Mathematical theory of compressible, (viscous), and heat conducting fluids

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Differential Equations, function spaces, approximation theory
Novosibirsk, 19 August - 24 August, 2013

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Basic field equations

State variables

$\varrho = \varrho(t,x)$ .... mass density
$\mathbf{u} = \mathbf{u}(t,x)$ .... velocity field
$\vartheta = \vartheta(t,x)$ .... absolute temperature

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \boxed{\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})}$$

Internal energy balance

$$\partial_t (\varrho e(\varrho, \vartheta)) + \mathrm{div}_x (\varrho e(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

Constitutive relations

Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \boxed{\mu \geq 0}, \ \eta \geq 0$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \ \boxed{\kappa > 0}$$

Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamics stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Local well posedness

Initial data

$$\varrho(0,\cdot) = \varrho_0 > 0,\ \vartheta(0,\cdot) = \vartheta_0 > 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

Regularity

$$\varrho,\ \vartheta,\ \mathbf{u} \in W^{m,2},\ m \geq 3$$

Local existence for viscous fluids - Navier-Stokes-Fourier system

A. Valli, W.Zajaczkowski [1982] - local existence for large data, A.Matsumura, T.Nishida [1980,1983] - global existence for small data

Local existence for ideal (inviscid) fluids - Euler-Fourier system

T. Alazard [2006] - local existence for large data

Several “equivalent” forms of energy balance

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \boxed{p \mathrm{div}_x \mathbf{u}}$$

Entropy production

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) \boxed{=} \frac{1}{\vartheta}\left(\boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Total energy balance

$$\partial_t \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e\right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e\right)\mathbf{u} + p\mathbf{u}\right] + \mathrm{div}_x \mathbf{q}$$

$$= -\boxed{\mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u})}$$

Weak formulation

Second law - entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \boxed{\geq} \frac{1}{\vartheta}\left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

First law - total energy balance

$$\partial_t \int \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right) \ \mathrm{d}x = 0$$

Relative entropy (energy)

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U}\right)$$

$$= \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right) \, \mathrm{d}x$$

Ballistic free energy

$$H_{\Theta}(\varrho, \vartheta) = \varrho \Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_{\Theta}(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_{\Theta}(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \, \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Weak solutions - summary

Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes-Fourier system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

Compatibility

Regular weak solutions are strong solutions

Weak-strong uniqueness

Weak and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

Conditional regularity

Sufficient condition for regularity

Suppose that a dissipative weak solution to the Navier-Stokes-Fourier system emanating from regular initial data satisfies

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} < \infty.$$

Then the solution is regular in $(0, T)$.

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\mathbf{u}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta\ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0, T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Application of the convex integration method

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi,\ \operatorname{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \Big(\partial_t (\varrho \vartheta) + \operatorname{div}_x \big(\vartheta (\mathbf{v} + \nabla_x \Psi) \big) \Big) - \Delta \vartheta = -\varrho \vartheta \operatorname{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

"Energy"

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}]$$

Dissipative solutions to the Euler-Fourier system

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$