

# Stability issues in complete fluid systems

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# Basic field equations

## Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

## Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

## Internal energy balance

$$\partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \operatorname{div}_x \mathbf{u}$$

# Constitutive relations

## Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}$$

## Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

## Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left( \frac{1}{\varrho} \right)$$

## Thermodynamics stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

# Local well posedness

## Initial data

$$\varrho(0, \cdot) = \varrho_0 > 0, \vartheta(0, \cdot) = \vartheta_0 > 0, \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

## Regularity

$$\varrho, \vartheta, \mathbf{u} \in W^{m,2}, m \geq 3$$

## Local existence for viscous fluids - Navier-Stokes-Fourier system

A. Valli, W.Zajaczkowski [1982] - local existence for large data,  
A.Matsumura, T.Nishida [1980,1983] - global existence for small data

## Local existence for ideal (inviscid) fluids - Euler-Fourier system

T. Alazard [2006] - local existence for large data

# Several “equivalent” forms of energy balance

## Internal energy balance

$$\partial_t(\rho e) + \operatorname{div}_x(\rho e \mathbf{u}) + \operatorname{div}_x \mathbf{q} = \boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \boxed{p \operatorname{div}_x \mathbf{u}}$$

## Entropy production

$$\partial_t(\rho s) + \operatorname{div}_x(\rho s \mathbf{u}) + \operatorname{div}_x \left( \frac{\mathbf{q}}{\vartheta} \right) \equiv \frac{1}{\vartheta} \left( \boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

## Total energy balance

$$\begin{aligned} \partial_t \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e \right) + \operatorname{div}_x \left[ \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e \right) \mathbf{u} + p \mathbf{u} \right] + \operatorname{div}_x \mathbf{q} \\ = - \boxed{\operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u})} \end{aligned}$$

# Weak formulation

## Second law - entropy inequality

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x \left( \frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left( \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

## First law - total energy balance

$$\partial_t \int \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) dx = 0$$

# Relative entropy (energy)

## Relative entropy functional

$$\begin{aligned} & \mathcal{E}(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \\ &= \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right) dx \end{aligned}$$

## Ballistic free energy

$$H_{\Theta}(\varrho, \vartheta) = \varrho \left( e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \right)$$

## Coercivity of the ballistic free energy

$$\varrho \mapsto H_{\Theta}(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_{\Theta}(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

# Dissipative solutions

## Relative entropy inequality

$$\begin{aligned} & \left[ \mathcal{E}(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \right]_{t=0}^{\tau} \\ & + \int_0^{\tau} \int_{\Omega} \frac{\Theta}{\vartheta} \left( \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) dx dt \\ & \leq \int_0^{\tau} \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) dt \end{aligned}$$

for any  $r > 0$ ,  $\Theta > 0$ ,  $\mathbf{U}$  satisfying relevant boundary conditions



# Remainder

$$\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})$$

$$\begin{aligned} &= \int_{\Omega} \left( \varrho \left( \partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) dx \\ &+ \int_{\Omega} \left[ \left( p(r, \Theta) - p(\varrho, \vartheta) \right) \operatorname{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] dx \\ &- \int_{\Omega} \left( \varrho \left( s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta + \varrho \left( s(\varrho, \vartheta) - s(r, \Theta) \right) \mathbf{u} \cdot \nabla_x \Theta \right. \\ &\quad \left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) dx \\ &+ \int_{\Omega} \frac{r - \varrho}{r} \left( \partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \right) dx \end{aligned}$$

# Weak solutions - summary

## Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes-Fourier system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

## Compatibility

Regular weak solutions are strong solutions

## Weak-strong uniqueness

Weak and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

# Conditional regularity

## Sufficient condition for regularity

Suppose that a dissipative weak solution to the Navier-Stokes-Fourier system emanating from regular initial data satisfies

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0, T) \times \Omega)} < \infty.$$

Then the solution is regular in  $(0, T)$ .

# Euler-Fourier system

## Mass conservation

$$\partial_t \rho + \operatorname{div}_x(\rho \mathbf{u}) = 0$$

## Momentum balance

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\rho \vartheta) = 0$$

## Internal energy balance

$$\frac{3}{2} \left[ \partial_t(\rho \vartheta) + \operatorname{div}_x(\rho \vartheta \mathbf{u}) \right] - \Delta \vartheta = -\rho \vartheta \operatorname{div}_x \mathbf{u}$$

# Existence of weak solutions

## Initial data

$$\varrho_0, \vartheta_0, \mathbf{u}_0 \in C^3, \varrho_0 > 0, \vartheta_0 > 0$$

## Global existence

For any (smooth) initial data  $\varrho_0, \vartheta_0, \mathbf{u}_0$  the Euler-Fourier system admits infinitely many weak solutions on a given time interval  $(0, T)$

## Regularity class

$$\varrho \in C^2, \partial_t \vartheta, \nabla_x^2 \vartheta \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\text{weak}}([0, T]; L^2) \cap L^\infty, \operatorname{div}_x \mathbf{u} \in C^1$$

# Application of the convex integration method

## Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \quad \operatorname{div}_x \mathbf{v} = 0$$

## Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left( \frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \left( \partial_t (\varrho \vartheta) + \operatorname{div}_x (\vartheta (\mathbf{v} + \nabla_x \Psi)) \right) - \Delta \vartheta = -\varrho \vartheta \operatorname{div}_x \left( \frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

## “Energy”

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}]$$

# Dissipative solutions to the Euler-Fourier system

## Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \varrho_0 > 0, \vartheta_0 > 0$$

## Infinitely many dissipative weak solutions

For any regular initial data  $\varrho_0, \vartheta_0$ , there exists a velocity field  $\mathbf{u}_0$  such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in  $(0, T)$