LOCALLY REGULAR GRAPHS

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(Received November 11, 1998)

Abstract. A graph G is called locally s-regular if the neighbourhood of each vertex of G induces a subgraph of G which is regular of degree s. We study graphs which are locally s-regular and simultaneously regular of degree r.

Keywords: regular graph, locally regular graph

MSC 2000: 05C75

At the Czechoslovak Symposium on Graph Theory in Smolenice in 1963 [1] A. A. Zykov suggested the problem to characterize graphs H with the property that there exists a graph G in which the neighbourhood of any vertex induces a subgraph isomorphic to H. This problem inspired many mathematical works and led also to a certain generalization, namely the study of local properties of graphs. A graph G is said to have locally a property P, if the neighbourhood of each vertex of G induces a subgraph having the property P. For locally connected graphs let us mention e.g. [2] and [4], for locally linear graphs e.g. [3]. A survey paper on local properties of graphs was written by J. Sedláček [5].

Here we will study locally s-regular graphs. A graph G is called locally s-regular, where s is a non-negative integer, if the neighbourhoods of all vertices of G induce subgraphs which are regular of degree s, shortly s-regular. We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$, the complement of G by $\overline{G}$. If $A \subseteq V(G)$, then $G\langle A\rangle$ is the subgraph of G induced by A. The symbol $G_1 + G_2$ denotes the disjoint union of two graphs G_1, G_2; the symbol $G_1 \oplus G_2$ denotes the Zykov sum of G_1 and G_2, i.e. the graph obtained from $G_1 + G_2$ by joining each vertex of G_1 with each vertex of G_2 by an edge. By $G_1 \times G_2$ the Cartesian product of G_1 and G_2 is denoted; its vertex set is $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2), (v_1, v_2) are adjacent in it if and only if either $u_1 = v_1$ and u_2, v_2 are adjacent in G_2, or u_1, v_1 are adjacent in G_1 and

$u_2 = v_2$. The symbol $N_G(v)$ denotes the (open) neighbourhood of a vertex v in a graph G, i.e. the set of all vertices which are adjacent to v in G. By C_n we denote the circuit of length n.

By $\operatorname{Locreg}(r, s)$, where r is a positive integer and s a non-negative integer, we denote the class of graphs which are simultaneously r-regular and locally s-regular.

Proposition 1. *If* $\operatorname{Locreg}(r, s) \neq \emptyset$, *then* $r \geqslant s+1$ *and at least one of the numbers* r, s *is even.*

This assertion is evident, because the conditions mentioned are the well-known necessary conditions for the existence of an s*-regular graph with* r *vertices.*

Proposition 2. *Let* s, k *be positive integers, let* k *be a divisor of* s. *Then* $\operatorname{Locreg}(s + k, s) \neq \emptyset$.

P r o o f. Let G be the complement of the disjoint union of $s + 2$ copies of the complete graph K_k with k vertices. Then $G \in \operatorname{Locreg}(s + k, s)$. □

Corollary 1. $\operatorname{Locreg}(s + 1, s) \neq \emptyset$ *for each integer* $s \geqslant 0$.

Corollary 2. $\operatorname{Locreg}(s + 2, s) \neq \emptyset$ *for each even integer* $s \geqslant 0$.

Proposition 3. *If* $\operatorname{Locreg}(r_1, s) \neq \emptyset$ *and* $\operatorname{Locreg}(r_2, s) \neq \emptyset$, *then also* $\operatorname{Locreg}(r_1 + r_2, s) \neq \emptyset$.

P r o o f. If $G_1 \in \operatorname{Locreg}(r_1, s)$ and $G_2 \in \operatorname{Locreg}(r_2, s)$, then the Cartesian product $G_1 \times G_2 \in \operatorname{Locreg}(r_1 + r_2, s)$. □

Now we state two lemmas.

Lemma 1. *Let* p, q *be positive integers such that* $q \geqslant p^2 - 1$. *Then there exist non-negative integers* a, b *such that*

$$q = ap + b(p + 1).$$

P r o o f. Let b be an integer such that $0 \leqslant b \leqslant p - 1$ and $b \equiv q (\operatorname{mod} p)$. Let $a = (q - b)/p - b$. We have $ap + b(p + 1) = q$. The number a is an integer, because $q \equiv b (\operatorname{mod} p)$. Further we have $q \geqslant p^2 - 1 = (p + 1)(p - 1) \geqslant b(p + 1)$ and thus $a = (q - b)/p - b \geqslant (b(p + 1) - b)/p - b = 0$. This proves the assertion. □

Lemma 2. *Let* p, q *be positive integers such that* $q \geqslant (2p - 1)(p - 1)$. *Then there exist non-negative integers* a, b *such that*

$$q = ap + b(2p - 1).$$

P r o o f. Let b be an integer such that $0 \leqslant b \leqslant p-1$ and $b+q \equiv 0 (\mathrm{mod}\ p)$. Let $a = (q+b)/p - 2b$. The proof that a is a non-negative integer is analogous to the proof of Lemma 1. $\square$

Now we prove some theorems.

Theorem 1. *Let r, s be positive integers such that s is even and $r \geqslant s(s+2)$. Then* $\mathrm{Locreg}(r,s) \neq \emptyset$.

P r o o f. According to Lemma 1 there exist non-negative integers a, b such that $r = a(s+1) + b(s+2)$. Then the assertion follows from the Corollaries 1 and 2 and from Proposition 3. $\square$

Theorem 2. *Let r, s be positive integers such that r is even, s is odd and $r \geqslant s(s-1)$. Then* $\mathrm{Locreg}(r,s) \neq \emptyset$.

P r o o f. Put $p = \frac{1}{2}(s = 1)$; then $\frac{1}{2}r \geqslant (2p-1)(p-1)$ and, as r is even, according to Lemma 2 there exist non-negative integers a, b such that $\frac{1}{2}r = ap + b(2p-1) = \frac{1}{2}a(s+1) + bs$ and thus $r = a(s+1) + 2bs$. According to Proposition 2 we have $\mathrm{Locreg}(s+1, s) \neq \emptyset$ and $\mathrm{Locreg}(2s, s) \neq 0$ and thus, by Proposition 3, also $\mathrm{Locreg}(r,s) \neq \emptyset$. $\square$

Now we turn our attention to small values of s.

Proposition 4. *Let $0 \leqslant s \leqslant 2$ and $r \geqslant s+1$ and in the case of $s = 1$ let r be even. Then* $\mathrm{Locreg}(r,s) \neq \emptyset$.

P r o o f. A graph from $\mathrm{Locreg}(r,0)$ is an arbitrary r-regular graph without triangles, e.g. the complete bipartite graph $K_{r,r}$. A graph from $\mathrm{Locreg}(2,1)$ is C_3 and $\mathrm{Locreg}(n,1) \neq \emptyset$ follows from Theorem 1. Examples of graphs from $\mathrm{Locreg}(3,2)$, $\mathrm{Locreg}(4,2)$ and $\mathrm{Locreg}(5,2)$ are successively the graphs of regular polyhedra tetrahedron, octahedron, icosahedron. Every $s \geqslant 6$ is a sum of numbers from $\{3,4,5\}$ and thus Proposition 3 implies $\mathrm{Locreg}(r,2)$ for every $r \geqslant 6$. $\square$

From these results it may seem that $\mathrm{Locreg}(r,s) \neq \emptyset$ for any r, s which satisfy the condition of Proposition 1. We will show an example for which this is not true.

Theorem 3. *The class* $\mathrm{Locreg}(7,4) = \emptyset$.

P r o o f. Suppose the contrary and let $G \in \mathrm{Locreg}(7,4)$. Let u be a vertex of G. The graph $G\langle N_G(u)\rangle$ is a 4-regular graph with seven vertices. Its complement is a 2-regular graph and therefore it is isomorphic either to $C_3 + C_4$, or to C_1. Hence $G\langle N_g(u)\rangle \cong \overline{C}_3 \oplus \overline{C}_4$ or $G\langle N_G(u)\rangle \cong \overline{C}_1$. Suppose the first case occurs. Denote the vertices of $G\langle N_G(u)\rangle$ by $v_1, v_2, v_3, w_1, w_2, w_3, w_4$ so that

$v_1v_2, v_2v_3, v_3v_1, w_1w_2, w_2w_3, w_3w_4, w_4w_1$ are edges of the complement of $G\langle N_a(u)\rangle$. Consider the graph $G\langle N_G(v_1)\rangle$. It contains the graph $\langle\{w_1, w_2, w_3, w_4\}\rangle \cong \overline{C}_4$ as an induced subgraph and therefore it cannot be isomorphic to $\overline{C}_7$. We have $G\langle N_G(v_1)\rangle \cong \overline{C}_3 \oplus \overline{C}_4$ and there exist vertices x_1, x_2, x_3 outside $N_G(u)$ which are pairwise non-adjacent and each of them is adjacent to v_1, w_1, w_2, w_3, w_4. (One of them is u.) But now $G\langle N_G(w_1)\rangle$ contains two disjoint independent triples $\{v_1, v_2, v_3\}$, $\{x_1, x_2, x_3\}$ and hence it is isomorphic neither to $\overline{C}_3 \oplus \overline{C}_4$ nor to $\overline{C}_7$, which is a contradiction. As u was chosen arbitrarily, we have proved that the neighbourhood of any vertex of G cannot induce $\overline{C}_3 \oplus \overline{C}_4$ and thus it must induce $\overline{C}_7$.

Thus let $G\langle N_G(u)\rangle \cong \overline{C}_7$. The vertices of $G\langle N_G(u)\rangle$ will be denoted by $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ in such a way that v_iv_{i+1} for $i = 1, \ldots, 7$ are edges of the complement of $G\langle N_G(u)\rangle$; here and everywhere in the sequel the subscripts are taken modulo 7. Consider the graph $G\langle N_G(v_i)\rangle$ for an arbitrary $i \in \{1, \ldots, 7\}$. It contains the vertices u, $v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}$ and does not contain v_{i+1} and v_{i+6}. As $G\langle N_G(v_i)\rangle \cong \overline{C}_7$, it contains vertices w_i, x_i outside $N_6(u)$ such that w_i is adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}, x_i$ and non-adjacent to u, v_{i+2} while x_i is adjacent to $v_i, v_{i+2}, v_{i+3}, v_{i+4}, w_i$ and non-adjacent to u, v_{i+5}. Consider the vertices w_i for $i = 1, \ldots, 7$. Suppose that $w_i = w_j$ for some i and j. This vertex is non-adjacent to v_{j+2} and adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}$ and thus $j + 2 \notin \{i, i+3, i+4, i+5\}$, which implies $j \notin \{i+1, i+2, i+3, i+5\}$. Further, this vertex is non-adjacent to v_{i+2} and adjacent to $v_j, v_{j+3}, v_{j+4}, v_{j+5}$; we have $i \notin \{j+1, j+2, j+3, j+5\}$, which implies $j \notin \{i+2, i+4, i+5, i+6\}$. Therefore $w_i = w_j$ implies $i = j$ and the vertices $w_1, \ldots, w_7$ are pairwise distinct. As w_i is adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}$ for $i = 1, \ldots, 7$, the vertex v_i is adjacent to $w_i, w_{i+2}, w_{i+3}, w_{i+4}$ for $i = 1, \ldots, 7$. Further, it is adjacent to $u, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}$ and thus its degree in G is at least 9, which is a contradiction. This proves the assertion. $\square$

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