta})} \left(\frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} [\varrho_0^{(1)}]_\eta + \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} [\vartheta_0^{(1)}]_\eta \right). \tag{5.22}$$

6 Convergence

In this section, we use the test functions (5.1) in the relative entropy inequality (2.6). Fixing $\eta > 0$ we perform the limit for $\varepsilon \to 0$. This will be carried over in several steps in the spirit of [7]. We omit the subscript η whenever no confusion arises.

6.1 Viscous and heat conducting terms

We show by direct calculation, splitting the terms in their essential and residual parts and using assumptions (2.14–2.15), uniform bounds (4.6–4.8), regularity (2.18), and estimates (5.14–5.18) that the dissipative terms related to the viscosity and to the heat conductivity on the right-hand side of (2.6) become negligible as $\varepsilon \to 0$. More precisely:

$$\varepsilon^a \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{U}_\varepsilon \to 0 \text{ in } L^2((0,T) \times \Omega) + L^2(0,T; L^{4/3}(\Omega; R^3)) \text{ as } \varepsilon \to 0,$$

and

$$\varepsilon^{b-2} \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \Theta_\varepsilon}{\vartheta_\varepsilon} \to 0 \text{ in } L^2((0,T) \times \Omega) + L^1((0,T) \times \Omega) \text{ as } \varepsilon \to 0.$$

Consequently, combining the previous observation with (5.9), we can write the relative entropy inequality (2.6) as

$$\mathcal{E}_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \middle| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right) (\tau) \tag{6.1}$$

$$\leq \chi(\varepsilon, \eta) + \int_0^\tau \int_\Omega \varrho_\varepsilon \Big(\partial_t \mathbf{U}_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{U}_\varepsilon \Big) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon) \ {\rm d}x \ {\rm d}t$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega \Big(\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{u}_\varepsilon\cdot\nabla_x T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t$$
$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big[\Big(p(r_\varepsilon,\Theta_\varepsilon)-p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}\mathbf{U}_\varepsilon+\frac{\varrho_\varepsilon}{r_\varepsilon}(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\nabla_x p(r_\varepsilon,\Theta_\varepsilon)\Big]\ \mathrm{d}x\mathrm{d}t$$
$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\frac{r_\varepsilon-\varrho_\varepsilon}{r_\varepsilon}\Big(\partial_t p(r_\varepsilon,\Theta_\varepsilon)+\mathbf{U}_\varepsilon\cdot\nabla_x p(r_\varepsilon,\Theta_\varepsilon)\Big)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon}\int_0^\tau\int_{R^3}\varrho_\varepsilon\nabla_x F\cdot(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\mathrm{d}x\mathrm{d}t,$$

where χ denotes a generic function satisfying

$$\lim_{\eta\to 0}\Big(\lim_{\varepsilon\to 0}\chi(\varepsilon,\eta)\Big)=0. \tag{6.2}$$

6.2 Velocity dependent terms

Our next goal is to handle the expression

$$\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\partial_t\mathbf{U}_\varepsilon+\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\otimes\mathbf{u}_\varepsilon:\nabla_x\mathbf{U}_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t=$$
$$\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\otimes(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon):\nabla_x\mathbf{U}_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$
$$+\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\Big(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v}\Big)\ \mathrm{d}x\ \mathrm{d}t+\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\partial_t\nabla_x\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$
$$+\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\otimes\nabla_x\Phi_\varepsilon:\nabla_x\mathbf{v}\ \mathrm{d}x+\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\otimes\mathbf{v}:\nabla_x^2\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$
$$+\frac{1}{2}\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\nabla_x|\nabla_x\Phi_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t.$$

Thanks to (2.18), (5.14), (5.15) and the energy bounds established in (4.4 - 4.8), the first integral on the right hand side can be dominated by the expression

$$\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}\Big(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\Big)\mathrm{d}t,$$

with c independent of ε, η.

The second term reads

$$\int_0^\tau\int_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Pi\ \mathrm{d}t-\int_0^\tau\int_\Omega\varrho_\varepsilon(\mathbf{v}+\nabla_x\Phi_\varepsilon)\cdot\nabla_x\Pi\ \mathrm{d}t$$
$$+\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x F\ \mathrm{d}t-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon(\mathbf{v}+\nabla_x\Phi_\varepsilon)\cdot\nabla_x F\ \mathrm{d}t$$
$$=\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{v}\cdot\nabla_x F\ \mathrm{d}t+\chi(\varepsilon,\eta)$$

where we have used the equations (1.11–1.12), formulas (4.12–4.13), the dispersive estimates (5.15), and relation (2.18).

Next, using the equation (5.3), we may write the third integral in the form

$$-\int_0^\tau\int_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla_x\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t-\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\mathbf{v}\cdot\nabla_x\left(\alpha R_\varepsilon+\beta T_\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\nabla_x\Phi_\varepsilon\cdot\nabla_x(\alpha R_\varepsilon+\beta T_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t+\frac{1}{2}\int_0^\tau\int_\Omega\overline{\varrho}\partial_t|\nabla_x\Phi_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$=-\int_0^\tau\int_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla_x\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t+\frac{1}{2}\int_0^\tau\int_\Omega\overline{\varrho}\partial_t|\nabla_x\Phi_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t+\chi(\varepsilon,\eta).$$

where we have used wave equation (5.2–5.3), estimates (4.4–4.6), (5.11), regularity of $\mathbf{v}$ stated (2.18), the relation (2.20), and dispersive estimates (5.15).

Finally, in view of the uniform bounds (2.18), (4.4 - 4.6), and the dispersive estimates stated in (5.15), the last three integrals tend to zero for $\varepsilon\to 0$, uniformly with respect to τ.

Resuming, we obtain

$$\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\partial_t\mathbf{U}_\varepsilon+\varrho_\varepsilon(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\otimes\mathbf{u}_\varepsilon:\nabla_x\mathbf{U}_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t$$
$$\le\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}\Big(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\Big)\mathrm{d}t+\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_xF\ \mathrm{d}x\ \mathrm{d}t$$
$$-\int_0^\tau\int_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla_x\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t+\frac{1}{2}\int_0^\tau\int_\Omega\overline{\varrho}\partial_t|\nabla_x\Phi_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{v}\cdot\nabla_xF\ \mathrm{d}t;$$

whence relation (6.1) becomes

$$\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)(\tau)\le\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)\ \mathrm{d}t \tag{6.3}$$
$$+\left[\int_\Omega\overline{\varrho}\frac{1}{2}|\nabla_x\Phi_\varepsilon|^2\ \mathrm{d}x\right]_{t=0}^{t=\tau}-\int_0^\tau\int_\Omega\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla_x\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$
$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_tT_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{u}_\varepsilon\cdot\nabla_xT_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t$$
$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\left((r_\varepsilon-\varrho_\varepsilon)\frac{1}{r_\varepsilon}\partial_tp(r_\varepsilon,\Theta_\varepsilon)-\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_xp(r_\varepsilon,\Theta_\varepsilon)\right)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$
$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\varrho_\varepsilon\nabla_xF\cdot(\mathbf{v}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t+\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_xF\mathrm{d}x\ \mathrm{d}t-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{v}\cdot\nabla_xF\ \mathrm{d}x\ \mathrm{d}t.$$

In the above, we have used the identity

$$\int_\Omega\left[\Big(p(r_\varepsilon,\Theta_\varepsilon)-p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x\mathbf{U}_\varepsilon+\left(1-\frac{\varrho_\varepsilon}{r_\varepsilon}\right)\mathbf{U}_\varepsilon\cdot\nabla_xp(r_\varepsilon,\Theta_\varepsilon)+\frac{\varrho_\varepsilon}{r_\varepsilon}(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\cdot\nabla_xp(r_\varepsilon,\Theta_\varepsilon)\right]\ \mathrm{d}x$$
$$=-\int_\Omega p(\varrho_\varepsilon,\vartheta_\varepsilon)\Delta\Phi_\varepsilon\ \mathrm{d}x-\int_\Omega\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_xp(r_\varepsilon,\Theta_\varepsilon)\ \mathrm{d}x,$$

together with

$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\varrho_\varepsilon\nabla_xF\cdot(\mathbf{U}_\varepsilon-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t=-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\varrho_\varepsilon\nabla_xF\cdot(\mathbf{v}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\varrho_\varepsilon\nabla_xF\cdot\nabla_x\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$
$$=\chi(\varepsilon,\eta)-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\varrho_\varepsilon\nabla_xF\cdot(\mathbf{v}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t+\frac{1}{\varepsilon^2}p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t.$$

Recall that $\nabla_x\Phi_\varepsilon(t,\cdot)$ decays fast as $|x|\to\infty$ and $\mathrm{div}_x\mathbf{v}=0$, which justifies the by-parts integration.

6.3 Pressure dependent terms

We write

$$\frac{1}{\varepsilon^2}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x p(r_\varepsilon,\Theta_\varepsilon)=\frac{1}{\varepsilon^2}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x\Big(p(r_\varepsilon,\Theta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)+\frac{1}{\varepsilon^2}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x p(\overline{\varrho}_\varepsilon,\overline{\vartheta})$$

$$=\frac{1}{\varepsilon^2}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x\left(p(r_\varepsilon,\Theta_\varepsilon)-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho}\varepsilon R_\varepsilon-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\vartheta}\varepsilon T_\varepsilon-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\right)$$

$$+\frac{1}{\varepsilon}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x\left(\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho}R_\varepsilon+\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\vartheta}T_\varepsilon\right)+\frac{1}{\varepsilon}\frac{\overline{\varrho}_\varepsilon}{r_\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x F.$$

Next, we use the decay properties of the equilibrium density profile $\overline{\varrho}_\varepsilon$ stated in (2.20), together with (5.19), (5.20) to observe that

$$\frac{1}{\varepsilon^2 r_\varepsilon}\nabla_x\left(p(r_\varepsilon,\Theta_\varepsilon)-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho}\varepsilon R_\varepsilon-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\vartheta}\varepsilon T_\varepsilon-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\right)\to\nabla_x H \text{ in } L^p(0,T;(L^2\cap L^q)(\Omega;R^3)),\ p\ge 1,\ q>2,$$

where the right-hand side is a gradient of a certain function H. Consequently, using (4.12), (4.13) we may infer that

$$\int_0^\tau\int_\Omega\frac{1}{\varepsilon^2}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x\left(p(r_\varepsilon,\Theta_\varepsilon)-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho}\varepsilon R_\varepsilon-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\vartheta}\varepsilon T_\varepsilon-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\right)\ \mathrm{d}x\ \mathrm{d}t=\chi(\varepsilon,\eta).$$

Moreover, by the same token, we obtain

$$\int_0^\tau\int_\Omega\frac{1}{\varepsilon}\frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x\left(\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho}R_\varepsilon+\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\vartheta}T_\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t=\eta(\varepsilon,\delta)+\int_0^\tau\int_\Omega\frac{1}{\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\left(\alpha R_\varepsilon+\beta T_\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t.$$

Making use of the identity

$$\int_0^\tau\int_\Omega\frac{1}{\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\left(\alpha R_\varepsilon+\beta T_\varepsilon\right)\ \mathrm{d}x\ \mathrm{d}t=-\int_0^\tau\int_{R^3}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla_x\Phi_\varepsilon\mathrm{d}x\mathrm{d}t$$

we may rewrite (6.3) in the form

$$\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)(\tau)\le\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)\ \mathrm{d}t+\left[\int_\Omega\overline{\varrho}\frac{1}{2}|\nabla_x\Phi_\varepsilon|^2\ \mathrm{d}x\right]_{t=0}^{t=\tau}\tag{6.4}$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{u}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega(r_\varepsilon-\varrho_\varepsilon)\frac{1}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\frac{R_\varepsilon}{r_\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t-\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\mathbf{v}\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t$$

$$+\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x F\mathrm{d}x\ \mathrm{d}t-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{v}\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t.$$

Finally, we use the fact that

$$\alpha R_\eta+\beta T_\eta=0,\tag{6.5}$$

and that T_η and θ satisfy the same equation (see (5.21) and (1.13)) with the initial data given by (5.22), (3.5), respectively, to deduce that

$$\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)(\tau)\leq\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)\,\mathrm{d}t+\left[\int_\Omega\overline{\varrho}\frac{1}{2}|\nabla_x\Phi_\varepsilon|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} \tag{6.6}$$

$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{u}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]\,\mathrm{d}x\,\mathrm{d}t$$

$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega(r_\varepsilon-\varrho_\varepsilon)\frac{1}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\mathbf{v}\cdot\nabla_x F\,\mathrm{d}x\,\mathrm{d}t-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{v}\cdot\nabla_x F\,\mathrm{d}x\,\mathrm{d}t.$$

6.4 Replacing velocity in the entropy convective term

Our intention in this section is to "replace" $\mathbf{u}_\varepsilon$ by $\mathbf{U}_\varepsilon$ in the remaining (last) convective term in (6.6). To this end, we write

$$\int_0^\tau\int_\Omega\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_x T_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$

$$=\int_0^\tau\int_\Omega\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega\varrho_\varepsilon\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon)\cdot\nabla_x T_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\left|\int_0^\tau\int_\Omega\varrho_\varepsilon\left[\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\right]_{\rm ess}(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon)\cdot\nabla_x T_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$\leq A(\eta)\int_0^\tau\int_\Omega\left(\varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon|^2+\left|\left[\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\right]_{\rm ess}\right|^2+\left|\left[\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}\right]_{\rm ess}\right|^2\right)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq c\int_0^\tau\mathcal{E}\Big(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\Big)\mathrm{d}t$$

and

$$+\int_0^\tau\int_\Omega\varrho_\varepsilon\left[\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\right]_{\rm res}(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon)\cdot\nabla_x T_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\chi(\varepsilon,\eta)\text{ provided }0<a<10/3.$$

When estimating the residual component, we have first deduced from (2.9 - 2.13) the inequality

$$\varrho|s(\varrho,\vartheta)|\leq c\left(\vartheta^3+\varrho|\log(\varrho)|+\varrho[\log(\vartheta)]^+\right) \tag{6.7}$$

and then employed the estimates (4.6–4.7) for ϱ_ε, ϑ_ε, together with the estimates (5.15–5.18) for R_ε, T_ε, $\nabla_x\Phi_\varepsilon$, and (2.18) for $\mathbf{v}$.

Consequently, we can can rewrite inequality (6.6) in the form

$$\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)(\tau)\leq\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)\,\mathrm{d}t+\left[\int_\Omega\overline{\varrho}\frac{1}{2}|\nabla_x\Phi_\varepsilon|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau} \tag{6.8}$$

$$
-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t
$$

$$
+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega(r_\varepsilon-\varrho_\varepsilon)\frac{1}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t
$$

$$
-\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\mathbf{v}\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\overline{\varrho}}\frac{\beta}{\alpha}\int_0^\tau\int_\Omega\theta\varrho_\varepsilon\mathbf{v}\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t.
$$

6.5 The entropy and the pressure

6.5.1 Handling the residual component

To begin, we observe that the residual components of all integrals on the second and third line of inequality (6.8) are negligible. To this end, we first use the estimates (5.15 - 5.18), (5.19), (5.20), together with the equations (5.2 - 5.5), to deduce

$$
\sup_{t\in[0,T]}\varepsilon\|\partial_t R_\varepsilon(t,\cdot)\|_{L^\infty(R^3)},\ \sup_{t\in[0,T]}\varepsilon\|\partial_t T_\varepsilon(t,\cdot)\|_{L^\infty(R^3)}\le A(\eta),\tag{6.9}
$$

$$
\varepsilon\|\partial_t R_\varepsilon(t,\cdot)\|_{L^\infty(R^3)}\to 0,\ \varepsilon\|\partial_t T_\varepsilon(t,\cdot)\|_{L^\infty(R^3)}\to 0 \text{ for any } t>0.\tag{6.10}
$$

Now, we employ these relations in combination with the uniform estimates (4.6); after a long but straightforward calculation, we finally get the desired result, namely

$$
-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]_{\mathrm{res}}\Big]\ \mathrm{d}x\ \mathrm{d}t\tag{6.11}
$$

$$
-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big[\frac{\varrho_\varepsilon-r_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\Big]_{\mathrm{res}}\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big[\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\Big]_{\mathrm{res}}\ \mathrm{d}x\ \mathrm{d}t=\chi(\varepsilon,\eta)
$$

6.5.2 Handling the essential component

In view of the preceding Section, we have to handle solely the essential part of the integrals at the first and second line of formula (6.8) whose integrands can be, roughly speaking, replaced by their linearization at $\overline{\varrho}_\varepsilon$, $\overline{\vartheta}$. Since we already know that the corresponding residual components are negligible, we may omit the symbol $[\cdot]_{\mathrm{ess}}$ in all integrands.

We check that

$$
-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t\tag{6.12}
$$

$$
-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-r_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t
$$

$$
=-\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}-\beta\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big)\Big(\partial_t T_\varepsilon+\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t
$$

$$
-\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\partial_t\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t+\int_0^\tau\int_\Omega\frac{\delta}{\beta^2+\alpha\delta}\Big(\alpha\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}+\beta\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\partial_t\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t
$$

$$
+\int_0^\tau\int_\Omega\frac{1}{\varepsilon}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\varrho}\right)\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t+\int_0^\tau\int_\Omega\frac{1}{\varepsilon}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}-\frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial\vartheta}\right)\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t+\chi(\varepsilon,\eta),
$$

where, in accordance with the dispersive estimates (5.14), (5.15) and (2.20),

$$\int_0^\tau \int_\Omega \frac{1}{\varepsilon}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \varrho} - \frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial \varrho}\right)\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon}\Delta\Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \frac{1}{\varepsilon}\left(\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta} - \frac{\partial p(\overline{\varrho}_\varepsilon,\overline{\vartheta})}{\partial \vartheta}\right)\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Delta\Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t = \chi(\varepsilon,\eta).$$

Consequently, we get

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^\tau \int_\Omega \Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon + \varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big] \ \mathrm{d}x \ \mathrm{d}t \\
&-\frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - r_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t - \frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \Big(p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho}_\varepsilon,\overline{\vartheta})\Big)\Delta\Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \\
&= \int_0^\tau \int_\Omega \Big(\delta T_\varepsilon - \beta R_\varepsilon\Big)\partial_t T_\varepsilon \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega R_\varepsilon \partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t \\
&-\Big[\int_0^\tau \int_\Omega \Big(\delta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \beta\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon}\Big)\partial_t T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \Big(\frac{\beta^2}{\beta^2+\alpha\delta}\frac{\varrho_\varepsilon - \overline{\varrho}_\varepsilon}{\varepsilon} - \frac{\beta\delta}{\beta^2+\alpha\delta}\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t\Big] \\
&- \int_0^\tau \int_\Omega \Big(\delta\frac{\vartheta_\varepsilon - \Theta_\varepsilon}{\varepsilon} - \beta\frac{\varrho_\varepsilon - r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \chi(\varepsilon,\eta).
\end{aligned} \tag{6.13}$$

In the next steps, we use the identities

$$(\beta^2+\alpha\delta)T = \beta(\alpha R + \beta T) + \alpha(\delta T - \beta R), \ (\beta^2+\alpha\delta)R = \delta(\alpha R + \beta T) - \beta(\delta T - \beta R), \tag{6.14}$$

to compute,

$$\begin{aligned}
&\int_0^\tau \int_\Omega \Big(\delta T_\varepsilon - \beta R_\varepsilon\Big)\partial_t T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega R_\varepsilon \partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t \\
&= \int_0^\tau \int_\Omega \Big[\frac{\beta}{\beta^2+\alpha\delta}\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big)\partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) + \frac{\alpha}{\beta^2+\alpha\delta}\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big)\partial_t\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big) \\
&+ \frac{\delta}{\beta^2+\alpha\delta}\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big)\partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) - \frac{\beta}{\beta^2+\alpha\delta}\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big)\partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big)\Big]\mathrm{d}x \ \mathrm{d}t \\
&= \frac{1}{2}\frac{\delta}{\beta^2+\alpha\delta}\left[\int_\Omega |\alpha R_\varepsilon + \beta T_\varepsilon|^2 \mathrm{d}x\right]_0^\tau + \frac{1}{2}\frac{\alpha}{\beta^2+\alpha\delta}\left[\int_\Omega |\delta T_\varepsilon - \beta R_\varepsilon|^2 \mathrm{d}x\right]_0^\tau,
\end{aligned} \tag{6.15}$$

where we have used (5.2).

Similarly, we get

$$\begin{aligned}
&-\int_0^\tau \int_\Omega \Big(\delta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \beta\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t - \int_0^\tau \int_\Omega \Big(\frac{\beta^2}{\beta^2+\alpha\delta}\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} - \frac{\beta\delta}{\beta^2+\alpha\delta}\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t \\
&= -\frac{\alpha}{\beta^2+\alpha\delta}\int_0^\tau \int_\Omega \Big(\delta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \beta\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t
\end{aligned} \tag{6.16}$$

Finally, the last line on the right-hand side of (6.13) reads

$$-\int_0^\tau \int_\Omega \Big(\delta\frac{\vartheta_\varepsilon - \Theta_\varepsilon}{\varepsilon} - \beta\frac{\varrho_\varepsilon - r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \tag{6.17}$$

$$
\begin{aligned}
&= -\frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}-\beta\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon\cdot\nabla_x\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad -\frac{\alpha}{\beta^2+\alpha\delta}\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}-\beta\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon\cdot\nabla_x\Big(\delta T_\varepsilon-\beta R_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&= -\frac{\alpha}{\beta^2+\alpha\delta}\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}-\beta\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon\cdot\nabla_x\Big(\delta T_\varepsilon-\beta R_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t+\chi(\varepsilon,\eta),
\end{aligned}
$$

where we have used the dispersive estimates (5.15).

Summing up the previous integrals and using equation (5.5) we may infer that

$$
\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t \qquad (6.18)\\
&-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-r_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho},\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t\\
&=\frac{1}{2}\frac{\delta}{\beta^2+\alpha\delta}\Big[\int_\Omega|\alpha R_\varepsilon+\beta T_\varepsilon|^2\ \mathrm{d}x\Big]_0^\tau+\frac{1}{2}\frac{\alpha}{\beta^2+\alpha\delta}\Big[\int_\Omega|\delta T_\varepsilon-\beta R_\varepsilon|^2\ \mathrm{d}x\Big]_0^\tau\\
&\quad-\frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}-\beta\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)\mathbf{v}\cdot\nabla_x F_\varepsilon\ \mathrm{d}x\mathrm{d}t+\chi(\varepsilon,\eta)
\end{aligned}
$$

Finally, we use relation (4.11) to obtain that

$$
-\frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}-\beta\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)\mathbf{v}\cdot\nabla_x F_\varepsilon\ \mathrm{d}x\mathrm{d}t=\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-\overline{\varrho}_\varepsilon}{\varepsilon}\mathbf{v}\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t+\chi(\varepsilon,\eta),
$$

while due to (5.5) and (6.5)

$$
\frac{1}{2}\frac{\alpha}{\beta^2+\alpha\delta}\Big[\int_\Omega|\delta T_\varepsilon-\beta R_\varepsilon|^2\ \mathrm{d}x\Big]_0^\tau=\frac{\beta}{\alpha}\int_0^\tau\int_\Omega T_\eta\mathbf{v}\cdot\nabla_x F\ \mathrm{d}x\ \mathrm{d}t+\chi(\varepsilon,\eta).
$$

As θ and T_η satisfy the *same* transport equation and the acoustic system (5.2), (5.3) conserves the total energy, we may use the previous estimates to rewrite (6.8) in the final form:

$$
\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)(\tau)\le\chi(\varepsilon,\eta)+c\int_0^\tau\mathcal{E}_\varepsilon\left(\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\Big|r_\varepsilon,\Theta_\varepsilon,\mathbf{U}_\varepsilon\right)\ \mathrm{d}t, \qquad (6.19)
$$

which, performing the limit (i) for $\varepsilon\to 0$, and then (ii) $\eta\to 0$, yields the conclusion of Theorem 3.1.

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