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Existence and uniqueness results for a class of dynamic elasto-plastic contact problems

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ABSTRACT

This paper focuses on a dynamical model for the motion of a visco-elasto-plastic body in contact with an elasto-plastic obstacle. The elastoplastic constitutive laws as well as the contact boundary condition are stated in terms of hysteresis operators. Under appropriate regularity assumptions on the initial data, we show that the resulting partial differential equation with hysteresis possesses a unique solution which is constructed by Galerkin approximations and the Minty trick. In the 1D case, the existence and uniqueness proof can be carried out without the viscosity assumption, and the necessary a priori estimates are derived from a hysteresis second order energy inequality.

1. Introduction

This paper aims to give some new mathematical results on existence and uniqueness for dynamic (visco)-elasto-plastic contact problems. The situations involving contact abound in industry, especially in engines or transmissions. For this reason a considerable engineering and mathematical literature deals with dynamic and quasi-static frictional contact problems. Note that the quasi-static contact problems arise when the forces applied to the system vary slowly in time and so that the accelerations are negligible. The first existence result was obtained for a quasi-static elasto-plastic frictionless contact problem in [12]. The author uses a Yosida regularization which leads to smooth systems and he shows that a priori estimates stay uniform in the regularization parameter. A time semi-discretization method for solving the variational inequality in the quasi-static frictional problem with normal compliance was successfully used in [3], the reader is also referred to [5,13] and the references therein. Note that the normal compliance may be considered as regularization of the usual, idealized, contact conditions, as was explained in [20]. The finite dimensional dynamic problem with friction and persistent contact has been solved via the theory of differential inclusions in [18]. Notice that contact problems with nonlinear viscoelastic or elasto-plastic materials were intensively studied in [24,2,21,22,1,23]. Finally, the uniqueness of the solution of contact problems in linearized elasto-statics with small Coulomb friction is discussed in [4]. Concerning the dynamic contact problems for (visco)-elasto-plastic bodies with friction, the mathematical problem is typically stated as a variational inequality in a Sobolev space, and the existence of solutions is established by a sophisticated choice of penalty method, see e.g. [10,11], and, in particular, the comprehensive monograph [8].

In this paper, we propose a completely different approach to dynamic contact problems. It is based on the theory of *hysteresis operators*. As an example, we consider the motion of a (visco)-elasto-plastic body in contact with an elasto-plastic obstacle. In particular, we are interested in the situation that the body penetrates into the obstacle as on Fig. 1. We derive

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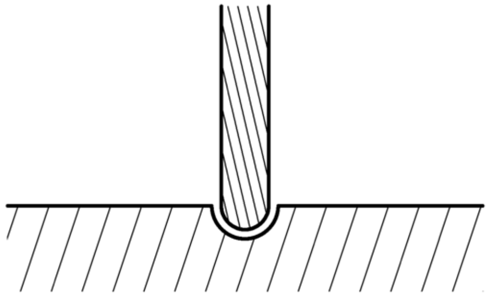

Fig. 1. Remanent deformation of an elasto-plastic obstacle.

a mathematical model valid in any dimension $d \geq 1$ and prove existence and uniqueness results under the condition that the moving body is visco-elasto-plastic with a constant positive viscosity coefficient. For $d = 1$, existence and uniqueness are obtained even if viscosity is absent, provided the contact condition is more regular.

The proofs rely substantially on the idea to solve the constitutive variational inequality as well as the (quasi)-variational inequality on the contact surface separately from the equation of motion, and to derive as much as possible information about the associated input–output operators. They are in fact hysteresis operators with very convenient analytical properties (Lipschitz continuity in suitable function spaces, monotonicity, first and second order energy inequalities). The momentum balance then is stated as a partial differential equation with hysteresis operators, which can be solved by conventional methods like Galerkin approximations and the Minty trick.

The paper is organized as follows. In Section 2, we present the mathematical models and we show the connection to hysteresis operators as solution operators of the underlying (quasi)-variational inequalities. Section 3 is devoted to proving existence and uniqueness results for the visco-elasto-plastic case. In Section 4, we treat the 1D case without viscosity and show how the hysteresis second order energy inequalities can be exploited for the existence and uniqueness proof.

2. Description of the model

Our aim here is to model the dynamic behavior of a visco-elasto-plastic body in contact with an elasto-plastic obstacle. On the contact surface, we observe remanent plastic deformations as on Figs. 1 and 2. The body itself, represented by a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is assumed to obey the constitutive relation

$$\sigma \stackrel{\text{def}}{=} \mathcal{P}[\varepsilon] + \nu\varepsilon_t \tag{2.1}$$

between the strain tensor $\varepsilon \stackrel{\text{def}}{=} \{\varepsilon_{ij}\}$ and stress tensor $\sigma \stackrel{\text{def}}{=} \{\sigma_{ij}\}$ in the space of symmetric tensors $\mathbb{T}_s^{d\times d}$, where $\mathcal{P}$ is a constitutive operator of elasto-plasticity satisfying the hypotheses of Section 3.1 below, $\nu \geq 0$ is a constant viscosity coefficient $(\cdot)_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t}(\cdot)$. Typically, we have in mind the Prandtl–Reuss model characterized by the following variational inequality

$$\begin{cases} \sigma^p(t) \in K & \text{for all } t \in [0, T], \\ \sigma^p(0) = \mathrm{Proj}_K(\mathbf{A}\varepsilon(0)), \\ (\varepsilon_t(t) - \mathbf{A}^{-1}\sigma_t^p(t)) : (\sigma^p(t) - y) \geq 0 & \text{a.e. for all } y \in K, \end{cases} \tag{2.2}$$

where $K \subset \mathbb{T}_s^{d\times d}$ is a convex closed admissible stress domain containing the origin, $\mathbf{A} \stackrel{\text{def}}{=} \{a_{ijkl}\}$ is a constant elasticity matrix, which is symmetric and positive definite with respect to the canonical scalar product ":" in $\mathbb{T}_s^{d\times d}$ and Proj_K denotes the projection onto a convex set K. Recall that this scalar product is defined as $\sigma : \varepsilon \stackrel{\text{def}}{=} \mathrm{tr}(\sigma^\top \varepsilon)$ for $\sigma, \varepsilon \in \mathbb{T}_s^{d\times d}$, and the corresponding norm is given by $|\sigma|^2 \stackrel{\text{def}}{=} \sigma : \sigma$. Here, $(\cdot)^\top$ denotes the transpose of the tensor, and $\mathrm{tr}(\cdot)$ is the trace of the matrix $(\cdot)$.

A unique solution $\sigma^p \in W^{1,1}(0, T; \mathbb{T}_s^{d\times d})$ to (2.2) exists for every $\varepsilon \in W^{1,1}(0, T; \mathbb{T}_s^{d\times d})$. We now define the operator $\mathcal{P}_0$ as the solution operator $\mathcal{P}_0[\varepsilon] \stackrel{\text{def}}{=} \sigma^p$. The extension of $\mathcal{P}_0$ to a continuous operator $\mathcal{P} : C^0([0, T]; \mathbb{T}_s^{d\times d}) \to C^0([0, T]; \mathbb{T}_s^{d\times d})$ if K has nonempty interior, is established in [15]. A canonical representative of the operator $\mathcal{P}$ in (2.1) is the elasto-plastic constitutive law with linear kinematic hardening

$$\mathcal{P}[\varepsilon] = \mathcal{P}_0[\varepsilon] + \mathbf{B}\varepsilon \tag{2.3}$$

with a constant symmetric positive semidefinite matrix $\mathbf{B}$. If the viscosity coefficient ν is positive, we allow also $\mathbf{B} = 0$, that is, no hardening.

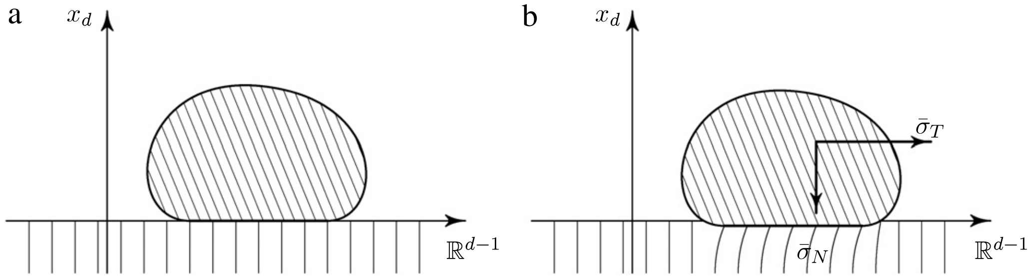

Fig. 2. (a) Referential state. (b) Deformation under the effect of normal and tangential load.

Let $u \stackrel{\text{def}}{=} \{u_i\}$ be the displacement vector. We assume small deformations, that is, $\varepsilon \stackrel{\text{def}}{=} \nabla_s u \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}}) \in \mathbb{T}_s^{d\times d}$. Let $\rho > 0$ be a constant mass density. The equation of motion in variational form reads as follows

$$\forall \phi \in W^{1,2}(\Omega; \mathbb{R}^d) : \int_\Omega (\rho u_{tt}\phi + \sigma : \nabla_s \phi)\,dx = \int_{\partial\Omega} (\sigma n) \cdot \phi\, dS, \tag{2.4}$$

where $n \stackrel{\text{def}}{=} \{n_i\}$ is the unit outward normal vector.

The main issue is to prescribe the boundary conditions. We assume that the boundary $\partial\Omega$ is divided into two nonintersecting parts Γ_C (contact surface) and Γ_E (surface subject to external load). Furthermore, the boundary Γ_C is flat in the reference state, and

$$\Gamma_C \subset \{x_d = 0\} \quad \text{and} \quad \Omega \subset \{x_d > 0\}. \tag{2.5}$$

We require

$$(\sigma n)_i \stackrel{\text{def}}{=} \sum_j \sigma_{ij} n_j = -p_i(t) \quad \text{on } \Gamma_E \quad \text{and} \quad (\sigma n)_i = -f_i[u] \quad \text{on } \Gamma_C. \tag{2.6}$$

Here, p is a given time dependent external force, and $f \stackrel{\text{def}}{=} \{f_i\}$ is a boundary contact operator satisfying the energy inequality

$$f[u] \cdot u_t - (e[u])_t \geq 0 \tag{2.7}$$

with a potential $e[u] \geq 0$. The corresponding mathematical problem for the unknown function u is stated as a partial differential equation of the form

$$\forall \phi \in W^{1,2}(\Omega; \mathbb{R}^d) : \int_\Omega (\rho u_{tt}\phi + (\mathcal{P}[\nabla_s u] + \nu \nabla_s u_t) : \nabla_s \phi)\,dx = -\int_{\Gamma_C} f[u] \cdot \phi\, dS - \int_{\Gamma_E} p \cdot \phi\, dS, \tag{2.8}$$

with operators $\mathcal{P}$ and f that we describe below. We couple (2.8) with initial conditions

$$u(x, 0) = u^0(x) \quad \text{and} \quad u_t(x, 0) = v^0(x) \tag{2.9}$$

where u^0 and v^0 are given data.

Let us start with describing the boundary contact operator f on Γ_C. We decompose the normal stress vector $\bar{\sigma} \stackrel{\text{def}}{=} \sigma n$ into the sum $\bar{\sigma} = \bar{\sigma}_N + \bar{\sigma}_T$ of a normal component $\bar{\sigma}_N \stackrel{\text{def}}{=} \sigma_N n$, where we denote $\sigma_N \stackrel{\text{def}}{=} \bar{\sigma} \cdot n$, and a tangential component $\bar{\sigma}_T \stackrel{\text{def}}{=} \bar{\sigma} - \bar{\sigma}_N$ orthogonal to n. Similarly, the displacement u is decomposed into the orthogonal sum $u = u_N n + u_T$ of a normal component $u_N n$ and a tangential component u_T. On Γ_C, we have $n = (0, \dots, 0, -1)$, $\sigma_N = \sigma_{dd}$, $u_N = -u_d$, and, omitting the last vanishing component, we can write

$$\bar{\sigma}_T = \begin{pmatrix} -\sigma_{1d} \\ \dots \\ -\sigma_{(d-1)d} \end{pmatrix} \quad \text{and} \quad u_T = \begin{pmatrix} u_1 \\ \dots \\ u_{d-1} \end{pmatrix}.$$

Our aim is to model the following phenomena:

(a) The obstacle touching the body Ω at Γ_C is elasto-plastic. We assume that initially at time $t = 0$, the obstacle is at the referential unperturbed state, that is, contact takes place if $u_N \geq 0$ and does not take place if $u_N < 0$.
(b) As a result of remanent plastic deformation in the normal direction, the next contact may occur for some positive value of u_N.
(c) If no contact takes place, we have $\sigma_N = \bar{\sigma}_T = 0$.
(d) In case of contact, there exists a bounded admissible domain $K(\sigma_N) \subset \mathbb{R}^{d-1}$ for tangential stresses $\bar{\sigma}_T$. For large normal stresses σ_N, the domain is larger, and it reduces to $\{0\}$ if $\sigma_N = 0$.

(e) If $\bar{\sigma}_T$ is in the interior of $K(\sigma_N)$, the contact is elastic in the tangential direction. If $\bar{\sigma}_T$ attains the boundary of $K(\sigma_N)$, sliding in the tangential direction takes place (cf. Fig. 2).

Mathematically, this can be achieved in the following way. Let $b > a > 0$ and $c > 0$ be given numbers, and let $\mathcal{S}$ be for any input function $v \in W^{1,1}(0, T)$ the solution operator $w = \mathcal{S}[v]$ of the variational inequality

$$\begin{cases} w(t) - av(t) \le c & \text{for every } t \in [0, T], \\ w(0) = \min\{av(0) + c, bv(0)\}, \\ (bv_t(t) - w_t(t))(w(t) - av(t) - z) \ge 0 & \text{a.e. for every } z \le c. \end{cases} \tag{2.10}$$

Let us mention a classical Lipschitz continuity result related to (2.10) that goes back to [15], namely that the operator $\mathcal{S}$ can be extended to the space $C^0([0, T])$ of continuous functions endowed with seminorms

$$|u|_{[0,t]} = \max\{|u(s)| : s \in [0, t]\}, \tag{2.11}$$

and that it is Lipschitz continuous in the sense

$$|w_1 - w_2|(t) \le L_1 \, |v_1 - v_2|_{[0,t]} \,, \tag{2.12}$$

for every $v_1, v_2 \in C^0([0, T])$ and every $t \in [0, T]$ with a constant $L_1 > 0$.

We give here a simple alternative proof of this fact in the following more general form. For a closed interval $[\alpha, \beta] \subset \mathbb{R}$, we denote by $Q_{[\alpha,\beta]}$ the projection of $\mathbb{R}$ onto $[\alpha, \beta]$, that is,

$$Q_{[\alpha,\beta]}(z) = \min\{\beta, \max\{\alpha, z\}\} \quad \text{for } z \in \mathbb{R}. \tag{2.13}$$

Lemma 2.1. *Let $\alpha, \beta, u \in W^{1,1}(0, T)$ be given functions, $\alpha(t) \le \beta(t)$ for all $t \in [0, T]$. Then there exists a unique solution $\xi \in W^{1,1}(0, T)$ of the variational inequality*

$$\begin{cases} u(t) - \xi(t) \in [\alpha(t), \beta(t)] & \text{for every } t \in [0, T], \\ \xi(0) = u(0) - Q_{[\alpha(0),\beta(0)]}(u(0)), \\ \xi_t(t)(u(t) - \xi(t) - z) \ge 0 & \text{a.e. for every } z \in [\alpha(t), \beta(t)]. \end{cases} \tag{2.14}$$

Moreover, if $\alpha_i, \beta_i, u_i \in W^{1,1}(0, T)$, $i = 1, 2$ are arbitrary input data, then the corresponding solutions satisfy for all $\tau \in [0, T]$ the inequality

$$|\xi_1(\tau) - \xi_2(\tau)| \le |u_1 - u_2|_{[0,\tau]} + \max\{|\alpha_1 - \alpha_2|_{[0,\tau]} \,, |\beta_1 - \beta_2|_{[0,\tau]}\}. \tag{2.15}$$

Proof. The existence result goes back to [19]. To prove (2.15) (which implies uniqueness), we fix $\tau \in [0, T]$, and for $t \in [0, \tau]$ define the function

$$V(t) = \max\{|\xi_1(t) - \xi_2(t)|^2, (|u_1 - u_2|_{[0,\tau]} + \max\{|\alpha_1 - \alpha_2|_{[0,\tau]} \,, |\beta_1 - \beta_2|_{[0,\tau]}\})^2\}.$$

We have $|\xi_1(0) - \xi_2(0)| \le |u_1(0) - u_2(0)| + \max\{|\alpha_1(0) - \alpha_2(0)|, |\beta_1(0) - \beta_2(0)|\}$. The statement will thus be proved if we check that $V_t(t) \le 0$ a.e. Assume that this is not the case, and that there exists a Lebesgue point $t \in (0, \tau)$ of the first derivatives of all functions appearing in (2.15) such that $V_t(t) > 0$. Then, interchanging possibly the indices, we have

$$\xi_1(t) - \xi_2(t) \quad > |u_1 - u_2|_{[0,\tau]} + \max\{|\alpha_1 - \alpha_2|_{[0,\tau]} \,, |\beta_1 - \beta_2|_{[0,\tau]}\}, \tag{2.16}$$

$$\xi_{1,t}(t) - \xi_{2,t}(t) > 0. \tag{2.17}$$

Hence, we necessarily have $\xi_{1,t}(t) > 0$ or $\xi_{2,t}(t) < 0$. In the former case, we obtain from (2.14) that

$$u_1(t) - \xi_1(t) - Q_{[\alpha_1(t),\beta_1(t)]}(u_2(t) - \xi_2(t)) \ge 0,$$

hence,

$$\xi_1(t) - \xi_2(t) \le u_1(t) - u_2(t) + \big(u_2(t) - \xi_2(t) - Q_{[\alpha_1(t),\beta_1(t)]}(u_2(t) - \xi_2(t))\big). \tag{2.18}$$

It is easy to see that for $z \in [\alpha', \beta']$, we have

$$|z - Q_{[\alpha,\beta]}(z)| \le \max\{|\alpha - \alpha'|, |\beta - \beta'|\}.$$

We conclude from (2.18) that

$$\xi_1(t) - \xi_2(t) \le |u_1(t) - u_2(t)| + \max\{|\alpha_1(t) - \alpha_2(t)|, |\beta_1(t) - \beta_2(t)|\},$$

which contradicts (2.16). Hence, V is nonincreasing, and the assertion of Lemma 2.1 holds. The other case $\xi_{2,t}(t) < 0$ is fully symmetric and we argue in a similar way. □

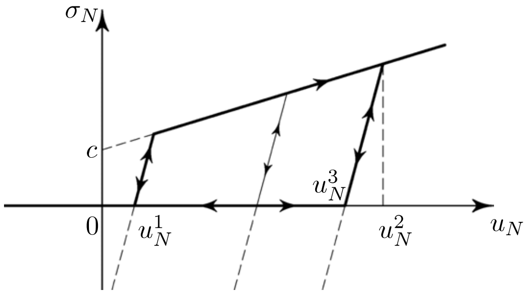

Fig. 3. Normal deformation of the obstacle.

Variational inequality (2.10) is indeed of the form (2.14), if we put $u(t) = (b-a)v(t)$, $\xi(t) = bv(t) - w(t)$, $\beta(t) = c$, $\alpha(t) = -R$ with $R > (b-a)\,|v|_{[0,T]}$. We thus obtain (2.12) with $L_1 = 2b - a$.

Let w^+ denote the positive part of $w \in \mathbb{R}$, that is, $w^+ \stackrel{\text{def}}{=} \max\{w, 0\}$. We claim that the normal components behave as described in the above items (a)–(b) provided we define

$$\sigma_N \stackrel{\text{def}}{=} (\mathcal{S}[u_N])^+. \tag{2.19}$$

Let the body touch at some time t_1 the obstacle for some value $u_N^1 \geq 0$ of the normal displacement u_N, see Fig. 3, and let it continue increasing. The response of the obstacle is first elastic with slope b, but if $u_N(t)$ attains the yield limit, $\sigma_N(t)$ follows the irreversible path with slope a (that is, plastic with kinematic hardening) till some value $\sigma_N(t_2)$ corresponding to an input value $u_N^2 = u_N(t_2)$. Let now $u_N(t)$ start decreasing after t_2. Then $\sigma_N(t)$ follows the reversible elastic path with slope b until reaching the value u_N^3, which represents a *remanent deformation* of the obstacle. In the next cycle, contact between the body and the obstacle is only established at the larger value u_N^3 of the normal displacement.

This can easily be seen if we examine more closely the variational inequality (2.10). Put $v \stackrel{\text{def}}{=} u_N$ and $w \stackrel{\text{def}}{=} \mathcal{S}[v]$. At time $t = t_2$, we have $w(t_2) - av(t_2) = c$ and $w(t_2) = b(v(t_2) - u_N^3)$. We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t) - b(v(t) - u_N^3))^+ \leq 0 \tag{2.20}$$

for almost every $t > t_2$. Indeed, by virtue of (2.10), we have $w_t(t) - bv_t(t) \leq 0$ almost everywhere, and (2.20) follows. In particular, $(w(t) - b(v(t) - u_N^3))^+ \leq (w(t_2) - b(v(t_2) - u_N^3))^+ = 0$ for $t > t_2$, hence $\sigma_N(t) = (w(t))^+ = 0$ if $v(t) \leq u_N^3$ for $t \geq t_2$. In particular, repeating the argument with $t_2 = 0$ and $u_N^3 = 0$, we have the inequality $w(t) \leq bv(t)$, which implies

$$u_N(t) \leq 0 \Longrightarrow \sigma_N(t) = 0 \tag{2.21}$$

for all $t \in [0, T]$.

The tangential "flow rule" in the case $d > 1$ is defined similarly. We consider the variational inequality

$$\begin{cases} W(t) \in K(r(t)) & \text{for all } t \in [0,T], \\ W(0) = \mathrm{Proj}_{K(r(0))}(\mu V(0)), \\ (\mu V_t(t) - W_t(t)) \cdot (W(t) - Z) \geq 0 & \text{a.e. for all } Z \in K(r(t)), \end{cases} \tag{2.22}$$

in $\mathbb{R}^{d-1}$ with a moving convex closed constraint $K(r)$ depending on a scalar function $r \in \mathrm{C}^0([0,T])$, with input $V \in \mathrm{W}^{1,1}(0,T;\mathbb{R}^{d-1})$ and output W. The parameter μ is assumed constant and represents the *shear modulus* of the obstacle. Furthermore, $\mathrm{Proj}_{K(r(0))}$ is the orthogonal projection of $\mathbb{R}^{d-1}$ onto $K(r(0))$. Let $\mathcal{Q}$ denote the solution operator $W \stackrel{\text{def}}{=} \mathcal{Q}[V, r]$ of (2.22). Assuming e.g.

$$K(r) \stackrel{\text{def}}{=} [-\gamma_1(r), \gamma_1(r)] \times \cdots \times [-\gamma_{d-1}(r), \gamma_{d-1}(r)] \tag{2.23}$$

with bounded Lipschitz continuous nondecreasing functions γ_i vanishing for $r \leq 0$, we can apply Lemma 2.1 componentwise and check that the operator $\mathcal{Q}$ admits an extension to the space $\mathrm{C}^0([0,T];\mathbb{R}^{d-1})$ of continuous functions and that it is Lipschitz continuous in the sense

$$|\mathcal{Q}(V_1, r_1) - \mathcal{Q}(V_2, r_2)|(t) \leq L_2(|V_1 - V_2|_{[0,t]} + |r_1 - r_2|_{[0,t]}) \tag{2.24}$$

for every $t \in [0, T]$, with a constant $L_2 > 0$. In fact, inequality (2.24) holds whenever $K(r(t))$ is a polyhedron with constant normal vectors and moving facets, see [17], but does not hold for general geometries, e.g. balls, see [15,16].

We now define the operators f_i, $i = 1, \ldots, d$, in (2.6) as

$$f_d[u] \stackrel{\text{def}}{=} -\mathcal{S}[-u_d]^+ \quad \text{and} \quad \begin{pmatrix} f_1[u] \\ \cdots \\ f_{d-1}[u] \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{Q}\left[\begin{pmatrix} u_1 \\ \cdots \\ u_{d-1} \end{pmatrix}, |f_d[u]|\right], \tag{2.25}$$

and we see that the behavior indicated in items (c)–(e) above is well reproduced.

It remains to construct the potential $e[u]$ such that the energy inequality (2.7) holds. This is easy for the first two components. Directly from (2.22) we obtain $(\mu V_t - W_t)W \geq 0$, hence

$$\mathcal{Q}\left[\begin{pmatrix} u_1 \\ \dots \\ u_{d-1} \end{pmatrix}, |f_d[u]|\right]\begin{pmatrix} u_{1,t} \\ \dots \\ u_{d-1,t} \end{pmatrix} - \frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathcal{Q}\left[\begin{pmatrix} u_1 \\ \dots \\ u_{d-1} \end{pmatrix}, |f_d[u]|\right]\right|^2 \geq 0.$$

For the last component, we first notice that $f_d[u]u_{d,t} = \mathcal{S}[u_N]^+(u_N)_t$. Set for simplicity $v \stackrel{\mathrm{def}}{=} u_N$, $w \stackrel{\mathrm{def}}{=} \mathcal{S}[v]$, and consider $\sigma_N \stackrel{\mathrm{def}}{=} g(w)$ with a nondecreasing function g. Indeed, this is in agreement with (2.19) if we choose $g(w) = w^+$. As a consequence of (2.10), we have $(bv_t - w_t)(w - av) \geq 0$, which we rewrite as

$$(bv_t - w_t)\left(w - \frac{a}{b-a}(bv - w)\right) \geq 0.$$

Since g is nondecreasing, we obtain

$$(bv_t - w_t)\left(g(w) - g\left(\frac{a}{b-a}(bv - w)\right)\right) \geq 0. \tag{2.26}$$

Put $G(s) \stackrel{\mathrm{def}}{=} \int_0^s g(z)\,\mathrm{d}z$. Then (2.26) can be written as

$$v_t g(w) - \frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}t}\left(G(w) + \frac{b-a}{a}G\left(\frac{a}{b-a}(bv - w)\right)\right) \geq 0. \tag{2.27}$$

Hence, for $g(w) = w^+$, inequality (2.7) holds for the following choice

$$e[u] = \frac{1}{2\mu}\left|\mathcal{Q}\left[\begin{pmatrix} u_1 \\ \dots \\ u_{d-1} \end{pmatrix}, |f_d[u]|\right]\right|^2 + \frac{1}{2b}(\mathcal{S}[u_N]^+)^2 + \frac{a}{2b(b-a)}\left((bu_N - \mathcal{S}[u_N])^+\right)^2.$$

3. Existence and uniqueness results

We state and prove here existence and uniqueness results for the equation (2.8) for the viscous problem $\nu > 0$. The case $\nu = 0$ and $d = 1$ without viscosity will be considered in Section 4.

3.1. Mathematical hypotheses

We consider general operators f and $\mathcal{P}$ in (2.8) with analytical properties specified below. The model derived in Section 2 is only a special case. In particular, multiyield plasticity and more complex hysteresis branching can be included.

(A1) The operator $f : \mathrm{C}^0([0,T];\mathbb{R}^d) \to \mathrm{C}^0([0,T];\mathbb{R}^d)$ is Lipschitz continuous in the following sense

$$|f[u_1] - f[u_2]|(t) \leq L_f\,|u_1 - u_2|_{[0,t]} \tag{3.1}$$

with a constant $L_f > 0$ for every $t \in [0,T]$ and every $u_1, u_2 \in \mathrm{C}^0([0,T];\mathbb{R}^d)$.

(A2) The operator $\mathcal{P} : \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d}) \to \mathrm{C}^0([0,T];\mathbb{T}_s^{d\times d})$ is Lipschitz continuous in the following sense

$$|\mathcal{P}[\varepsilon_1](t) - \mathcal{P}[\varepsilon_2](t)| \leq L_{\mathcal{P}}\left(\int_0^t |\varepsilon_{1,t}(\tau) - \varepsilon_{2,t}(\tau)|\,\mathrm{d}\tau + |\varepsilon_1(0) - \varepsilon_2(0)|\right) \tag{3.2}$$

with a constant $L_{\mathcal{P}} > 0$ for every $t \in (0,T)$ and every input $\varepsilon_1, \varepsilon_2 \in \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d})$.

(A3) There exist two potential energy operators $\mathcal{V} : \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d}) \to \mathrm{W}^{1,1}(0,T;\mathbb{R}_+)$ and $e : \mathrm{W}^{1,1}(0,T;\mathbb{R}^d) \to \mathrm{W}^{1,1}(0,T;\mathbb{R}_+)$, where $\mathbb{R}_+ \stackrel{\mathrm{def}}{=} [0,\infty)$, such that

$$\mathcal{V}[\varepsilon](t) \geq c_0|\mathcal{P}[\varepsilon](t)|^2, \qquad \mathcal{V}[\varepsilon](0) \leq c_1|\varepsilon(0)|^2 \quad \text{and} \quad e[u](0) \leq c_1|u(0)|^2,$$

with constants $c_0, c_1 > 0$, and the inequality (2.7) together with

$$\mathcal{P}[\varepsilon](t)\varepsilon_t(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}[\varepsilon](t) \geq 0 \tag{3.3}$$

hold for almost every $t \in (0,T)$ and for arbitrary inputs $u \in \mathrm{W}^{1,1}(0,T;\mathbb{R}^d)$ and $\varepsilon \in \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d})$.

(A4) The operator $\mathcal{P}$ is monotone in the sense that there exists a mapping $\widetilde{\mathcal{V}} : \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d}) \times \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d}) \to \mathrm{W}^{1,1}(0,T;\mathbb{R}_+)$ such that for all $\varepsilon_1, \varepsilon_2 \in \mathrm{W}^{1,1}(0,T;\mathbb{T}_s^{d\times d})$ we have

$$(\mathcal{P}[\varepsilon_1](t) - \mathcal{P}[\varepsilon_2](t)) : (\varepsilon_{1,t}(t) - \varepsilon_{2,t}(t)) \geq \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{V}}[\varepsilon_1,\varepsilon_2](t) \quad \text{a.e.,} \tag{3.4}$$

with the upper bound

$$\widetilde{\mathcal{V}}[\varepsilon_1,\varepsilon_2](0) \leq c_1|\varepsilon_1(0) - \varepsilon_2(0)|^2. \tag{3.5}$$

We now check that the operator $\mathcal{P}$ defined in (2.3) has the above properties. By (2.2), we have

$$(\varepsilon_{1,t} - \varepsilon_{2,t}) : (\mathcal{P}_0[\varepsilon_1] - \mathcal{P}_0[\varepsilon_2]) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^{-1}(\mathcal{P}_0[\varepsilon_1] - \mathcal{P}_0[\varepsilon_2]) : (\mathcal{P}_0[\varepsilon_1] - \mathcal{P}_0[\varepsilon_2]),$$

hence

$$(\varepsilon_{1,t} - \varepsilon_{2,t}) : (\mathcal{P}[\varepsilon_1] - \mathcal{P}[\varepsilon_2]) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}^{-1}(\mathcal{P}_0[\varepsilon_1] - \mathcal{P}_0[\varepsilon_2]) : (\mathcal{P}_0[\varepsilon_1] - \mathcal{P}_0[\varepsilon_2]) + \mathbf{B}(\varepsilon_1 - \varepsilon_2) : (\varepsilon_1 - \varepsilon_2)\right),$$

and Assumptions (A2)–(A4) easily follow. Operators $\mathcal{P}$ from the *Prandtl–Ishlinskii class* also satisfy the hypotheses, see [9]. We may infer from (A1) and (A2) that the initial values of the operators $\mathcal{P}$ and f can be represented by mappings $\widehat{f} : \mathbb{R}^d \to \mathbb{R}^d$ and $\widehat{\mathcal{P}} : \mathbb{T}_s^{d\times d} \to \mathbb{T}_s^{d\times d}$ in the sense that $\mathcal{P}[\varepsilon](0) = \widehat{\mathcal{P}}(\varepsilon(0)), f[u](0) = \widehat{f}(u(0))$ for all time dependent inputs ε, u, and that these functions satisfy the Lipschitz conditions

$$|\widehat{f}(u_1(0)) - \widehat{f}(u_2(0))| \leq L_f|u_1(0) - u_2(0)|, \tag{3.6}$$

and

$$|\widehat{\mathcal{P}}(\varepsilon_1(0)) - \widehat{\mathcal{P}}(\varepsilon_2(0))| \leq L_{\mathcal{P}}|\varepsilon_1(0) - \varepsilon_2(0)|. \tag{3.7}$$

As usual, Korn's inequality will play a role in the mathematical analysis developed below. We have assumed that $\partial\Omega$ is smooth enough, so we have

$$\exists c^{\mathrm{Korn}} > 0, \quad \forall u \in \mathrm{W}_0^{1,2}(\Omega) : \|\varepsilon(u)\|^2_{\mathrm{L}^2(\Omega)} \geq c^{\mathrm{Korn}}\|u\|^2_{\mathrm{W}^{1,2}(\Omega)}, \tag{3.8}$$

for further details on Korn's inequality, the reader is referred to [14,7].

3.2. General existence and uniqueness theorem

For simplicity, we denote here and in the sequel $Q_t = \Omega \times (0,t)$ for $t \in [0,T]$.

Theorem 3.1. *Let $d \geq 1, \nu > 0$, and assume that Hypotheses* (A1)–(A4) *hold. Let $p \in \mathrm{L}^2(0,T;\Gamma_E)$, and initial conditions $u^0 \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)$ and $v^0 \in \mathrm{L}^2(\Omega;\mathbb{R}^d)$ be given. Then there exists a unique $u \in \mathrm{L}^2(Q_T;\mathbb{R}^d)$ such that $\nabla_s u_t \in \mathrm{L}^2(Q_T;\mathbb{T}_s^{d\times d})$, $u_{tt} \in \mathrm{L}^2(0,T;\mathrm{W}^{-1,2}(\Omega;\mathbb{R}^d))$, and satisfying (2.8) with initial conditions (2.9) almost everywhere.*

Proof. We construct approximate solutions by Galerkin approximations. Let $\{\omega_k\}_{k\in\mathbb{N}}$ be the complete system of eigenfunctions $\omega_k \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^d)$, orthonormal in $\mathrm{L}^2(\Omega;\mathbb{R}^d)$, of the problem

$$\forall\phi \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^d) : \int_\Omega \nabla_s\omega_k : \nabla_s\phi \,\mathrm{d}x = \lambda_k\int_\Omega \omega_k\cdot\phi\,\mathrm{d}x. \tag{3.9}$$

We have indeed $\lambda_0 = 0$ and $\lambda_k > 0$ for all $k > 0$. For a given integer $m > 0$, we consider the approximation for u of the form

$$u^{(m)}(x,t) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{m} u_k(t)\omega_k(x) \tag{3.10}$$

with functions $u_k \in \mathrm{W}^{2,2}(0,T)$ satisfying the system

$$\int_\Omega\left(\rho u_{tt}^{(m)}\cdot\omega_k + \left(\mathcal{P}[\nabla_s u^{(m)}] + \nu\nabla_s u_t^{(m)}\right) : \nabla_s\omega_k\right)\mathrm{d}x = -\int_{\Gamma_C} f[u^{(m)}]\cdot\omega_k\,\mathrm{d}S - \int_{\Gamma_E} p\cdot\omega_k\,\mathrm{d}S \tag{3.11}$$

for every eigenfunction ω_k, $k = 0, 1, \ldots, m$. This can be rewritten as

$$\rho u_{k,tt}(t) + \nu\lambda_k u_{k,t} + \int_\Omega \mathcal{P}[\nabla_s u^{(m)}] : \nabla_s\omega_k\,\mathrm{d}x = -\int_{\Gamma_C} f[u^{(m)}]\cdot\omega_k\,\mathrm{d}S - \int_{\Gamma_E} p\cdot\omega_k\,\mathrm{d}S \tag{3.12}$$

for every $k = 0, 1, \ldots, m$, with initial conditions given by

$$u_k(0) = \int_\Omega u^0 \cdot \omega_k \,\mathrm{d}x \quad \text{and} \quad u_{k,t}(0) = \int_\Omega v^0 \cdot \omega_k \,\mathrm{d}x. \tag{3.13}$$

This is a system of ordinary differential equations with a Lipschitz continuous right hand side, hence it admits a unique solution on $[0, T]$ for every $m \in \mathbb{N}$.

We test (3.11) by $u_{k,t}$ and sum over k which yields

$$\int_\Omega \left(\rho u_{tt}^{(m)} \cdot u_t^{(m)} + \left(\mathcal{P}[\nabla_s u^{(m)}] + \nu \nabla_s u_t^{(m)}\right) : \nabla_s u_t^{(m)}\right) \mathrm{d}x + \int_{\Gamma_C} f[u^{(m)}] \cdot u_t^{(m)} \,\mathrm{d}S = -\int_{\Gamma_E} p \cdot u_t^{(m)} \,\mathrm{d}S. \tag{3.14}$$

Using (3.3) and (2.7) into (3.14), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \left(\rho |u_t^{(m)}|^2 + \mathcal{V}[\nabla_s u^{(m)}]\right) \mathrm{d}x + \int_{\Gamma_C} e[u^{(m)}] \,\mathrm{d}S\right) + \nu \int_\Omega |\nabla_s u_t^{(m)}|^2 \,\mathrm{d}x \le -\int_{\Gamma_E} p \cdot u_t^{(m)} \,\mathrm{d}S. \tag{3.15}$$

For $v = u_t^{(m)}$, Korn's inequality (3.8) and the trace embedding inequality imply that there exists $C_1 > 0$ such that

$$\int_{\partial\Omega} |v|^2 \,\mathrm{d}S \le C_1 \int_\Omega \left(|v|^2 + |\nabla_s v|^2\right) \mathrm{d}x. \tag{3.16}$$

Hence, (3.15) yields a uniform bounds independent of m for $u_t^{(m)}$ in $\mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega; \mathbb{R}^d))$, for $\mathcal{P}[\nabla_s u^{(m)}]$ in $\mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega; \mathbb{T}_s^{d\times d}))$, and for $\nabla_s u_t^{(m)}$ in $\mathrm{L}^2(Q_T; \mathbb{T}_s^{d\times d})$. By comparison, we obtain from (3.11) a bound for $u_{tt}^{(m)}$ independent of m in $\mathrm{L}^2(0, T; \mathrm{W}^{-1,2}(\Omega; \mathbb{R}^d))$.

We now let m tend to $+\infty$, and denote by the symbol $\rightharpoonup$ the weak convergence, by $\overset{*}{\rightharpoonup}$ the weak-* convergence, and by $\to$ the strong convergence. Selecting a subsequence if necessary, still indexed by m, we find elements $u \in \mathrm{L}^2(Q_T; \mathbb{R}^d)$ with $\nabla_s u_t \in \mathrm{L}^2(Q_T; \mathbb{T}_s^{d\times d})$, $u_{tt} \in \mathrm{L}^2(0, T; \mathrm{W}^{-1,2}(\Omega; \mathbb{R}^d))$, and $\xi \in \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega; \mathbb{T}_s^{d\times d}))$, such that

$$\nabla_s u_t^{(m)} \rightharpoonup \nabla_s u_t \quad \text{in } \mathrm{L}^2(Q_T; \mathbb{T}_s^{d\times d}), \tag{3.17a}$$

$$u_t^{(m)} \overset{*}{\rightharpoonup} u_t \quad \text{in } \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega; \mathbb{R}^d)), \tag{3.17b}$$

$$\mathcal{P}[\nabla_s u^{(m)}] \overset{*}{\rightharpoonup} \xi \quad \text{in } \mathrm{L}^\infty(0, T; \mathrm{L}^2(\Omega; \mathbb{T}_s^{d\times d})), \tag{3.17c}$$

$$u_t^{(m)}|_{\partial\Omega} \rightharpoonup u_t|_{\partial\Omega} \quad \text{in } \mathrm{L}^2(0, T; \mathrm{L}^2(\partial\Omega; \mathbb{R}^d)), \tag{3.17d}$$

$$\int_\Omega u_{tt}^{(m)} \cdot \omega_k \,\mathrm{d}x \rightharpoonup \int_\Omega u_{tt} \cdot \omega_k \,\mathrm{d}x \quad \text{in } \mathrm{L}^2(0, T) \text{for every } k, \tag{3.17e}$$

$$u^{(m)}|_{\partial\Omega} \to u|_{\partial\Omega} \quad \text{in } \mathrm{L}^2(\partial\Omega; \mathrm{C}^0([0, T]; \mathbb{R}^d)). \tag{3.17f}$$

Furthermore, we claim that $f[u^{(m)}] \to f[u]$ in $\mathrm{L}^2(\Gamma_C; \mathrm{C}^0([0, T]; \mathbb{R}^d))$ strongly. Indeed, by (3.1) we have

$$\int_{\Gamma_C} \left|f[u^{(m)}](x, \cdot) - f[u](x, \cdot)\right|_{[0,T]}^2 \,\mathrm{d}S \le L_f^2 \int_{\Gamma_C} \left|u^{(m)}(x, \cdot) - u(x, \cdot)\right|_{[0,T]}^2 \,\mathrm{d}S \to 0.$$

We can pass to the limit in (3.11) as m tends to $+\infty$ and obtain

$$\int_\Omega (\rho u_{tt} \cdot \phi + (\xi + \nu \nabla_s u_t) : \nabla_s \phi) \,\mathrm{d}x = -\int_{\Gamma_C} f[u] \cdot \phi \,\mathrm{d}S - \int_{\Gamma_E} p \cdot \phi \,\mathrm{d}S \tag{3.18}$$

first for $\phi = \omega_k$ for every $k = 0, 1, 2, \ldots$ and then, by density, for every $\phi \in \mathrm{W}^{1,2}(\Omega; \mathbb{R}^d)$. Furthermore, by using a standard argument, we prove that u satisfies the initial conditions (2.9) almost everywhere. Indeed, by (3.10) and (3.13), $u^{(m)}(\cdot, 0) \to u^0$ and $u_t^{(m)}(\cdot, 0) \to v^0$ strongly in $\mathrm{L}^2(\Omega)$. Using the fact that the function $t \mapsto u_t(\cdot, t)$ is weakly continuous in $\mathrm{L}^2(\Omega)$ and $t \mapsto \|u_t(\cdot, t)\|_{\mathrm{L}^2(\Omega)}^2$ is continuous (a proof of the latter statement in the context of general convex lower semi-continuous functionals can be found in [6, Proposition 4.2]), we conclude that both functions $t \mapsto u(\cdot, t)$, $t \mapsto u_t(\cdot, t)$ are continuous in $\mathrm{L}^2(\Omega)$, hence the initial conditions are meaningful.

It remains to check that ξ can be replaced with $\mathcal{P}[\nabla_s u]$ in (3.18). This will be done by a variant of the classical Minty trick. Testing (3.18) by u_t and integrating over $(0, \tau)$, we find

$$\begin{aligned}\frac{\rho}{2}\int_\Omega (|u_t(\cdot, \tau)|^2 - |u_t(\cdot, 0)|^2) \,\mathrm{d}x + \int_0^\tau \int_\Omega (\xi + \nu \nabla_s u_t) : \nabla_s u_t \,\mathrm{d}x \,\mathrm{d}t \\ + \int_0^\tau \int_{\Gamma_C} f[u] \cdot u_t \,\mathrm{d}S \,\mathrm{d}t = -\int_0^\tau \int_{\Gamma_E} p \cdot u_t \,\mathrm{d}S \,\mathrm{d}t.\end{aligned} \tag{3.19}$$

We integrate (3.14) over $(0, \tau)$, we obtain

$$\frac{\rho}{2}\int_\Omega (|u_t^{(m)}(\cdot,\tau)|^2 - |u_t^{(m)}(\cdot,0)|^2)\,\mathrm{d}x + \int_0^\tau\int_\Omega (\mathcal{P}[\nabla_s u^{(m)}] + \nu\nabla_s u_t^{(m)}) : \nabla_s u_t^{(m)}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau\int_{\Gamma_C} f[u^{(m)}]\cdot u_t^{(m)}\,\mathrm{d}S\,\mathrm{d}t = -\int_0^\tau\int_{\Gamma_E} p\cdot u_t^{(m)}\,\mathrm{d}S\,\mathrm{d}t. \tag{3.20}$$

We now pass to the limit in (3.20) as m tends to $+\infty$. The initial and boundary terms converge to the corresponding ones in (3.19) by virtue of (3.17). Consequently, for every $\tau \in [0, T]$, we have

$$\frac{\rho}{2}\int_\Omega |u_t(\cdot,\tau)|^2\,\mathrm{d}x + \int_0^\tau\int_\Omega (\xi + \nu\nabla_s u_t) : \nabla_s u_t\,\mathrm{d}x\,\mathrm{d}t$$
$$= \lim_{m\to\infty}\frac{\rho}{2}\int_\Omega |u_t^{(m)}(\cdot,0)|^2\,\mathrm{d}x + \int_0^\tau\int_\Omega (\mathcal{P}[\nabla_s u^{(m)}] + \nu\nabla_s u_t^{(m)}) : \nabla_s u_t^{(m)}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.21}$$

The weak convergence in (3.17) yields

$$\int_\Omega |u_t(\cdot,\tau)|^2\,\mathrm{d}x \le \liminf_{m\to+\infty}\int_\Omega |u_t^{(m)}(\cdot,\tau)|^2\,\mathrm{d}x,$$

and

$$\int_0^\tau\int_\Omega |\nabla_s u_t|^2\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{m\to+\infty}\int_0^\tau\int_\Omega |\nabla_s u_t^{(m)}|^2\,\mathrm{d}x\,\mathrm{d}t.$$

In particular, we have

$$\int_0^T\int_\Omega \xi : \nabla_s u_t\,\mathrm{d}x\,\mathrm{d}t \ge \liminf_{m\to+\infty}\int_0^T\int_\Omega \mathcal{P}[\nabla_s u^{(m)}] : \nabla_s u_t^{(m)}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.22}$$

For each v with $\nabla_s v_t \in \mathrm{L}^2(Q_T; \mathbb{T}_s^{d\times d})$, we have $\mathcal{P}[\nabla_s v] \in \mathrm{L}^2(\Omega; \mathrm{C}^0([0,T]; \mathbb{T}_s^{d\times d}))$. Indeed, for almost every $x \in \Omega$, the function $t \mapsto \mathcal{P}[\nabla_s v(x,t)]$ is continuous, and denoting $P_0(t) \overset{\text{def}}{=} \mathcal{P}[0](t) \in \mathrm{C}^0([0,T]; \mathbb{T}_s^{d\times d})$ (the image of the constant zero function), we obtain from (3.2) that

$$\int_\Omega |\mathcal{P}[\nabla_s v] - P_0|_{[0,T]}^2\,\mathrm{d}x \le 2L_{\mathcal{P}}^2\left(T\int_\Omega\int_0^T |\nabla_s v_t|^2\,\mathrm{d}t\,\mathrm{d}x + \int_\Omega |\nabla_s v(x,0)|^2\,\mathrm{d}x\right).$$

We now choose $v = u - \kappa t\omega_k$ with some constant κ and any k, and use (3.4) to obtain

$$\int_0^T\int_\Omega (\mathcal{P}[\nabla_s u^{(m)}] - \mathcal{P}[\nabla_s v]) : (\nabla_s u_t^{(m)} - \nabla_s v_t)\,\mathrm{d}x\,\mathrm{d}t \ge -\int_\Omega \widetilde{\mathcal{V}}[\nabla_s u^{(m)}, \nabla_s u](x,0)\,\mathrm{d}x. \tag{3.23}$$

By (3.5), (3.10), and (3.13), the right hand side of (3.23) converges to 0. For $m \to +\infty$ this yields by virtue of (3.22) that

$$\int_0^T\int_\Omega (\xi - \mathcal{P}[\nabla_s v]) : (\nabla_s u_t - \nabla_s v_t)\,\mathrm{d}x\,\mathrm{d}t \ge 0, \tag{3.24}$$

or, equivalently,

$$\kappa\int_0^T\int_\Omega (\xi - \mathcal{P}[\nabla_s u - \kappa t\nabla_s\omega_k]) : \nabla_s\omega_k\,\mathrm{d}x\,\mathrm{d}t \ge 0. \tag{3.25}$$

Choosing $\kappa > 0$ and $\kappa < 0$, letting κ tend to 0, and using (3.2), we conclude that

$$\int_0^T\int_\Omega (\xi - \mathcal{P}[\nabla_s u]) : \nabla_s\omega_k\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.26}$$

for all $k = 0, 1, 2, \ldots$. Comparing (3.26) with (3.18), we see that u is the desired solution of (2.8).

Assume now that u^1 and u^2 are two solutions with the prescribed regularity. We denote $\bar{u} \overset{\text{def}}{=} u^1 - u^2$, and test the difference of (2.8) by $\bar{u}_t$. We obtain

$$\int_\Omega (\rho\bar{u}_{tt}\cdot\bar{u}_t + (\mathcal{P}[\nabla_s u^1] - \mathcal{P}[\nabla_s u^2]) : \nabla_s\bar{u}_t + \nu|\nabla_s\bar{u}_t|^2)\,\mathrm{d}x = -\int_{\Gamma_C} (f[u^1] - f[u^2])\cdot\bar{u}_t\,\mathrm{d}S. \tag{3.27}$$

Using assumptions (A1) and (A4) produces

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho|\bar{u}_t|^2\,\mathrm{d}x+\int_{\Omega}\nu|\nabla_s\bar{u}_t|^2\,\mathrm{d}x\leq L_f\int_{\Gamma_C}|\bar{u}|_{[0,t]}\,|\bar{u}_t|\,\mathrm{d}S$$
$$\leq L_f\left(\tau\int_{\Gamma_C}\int_0^{\tau}|\bar{u}_t|^2\,\mathrm{d}t\,\mathrm{d}S\right)^{1/2}\left(\int_{\Gamma_C}|\bar{u}_t|^2\,\mathrm{d}S\right)^{1/2}. \tag{3.28}$$

We now use embedding inequality (3.16) and Gronwall's argument to conclude that $\bar{u}_t=0$, and the proof is complete. □

4. The 1D case

As mentioned earlier, in the 1D case, we can obtain the existence and uniqueness of solutions without the assumption that viscosity is present in the constitutive law under further assumptions on the contact boundary condition and on the regularity of the data.

We consider an elasto-plastic bar of length L which vibrates longitudinally. The bar is free to move on the one end as long as it does not hit a material obstacle, while on the other end a force is applied. Let $u(x,t)$ be the displacement at time t of the material point of spatial coordinate $x\in\Omega$ with $\Omega\overset{\mathrm{def}}{=}(0,L)$. The motion is governed by the equation

$$\rho u_{tt}-(\mathcal{P}[u_x])_x=0, \tag{4.1}$$

with Cauchy initial data

$$u(x,0)=u^0(x)\quad\text{and}\quad u_t(x,0)=v^0(x), \tag{4.2}$$

and boundary conditions at $x=0$ and $x=L$, $t>0$, given by

$$\mathcal{P}[u_x(0,\cdot)](t)=-p(t)\quad\text{and}\quad\mathcal{P}[u_x(L,\cdot)](t)=-f[u(L,\cdot)](t), \tag{4.3}$$

where $(\cdot)_x\overset{\mathrm{def}}{=}\frac{\partial(\cdot)}{\partial x}$ and $\mathcal{P}$ and f are the constitutive operator of elastoplasticity and the boundary contact operator, respectively. In order to simplify the notation, we have, indeed, reversed the roles of the boundary points with respect to (2.5), that is, contact with the obstacle takes place at $x=L$. For simplicity, we assume the canonical form of $\mathcal{P}$ defined by the scalar version of (2.3) that we write in the form

$$\mathcal{P}[\varepsilon]=\lambda\varepsilon+\mathcal{P}_0[\varepsilon], \tag{4.4}$$

where $\mathcal{P}_0$ is the solution operator $\sigma^p=\mathcal{P}_0[\varepsilon]$ of the scalar counterpart of (2.2), that is,

$$\begin{cases}\sigma^p(t)\in K & \text{for all } t\in[0,T],\\ \sigma^p(0)=\mathrm{Proj}_K(E\varepsilon(0)),\\ (E\varepsilon_t(t)-\sigma^p_t(t))(\sigma^p(t)-y)\geq 0 & \text{a.e. for all } y\in K,\end{cases} \tag{4.5}$$

where σ^p corresponds to the plastic stress component with yield point $r>0$ and elasticity domain $K\overset{\mathrm{def}}{=}[-r,r]$, the constant $E>0$ is the elasticity modulus, and $\lambda>0$ is the kinematic hardening modulus.

Again, for simplicity, we choose the boundary contact operator f in the form $f[u]=g(\mathcal{S}[u])$ as in Section 2, where $\mathcal{S}$ is the solution operator of (2.10), and g is a twice continuously differentiable nondecreasing function, which vanishes for negative arguments. The function $g(w)=w^+$, which appears in (2.19), does not allow to use the second order energy inequality which we need here, and has to be regularized. Physically, this corresponds to the existence of a thin contact layer with progressive elasticity modulus before full contact is established.

The initial value mappings $\widehat{\mathcal{P}}$ and $\widehat{f}$ from $\mathbb{R}$ to $\mathbb{R}$ associated with the operators $\mathcal{P}$ and f (cf. (3.6)–(3.7)) can be written explicitly here in the form

$$\widehat{\mathcal{P}}(\varepsilon)\overset{\mathrm{def}}{=}\lambda\varepsilon+\min\{r,\max\{-r,E\varepsilon\}\},\qquad\widehat{f}(u)\overset{\mathrm{def}}{=}g(\max\{au+c,bu\}). \tag{4.6}$$

Putting $u_N(t)=u(L,t)$, we rewrite (4.1)–(4.3) in variational form as

$$\int_{\Omega}(\rho u_{tt}\phi(x)+\mathcal{P}[u_x]\phi'(x))\,\mathrm{d}x+g(\mathcal{S}[u_N](t))\phi(L)=p(t)\phi(0). \tag{4.7}$$

We first establish some preliminary results that will be used in the proof of existence and uniqueness for (4.7) and concerning a higher order energy estimate (related to the convexity of hysteresis loops, cf. [16]). Here, we use a simplified version which follows from the monotone character of the variational inequalities (2.10) and (4.5).

Let $v_1,v_2\in W^{1,1}(0,T)$ be given, and let $w_i\overset{\mathrm{def}}{=}\mathcal{S}[v_i]$, $i=1,2$, be solutions of (2.10). Then we have

$$((bv_{1,t}-w_{1,t})-(bv_{2,t}-w_{2,t}))((w_1-av_1)-(w_2-av_2))\geq 0\quad\text{a.e.},$$

hence,

$$(w_1 - w_2)(v_{1,t} - v_{2,t}) \geq \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2(b-a)}((w_1 - av_1) - (w_2 - av_2))^2 + \frac{a}{2}(v_1 - v_2)^2\right) \quad \text{a.e.} \tag{4.8}$$

Similarly, if $\sigma_i = \mathcal{P}[\varepsilon_i]$, $i = 1, 2$, are as in (4.4), then

$$(\sigma_1 - \sigma_2)(\varepsilon_{1,t} - \varepsilon_{2,t}) \geq \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{2}(\varepsilon_1 - \varepsilon_2)^2 + \frac{1}{2E}(\mathcal{P}_0[\varepsilon_1] - \mathcal{P}_0[\varepsilon_2])^2\right) \quad \text{a.e.} \tag{4.9}$$

Lemma 4.1. *Let $v \in \mathrm{W}^{2,2}(0,T)$ be given, and let $w = \mathcal{S}[v]$ be as in (2.10). Then for every nonnegative test function $\psi \in \mathrm{W}^{1,2}(0,T)$ and for almost every $\tau \in (0,T)$, we have*

$$\int_0^\tau w_t v_{tt}\psi \,\mathrm{d}t + \frac{1}{2}\int_0^\tau \left(\frac{1}{b-a}(w_t - av_t)^2 + av_t^2\right)\psi_t \,\mathrm{d}t$$
$$\geq \frac{1}{2}\left(\frac{1}{b-a}(w_t - av_t)^2 + av_t^2\right)(\tau)\psi(\tau) - \left(2b - \frac{3a}{2}\right)v_t(0)^2\psi(0). \tag{4.10}$$

Proof. Let $\tau < T$ be such that $\lim_{h\to 0+} \frac{1}{h}(w(\tau+h) - w(\tau)) = w_t(\tau)$. We now use (4.8) with $v_1(\tau) = v(\tau)$ and $v_2(\tau) = v(\tau+h)$. Put $v_h(\tau) \stackrel{\text{def}}{=} \frac{1}{h}(v(\tau+h) - v(\tau))$ and $w_h(\tau) \stackrel{\text{def}}{=} \frac{1}{h}(w(\tau+h) - w(\tau))$. From (4.8) we obtain, integrating by parts, that

$$\int_0^\tau w_h v_{h,t}\psi \,\mathrm{d}t + \frac{1}{2}\int_0^\tau \left(\frac{1}{b-a}(w_h - av_h)^2 + av_h^2\right)\psi_t \,\mathrm{d}t$$
$$\geq \frac{1}{2}\left(\frac{1}{b-a}(w_h - av_h)^2 + av_h^2\right)(\tau)\psi(\tau) - \frac{1}{2}\left(\frac{1}{b-a}(w_h - av_h)^2 + av_h^2\right)(0)\psi(0). \tag{4.11}$$

We have

$$|(bv - w)(h) - (bv - w)(0)| \leq (b-a)\max_{t\in[0,h]} |v(t) - v(0)|,$$

hence,

$$|(av_h - w_h)(0)| \leq \frac{2}{h}(b-a)\max_{t\in[0,h]} |v(t) - v(0)|,$$

which yields

$$\limsup_{h\to 0+}\left(\frac{1}{b-a}(w_h - av_h)^2 + av_h^2\right)(0) \leq (4b - 3a)v_t(0)^2,$$

and the assertion is obtained from (4.11) when passing to the limit as $h \to 0+$. □

We have a similar result for the operator $\mathcal{P}$ in (4.4).

Lemma 4.2. *Let $\varepsilon \in \mathrm{W}^{2,2}(0,T)$ be given, and let $\sigma^p = \mathcal{P}_0[\varepsilon]$, $\sigma = \mathcal{P}[\varepsilon] = \lambda\varepsilon + \sigma^p$. Then for every nonnegative test function $\psi \in \mathrm{W}^{1,2}(0,T)$ and for almost every $\tau \in (0,T)$, we have*

$$\int_0^\tau \sigma_t \varepsilon_{tt}\psi \,\mathrm{d}t + \frac{1}{2}\int_0^\tau \left(\frac{1}{E}(\sigma_t^p)^2 + \lambda\varepsilon_t^2\right)\psi_t \,\mathrm{d}t$$
$$\geq \frac{1}{2}\left(\frac{1}{E}(\sigma_t^p)^2 + \lambda\varepsilon_t^2\right)(\tau)\psi(\tau) - \left(2E + \frac{\lambda}{2}\right)\varepsilon_t(0)^2\psi(0). \tag{4.12}$$

Proof. We proceed as in the proof of Lemma 4.1, using (4.9) and the inequality

$$|\sigma^p(h) - \sigma^p(0)| \leq 2E\max_{t\in[0,h]} |\varepsilon(t) - \varepsilon(0)|,$$

and the desired result follows. □

We are now ready to prove the main result of this section.

Theorem 4.3. *Assume that* (A1)–(A4) *hold, and let* $p \in W^{2,2}(0,T)$, $u^0 \in W^{2,2}(\Omega)$, *and* $v^0 \in W^{1,2}(\Omega)$ *be given. Assume that the compatibility conditions*

$$\widehat{\mathcal{P}}(u^0_x(0)) + p(0) = 0 \quad \text{and} \quad \widehat{\mathcal{P}}(u^0_x(L)) + \widehat{f}(u^0(L)) = 0$$

hold, where $\widehat{\mathcal{P}}$ *and* $\widehat{f}$ *are as in* (4.6). *Then there exists a unique* $u \in L^2(Q_T)$ *such that* u_{xt} *and* u_{tt} *belong to* $L^\infty(0,T;L^2(\Omega))$ *and satisfy* (4.7) *with initial conditions* $u(\cdot,0) = u^0$ *and* $u_t(\cdot,0) = v^0$ *almost everywhere.*

Proof. Once again we proceed by Galerkin approximations to prove the existence result. To this aim, we denote by $\{\omega_k : k = 0,1,\ldots\}$ the complete system of eigenfunctions $\omega_k \in W^{1,2}(\Omega)$, orthonormal in $L^2(\Omega)$, of the problem

$$\forall \phi \in W^{1,2}(\Omega): \int_\Omega \omega_k' \phi' \,dx = \lambda_k \int_\Omega \omega_k \phi \,dx. \tag{4.13}$$

For any integer $m > 0$, we define $u^{(m)}$ similarly as in (3.10), taking into account the compatibility conditions. More specifically, we define real numbers α and β by the formula

$$\alpha \overset{\text{def}}{=} \widehat{\mathcal{P}}^{-1}(-p(0)) \quad \text{and} \quad \beta \overset{\text{def}}{=} \frac{1}{2L}\big(\widehat{\mathcal{P}}^{-1}(-\widehat{f}(u^0(L))) - \widehat{\mathcal{P}}^{-1}(-p(0))\big). \tag{4.14}$$

This is indeed admissible, as by virtue of (4.6), the function $\widehat{\mathcal{P}}$ is Lipschitz continuously invertible. Put

$$\forall x \in [0,L]: \ \widehat{u}^0(x) \overset{\text{def}}{=} u^0(x) - \alpha x - \beta x^2. \tag{4.15}$$

The motivation for this choice is to guarantee that $\widehat{u}^0$ satisfies the homogeneous Neumann boundary conditions $\widehat{u}^0_x(0) = \widehat{u}^0_x(L) = 0$. We define the Fourier coefficients of $\widehat{u}^0$ by

$$\widehat{u}_j = \int_\Omega \widehat{u}^0(x) w_j(x)\,dx \quad \text{for } j = 0,1,\ldots.$$

Let us observe that for all $x \in \Omega$, the series $\widehat{u}^0(x) = \sum_{j=0}^{+\infty} \widehat{u}_j \omega_j(x)$ and $\widehat{u}^0_x(x) = \sum_{j=1}^{+\infty} \widehat{u}_j \omega_j'(x)$ are uniformly convergent and $\sum_{j=0}^{+\infty} \lambda_j^2 \widehat{u}_j^2 = \|\widehat{u}^0_{xx}\|^2_{L^2(\Omega)} < +\infty$. Formula (3.10) is replaced here by

$$u^{(m)}(x,t) \overset{\text{def}}{=} \sum_{k=0}^{m} u_k(t)\omega_k(x) + \alpha x + \beta x^2, \tag{4.16}$$

with functions $u_k \in W^{2,2}(0,T)$ satisfying the system

$$\int_\Omega (\rho u^{(m)}_{tt}\omega_k + \mathcal{P}[u^{(m)}_x]\omega_k')(x,t)\,dx + g(\mathcal{S}[u^{(m)}_N](t))\omega_k(L) = p(t)\omega_k(0), \tag{4.17}$$

with initial conditions $u_k(0) = \widehat{u}_k$ and $u_{k,t}(0) = \int_\Omega v^0 \omega_k\,dx$, for every $k = 0,1,\ldots,m$, where $u^{(m)}_N(t) = u^{(m)}(L,t)$. The existence of a unique solution to (4.17) is again obvious.

The energy estimate is obtained by testing (4.17) with $u_{k,t}$ and summing over $k = 0,1,\ldots,m$, we get

$$\int_\Omega (\rho u^{(m)}_{tt} u^{(m)}_t + \mathcal{P}[u^{(m)}_x] u^{(m)}_{xt})\,dx + g(\mathcal{S}[u^{(m)}_N](t)) u^{(m)}_{N,t}(t) = p(t) u^{(m)}_t(0,t). \tag{4.18}$$

Note that the inequality (3.3) holds with $\mathcal{V}[\varepsilon] = \frac{\lambda}{2}\varepsilon^2 + \frac{1}{2E}\mathcal{P}_0[\varepsilon]^2$ as in (4.9), and

$$g(\mathcal{S}[v](t))v_t - \frac{d}{dt}e[v] \geq 0 \tag{4.19}$$

holds with $e[v] = \frac{1}{b}\big(G(w) + \frac{b-a}{a} G\big(\frac{a}{b-a}(bv - w)\big)\big)$ as in (2.27). We integrate (3.14) over $(0,\tau)$, thus according to (4.19), we may deduce that there exists $C_1 > 0$ independent of m such that

$$\frac{1}{2}\int_\Omega (\rho |u^{(m)}_t|^2 + \lambda |u^{(m)}_x|^2)(x,\tau)\,dx \leq C_1 + p(\tau)u^{(m)}(0,\tau) - \int_0^\tau p_t(t) u^{(m)}(0,t)\,dt. \tag{4.20}$$

It follows that there exists $C_2 > 0$ independent of m such that

$$\rho\|u_t^{(m)}(\cdot,\tau)\|^2_{L^2(\Omega)} + \lambda\|u_x^{(m)}(\cdot,\tau)\|^2_{L^2(\Omega)} \le C_2. \tag{4.21}$$

We now differentiate (4.17) with respect to t and we test by $u_{k,tt}(t)$ to get

$$\int_\Omega (\rho u_{ttt}^{(m)} u_{tt}^{(m)} + \mathcal{P}[u_x^{(m)}]_t u_{xtt}^{(m)})(x,t)\,dx + g(\mathcal{S}[u_N^{(m)}](t))_t u_{N,tt}^{(m)}(t) = p_t(t) u_{tt}^{(m)}(0,t). \tag{4.22}$$

We integrate (4.22) over $(0,\tau)$, it comes by using Lemma 4.2 with $\psi \equiv 1$, and Lemma 4.1 with $v = u_N^{(m)}$, $w = \mathcal{S}[v]$ and $\psi = g'(w)$, together with the elementary inequality $|w_t(t)| \le b|v_t(t)|$, that there exists $C_3 > 0$ independent of m such that

$$\begin{aligned}\frac{1}{2}\int_\Omega (\rho|u_{tt}^{(m)}|^2 + \lambda|u_{xt}^{(m)}|^2)(x,\tau)\,dx \le{}& \frac{1}{2}\int_\Omega (\rho|u_{tt}^{(m)}|^2 + \lambda|u_{xt}^{(m)}|^2)(x,0)\,dx \\ &+ C_3\int_0^\tau (1 + |u_t^{(m)}(L,t)|^3)\,dt + p_t(\tau)u_t^{(m)}(0,\tau) \\ &- \int_0^\tau p_{tt}(t)u_t^{(m)}(0,t)\,dt. \end{aligned}\tag{4.23}$$

We observe that

$$\|u_{xt}^{(m)}(\cdot,0)\|_{L^2(\Omega)} \le \|v_x^0\|_{L^2(\Omega)}, \tag{4.24}$$

$$\|u_{tt}^{(m)}(\cdot,0)\|_{L^2(\Omega)} \le C_4(|p(0)| + \|u^0\|_{L^2(\Omega)} + \|u_{xx}^0\|_{L^2(\Omega)}), \tag{4.25}$$

with a constant $C_4 > 0$ independent of m. Note that (4.25) can be derived from (4.17) and from the compatibility conditions in the following way. We first rewrite (4.17) for $t = 0$ after integration by parts as

$$\rho u_{k,tt}(0) = \int_\Omega [\widehat{\mathcal{P}}(u_x^{(m)}(x,0))]_x \omega_k(x)\,dx - [\widehat{\mathcal{P}}(u_x^{(m)}(x,0))\omega_k]_0^L - \widehat{f}(u^{(m)}(L,0))\omega_k(L) + p(0)\omega_k(0). \tag{4.26}$$

We have by (4.13) that $\omega_k'(0) = \omega_k'(L) = 0$ for all $k = 0,1,\ldots,$ hence $u_x^{(m)}(0,0) = \widehat{\mathcal{P}}^{-1}(-p(0)) = \alpha$ and $u_x^{(m)}(L,0) = \alpha + 2L\beta = \widehat{\mathcal{P}}^{-1}(-\widehat{f}(u^0(L)))$, see (4.14). In particular, we have

$$-\widehat{\mathcal{P}}(u_x^{(m)}(L,0))\omega_k(L) - \widehat{f}(u^{(m)}(L,0))\omega_k(L) = (\widehat{f}(u^0(L)) - \widehat{f}(u^{(m)}(L,0)))\omega_k(L),$$
$$\widehat{\mathcal{P}}(u_x^{(m)}(0,0))\omega_k(0) + p(0)\omega_k(0) = 0.$$

Then it follows by using (4.15) and (4.16) that

$$\begin{aligned}\rho|u_{k,tt}(0)| &\le \left|\int_\Omega [\widehat{\mathcal{P}}(u_x^{(m)}(x,0))]_x\omega_k(x)\,dx\right| + |(\widehat{f}(u^0(L)) - \widehat{f}(u^{(m)}(L,0)))\omega_k(L)| \\ &\le \left|\int_\Omega [\widehat{\mathcal{P}}(u_x^{(m)}(x,0))]_x\omega_k(x)\,dx\right| + L_f\left|\widehat{u}^0(L) - \sum_{j=0}^m \widehat{u}_j w_j(L)\right|. \end{aligned}\tag{4.27}$$

We estimate now the two terms on the right hand side of (4.27). On the one hand, we define $\widehat{v}^{(m)}(x) \stackrel{\text{def}}{=} [\widehat{\mathcal{P}}(u_x^{(m)}(x,0))]_x$ and $\widehat{v}_k \stackrel{\text{def}}{=} \int_\Omega \widehat{v}^{(m)}(x)\omega_k(x)\,dx$ for $k = 0,1,2,\ldots$. Clearly, we have

$$\sum_{k=0}^m |\widehat{v}_k|^2 \le \sum_{k=0}^{+\infty} |\widehat{v}_k|^2 = \|\widehat{v}^m\|^2_{L^2(\Omega)} \le L^2_{\mathcal{P}}\|u_{xx}^{(m)}(\cdot,0)\|^2_{L^2(\Omega)}. \tag{4.28}$$

Furthermore we observe that $u_{xx}^{(m)}(\cdot,0) = -\sum_{k=0}^m \widehat{u}_k\lambda_k\omega_k + 2\beta$, then

$$\|u_{xx}^{(m)}(\cdot,0)\|_{L^2(\Omega)} \le \left\|\sum_{k=0}^{+\infty}\widehat{u}_k\lambda_k\omega_k\right\|_{L^2(\Omega)} + 2|\beta|L \le \|u_{xx}^0\|_{L^2(\Omega)} + 4|\beta|L.$$

Therefore by employing (4.14), we may infer that there exists $C_5 > 0$ such that

$$\|u_{xx}^{(m)}(\cdot,0)\|_{L^2(\Omega)} \le C_5(|p(0)| + \|u^0\|_{L^2(\Omega)} + \|u_{xx}^0\|_{L^2(\Omega)}). \tag{4.29}$$

On the other hand, there exists $C_\omega > 0$ such that $\omega_k(x) \le C_\omega$ for all k and all x. Hence by using Cauchy–Schwarz's inequality, we find

$$\left|\left(\widehat{u}^0(L) - \sum_{j=0}^{m} \widehat{u}_j \omega_j(L)\right)\omega_k(L)\right| \le C_\omega^2 \sum_{j=m+1}^{+\infty} |\widehat{u}_j|$$
$$\le C_\omega^2 \left(\sum_{j=m+1}^{+\infty} \lambda_j^2 \widehat{u}_j^2\right)^{1/2} \left(\sum_{j=m+1}^{+\infty} \frac{1}{\lambda_j^2}\right)^{1/2} \le C_\omega^2 \|\widehat{u}_{xx}^0\|_{L^2(\Omega)} \left(\sum_{j=m+1}^{+\infty} \frac{1}{\lambda_j^2}\right)^{1/2},$$

and the estimate for $\|u_{tt}^{(m)}(\cdot,0)\|_{L^2(\Omega)}^2 = \sum_{k=0}^m |u_{k,tt}(0)|^2$ in (4.25) follows from the facts that $\lim_{m\to+\infty} m \sum_{j=m+1}^{+\infty} \frac{1}{\lambda_j^2} = 0$ and (4.29) holds.

The remaining terms on the right hand side of (4.23) are estimated using the identity

$$|u_t^{(m)}(L,t)|^2 - |u_t^{(m)}(x,t)|^2 = 2\int_x^L u_t^{(m)}(x',t)u_{xt}^{(m)}(x',t)\,dx'. \tag{4.30}$$

Hence integrating again over x and using Cauchy–Schwarz's inequality, we deduce that there exists $C_6 > 0$ such that

$$|u_t^{(m)}(L,t)|^2 \le \frac{1}{L}\|u_t^{(m)}(\cdot,t)\|_{L^2(\Omega)}^2 + 2\|u_t^{(m)}(\cdot,t)\|_{L^2(\Omega)}\|u_{xt}^{(m)}(\cdot,t)\|_{L^2(\Omega)}$$
$$\le C_6(1 + \|u_{xt}^{(m)}(\cdot,t)\|_{L^2(\Omega)}) \tag{4.31}$$

as a consequence of (4.21). The left hand side of (4.23) is thus dominating the right hand side, it follows that there exists $C_7 > 0$ such that

$$\forall \tau \in [0,T]: \ \rho\|u_{tt}^{(m)}(\cdot,\tau)\|_{L^2(\Omega)}^2 + \lambda\|u_{xt}^{(m)}(\cdot,\tau)\|_{L^2(\Omega)}^2 \le C_7. \tag{4.32}$$

We now proceed as in the proof of Theorem 3.1, passing to the limit and using the Minty trick, to construct a solution with the regularity as in Theorem 4.3. We leave the details to the reader. Let us just point out that the stresses $\sigma^{(m)} = \mathcal{P}[u_x^{(m)}]$ do not converge uniformly, and the boundary conditions are satisfied only in the limit as m tends to $+\infty$.

It remains to check that the solution is unique. Assume that u^1 and u^2 are two solutions of (4.7) with the same initial conditions, and with the properties as in Theorem 4.3. In particular, u_t^1 and u_t^2 are uniformly bounded in $\bar{Q}_T$. We subtract (4.7) for u^1 and u^2, denote by $\bar{u} = u^1 - u^2$ and $\bar{u}_N = u_N^1 - u_N^2$, and test with $\bar{u}_t$ to obtain

$$\int_\Omega (\rho\bar{u}_{tt}\bar{u}_t + (\mathcal{P}[u_x^1] - \mathcal{P}[u_x^2])\bar{u}_{xt})\,dx + (g(\mathcal{S}[u_N^1](t)) - g(\mathcal{S}[u_N^2](t)))\bar{u}_{N,t}(t) = 0. \tag{4.33}$$

We integrate (4.33) over $(0,\tau)$, and using (4.9), we get

$$\frac{1}{2}\int_\Omega (\rho\bar{u}_t^2 + \lambda\bar{u}_x^2)(\cdot,\tau)\,dx + \int_0^\tau (g(\mathcal{S}[u_N^1]) - g(\mathcal{S}[u_N^2]))(t)\bar{u}_{N,t}(t)\,dt \le 0. \tag{4.34}$$

Put $w_N^1 = \mathcal{S}[u_N^1]$, $w_N^2 = \mathcal{S}[u_N^2]$, and $\bar{w}_N = w_N^1 - w_N^2$. We have

$$\int_0^\tau (g(\mathcal{S}[u_N^1]) - g(\mathcal{S}[u_N^2]))(t)\bar{u}_{N,t}(t)\,dt = (g(w_N^1) - g(w_N^2))(\tau)\bar{u}_N(\tau) - g'(w_N^1)(\tau)\bar{w}_N(\tau)\bar{u}_N(\tau)$$
$$+ \int_0^\tau g'(w_N^1)(t)\bar{w}_N(t)\bar{u}_{N,t}(t)\,dt$$
$$- \int_0^\tau (g'(w_N^1) - g'(w_N^2))(t)w_{N,t}^2(t)\bar{u}_N(t)\,dt$$
$$+ \int_0^\tau g''(w_N^1)(t)w_{N,t}^1(t)\bar{w}_N(t)\bar{u}_N(t)\,dt.$$

In the term $\int_0^\tau g'(w_N^1)(t)\bar{w}_N(t)\bar{u}_{N,t}(t)\,dt$ we use inequality (4.8), integrate by parts, and using the L^∞-boundedness of $w_{N,t}^1$ and $w_{N,t}^2$, we conclude from (2.12) that there exists $C_8 > 0$ such that

$$\int_0^\tau (g(\mathcal{S}[u_N^1]) - g(\mathcal{S}[u_N^2]))(t)\bar{u}_{N,t}(t)\,dt \ge -C_8\,|\bar{u}_N|_{[0,\tau]}^2$$

for all $\tau \in [0,T]$. An analogous argument to (4.30) leads to

$$\frac{\rho}{2}\|\bar{u}_t(\cdot,\tau)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2}\|\bar{u}_x(\cdot,\tau)\|_{L^2(\Omega)}^2 \le C_8\,|\bar{u}_N|_{[0,\tau]}^2 \le C_\delta \left|\|\bar{u}\|_{L^2(\Omega)}^2\right|_{[0,\tau]} + \delta\left|\|\bar{u}_x\|_{L^2(\Omega)}^2\right|_{[0,\tau]} \tag{4.35}$$

for all $\tau \in [0, T]$, with any (small) constant $\delta > 0$ and some suitable $C_\delta > 0$. Choosing $\delta < \lambda/2$, we obtain from Gronwall's lemma that $\bar{u} = 0$, which completes the proof. □

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