Scale interactions in compressible rotating fluids

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Scaled Navier-Stokes system

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \varrho \mathbf{f} \times \mathbf{u} + \boxed{\frac{1}{\varepsilon^{2m}}} \nabla_x p(\varrho)$$

$$= \boxed{\varepsilon^{\alpha}} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon^{2n}}} \varrho \nabla_x G$$

Newtonian viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0$$

f-plane approximation

$$\mathbf{f} = [0, 0, 1], \ \nabla_x G = [0, 0, -1]$$

Spatial domain and boundary conditions

Infinite slab

$$\Omega = R^2 \times (0,1)$$

Complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = u_3|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}]_{\mathrm{tan}}|_{\partial\Omega} = 0$$

Far field conditions

$$\varrho \to \tilde{\varrho}_\varepsilon, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Static density distribution

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-n)} \tilde{\varrho}_\varepsilon \nabla_x G, \ \tilde{\varrho}_\varepsilon \to 1 \text{ as } \varepsilon \to 0$$

Singular limits

Low Rossby number

Rossby number $\approx \varepsilon$:

$$3\text{D flow} \rightarrow 2\text{D flow}$$

Low Mach number

Mach number $\approx \varepsilon^m$:

$$\text{compressible} \rightarrow \text{incompressible}$$

High Reynolds number

Reynolds number $\approx \varepsilon^{-\alpha}$:

$$\text{viscous (Navier-Stokes)} \rightarrow \text{inviscid (Euler)}$$

Low stratification

$$\frac{m}{2} > n \geq 1$$

Uniform bounds

Energy inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\varrho) - H'(\tilde{\varrho}_\varepsilon)(\varrho - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon) \right) \right] (\tau, \cdot) \, \mathrm{d}x$$

$$+\varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\varrho_{0,\varepsilon}) - H'(\tilde{\varrho}_\varepsilon)(\varrho_{0,\varepsilon} - \tilde{\varrho}_\varepsilon) - H(\tilde{\varrho}_\varepsilon) \right) \right] \, \mathrm{d}x$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z, \; p(\varrho) \approx a\varrho^\gamma, \; \gamma > \frac{3}{2}$$

Ill-prepared initial data

$$\varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon^m \varrho_{0,\varepsilon}^{(1)}, \; \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ in } L^2(\Omega), \; \|\varrho_{0,\varepsilon}^{(1)}\|_{L^\infty} \leq c,$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3)$$

Limit system

Limit density deviation

$$\text{ess} \sup_{t \in (0,T)} \|\varrho_\varepsilon(t, \cdot) - 1\|_{L^\gamma_{\text{loc}}(\Omega)} \leq \varepsilon^n c$$

Limit velocity

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{v} \begin{cases} \text{weakly-(*) in } L^\infty(0, T; L^2(\Omega; R^3)), \\[2mm] \boxed{\text{strongly in } L^1_{\text{loc}}((0, T) \times \Omega; R^3)}, \end{cases}$$

Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0 \text{ in } (0, T) \times R^2$$

$$\mathbf{v}_0 = \mathbf{H}\left[\int_0^1 \mathbf{u}_0 \, \mathrm{d}x_3\right]$$

Relative entropy inequality

Relative entropy

$$\mathcal{E}_\varepsilon \left[\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U} \right]$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^{2m}} \Big(H(\varrho) - H'(r)(\varrho - r) - H(r) \Big) \right] \, \mathrm{d}x$$

Relative entropy inequality

$$\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U} \right)(\tau) + \varepsilon^\alpha \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \mathcal{E}_\varepsilon \left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \,\Big|\, r(0, \cdot), \mathbf{U}(0, \cdot) \right) + \int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t$$

Test functions

$$r > 0, \ \mathbf{U} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ (r - \tilde{\varrho}_\varepsilon), \ \mathbf{U} \to 0 \text{ as } |x| \to \infty$$

Remainder

$$\int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho (\mathbf{f} \times \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \left[(r - \varrho) \partial_t H'(r) + \nabla_x \left(H'(r) - H'(\tilde{\varrho}_\varepsilon) \right) \cdot (r\mathbf{U} - \varrho\mathbf{u}) \right] \mathrm{d}x \, \mathrm{d}t$$

$$- \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \left(p(\varrho) - p(r) \right) \mathrm{d}x \, \mathrm{d}t + \frac{1}{\varepsilon^{2n}} \int_0^\tau \int_\Omega (\varrho - r) \nabla_x G \cdot \mathbf{U} \, \mathrm{d}x \, \mathrm{d}t$$

Test function ansatz

Density deviation

$$r = \tilde{\varrho}_\varepsilon + \varepsilon^m(q_\varepsilon + s_\varepsilon)$$

Velocity decomposition

$$\mathbf{U} = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

Initial data

$$\varrho_{0,\varepsilon}^{(1)} = (q_\varepsilon + s_\varepsilon)(0, \cdot), \ \mathbf{u}_{0,\varepsilon} = (\mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon)(0, \cdot)$$

Non-oscillatory - limit part

Diagnostic equation

$$\omega \mathbf{f} \times \mathbf{v}_\varepsilon + \nabla_x q_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

Perturbed Euler system

$$\partial_t \left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right) - \frac{1}{\omega} \nabla^t q_\varepsilon \cdot \nabla \left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right) = 0$$

Initial data

$$\left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right)(0, \cdot) = \omega \mathrm{curl} \left[\int_0^1 \mathbf{u}_{0,\varepsilon} \ \mathrm{d}x_3 \right] - \omega^2 \int_0^1 r_{0,\varepsilon} \ \mathrm{d}x_3$$

Oscillatory - vanishing part

Acoustic - Rossby waves

$$\varepsilon^m \partial_t s_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon^m \partial_t \mathbf{V}_\varepsilon + \omega \mathbf{f} \times \mathbf{V}_\varepsilon + \nabla_x s_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

Antisymmetric acoustic propagator

$$\mathcal{B}(\omega) : \left[\begin{array}{c} s \\ \mathbf{V} \end{array} \right] \mapsto \left[\begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \omega \mathbf{f} \times \mathbf{V} + \nabla_x s \end{array} \right].$$

Fourier representation

Hermitian matrix

$$\mathcal{A}(\xi, k, \omega) = \begin{bmatrix} 0 & \xi_1 & \xi_2 & k \\ \xi_1 & 0 & \omega\mathrm{i} & 0 \\ \xi_2 & -\omega\mathrm{i} & 0 & 0 \\ k & 0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues

$$\lambda_{1,2}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 + \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

$$\lambda_{3,4}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 - \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

Fourier analysis

Frequency cut-off

$$Z(t, x_h, k, \omega) = \mathcal{F}^{-1}_{\xi \to x_h} \left[\exp \left(\pm i \lambda_j(|\xi|, k, \omega) t \right) \psi(|\xi|) \hat{h}(\xi) \right]$$

$$\|Z(t, \cdot, k, \omega)\|_{L^\infty(R^2)}$$

$$\leq \left\| \mathcal{F}^{-1}_{\xi \to x_h} \left[\exp \left(\pm i \lambda_j(|\xi|, k, \omega) t \right) \psi(|\xi|) \right] \right\|_{L^\infty(R^2)} \|h\|_{L^1(R^2)}$$

Fourier transform of radially symmetric function

$$\mathcal{F}^{-1}_{\xi \to x_h} \left[\exp \left(\pm i \lambda_j(|\xi|, k, \omega) t \right) \psi(|\xi|) \right] (x_h)$$

$$= \int_0^\infty \exp \left(\pm i \lambda_j(r, k, \omega) t \right) \psi(r) r J_0(r|x_h|) \, dr,$$

van Corput's lemma

Lemma

Let $\Lambda = \Lambda(z)$ be a smooth function away from the origin,

$$\partial_z \Lambda(z) \text{ monotone, } |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

for all $z \in [a, b]$, $0 < a < b < \infty$. Let Φ be a smooth function on $[a, b]$. Then

$$\left| \int_a^b \exp\left(i\Lambda(z)t\right) \Phi(z) \, dz \right| \leq c \frac{1}{t\Lambda_0} \left[|\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \, dz \right],$$

where c is an absolute constant indpendent of Λ and Φ.

Decay estimates

$L^p - L^q$ estimates

$$\|Z(t,\cdot,k,\omega)\|_{L^p(R^2)} \leq c(\psi,p,k) \max\left\{\frac{1}{\omega t^{1-\beta/2}}; \frac{1}{t^{\beta/2}}\right\}^{1-\frac{2}{p}} \|h\|_{L^{p'}(R^2)}$$

$$\text{for } p \geq 2, \ \frac{1}{p} + \frac{1}{p'} = 1, \ \beta > 0, \ \lambda_j \neq 0.$$

Scaling

$$\omega \approx \varepsilon^{m-1}, \ t \approx \varepsilon^{-m}$$