SEQUENTIAL CONVERGENCES IN A VECTOR LATTICE

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Abstract. In the present paper we deal with sequential convergences on a vector lattice L which are compatible with the structure of L.

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In this paper we will investigate the system $\operatorname{Conv} L$ of all sequential convergences in a vector lattice L. The analogously defined notions of sequential convergences in a lattice ordered group or in a Boolean algebra were studied in [3]–[12].

The following results will be established.

The set $\operatorname{Conv} L$ is nonempty if and only if L is archimedean. Let L be archimedean. Then $\operatorname{Conv} L$ has the least element (it need not have, in general, a greatest element). Each interval of $\operatorname{Conv} L$ is a Brouwerian lattice. If L is $(\aleph_0, 2)$-distributive, then $\operatorname{Conv} L$ is a complete lattice. There is a convex vector sublattice L_1 of L such that (i) $\operatorname{Conv} L_1$ is a complete lattice; (ii) if L_2 is a convex vector sublattice of L such that $\operatorname{Conv} L_2$ is a complete lattice, then $L_2 \subseteq L_1$. Let X_i $(i = 1, 2)$ be archimedean vector lattices; if X_1 and X_2 are isomorphic as lattices and if $\operatorname{Conv} X_1$ is a complete lattice, then $\operatorname{Conv} X_2$ is a complete lattice as well. If L is a direct sum of linearly ordered vector lattices, then $\operatorname{Conv} L$ is a complete lattice and has no atom. Some further results (concerning orthogonal sequences and strong units) are also proved.

1. Preliminaries

The notion of a vector lattice is applied here in the same sense as in [1], Chap. XV. (In [16], the term "Riesz space" is used; in [13] vector lattices are called K-lineals.)

Let L be a vector lattice and let $\mathbb{N}$ be the set of all positive integers. The direct product $\prod\limits_{n\in\mathbb{N}} L_n$, where $L_n = L$ for each $n \in \mathbb{N}$, will be denoted by $L^{\mathbb{N}}$. The elements of $L^{\mathbb{N}}$ are denoted, e.g., as $(x_n)_{n\in\mathbb{N}}$, or simply (x_n); instead of n, sometimes other indices will be applied. (x_n) is said to be a sequence in L. If $x \in L$ and $x_n = x$ for each $n \in \mathbb{N}$, then we denote $(x_n) = \operatorname{const} x$. The notion of a subsequence has the usual meaning.

If $\alpha \subseteq L^{\mathbb{N}} \times L$, then instead of $((x_n), x) \in \alpha$ we also write $x_n \to_\alpha x$.

If the partial order (as defined in L) is not taken into account, then we obtain a linear space which will be denoted by $\ell(L)$; similarly, if we disregard the multiplication of elements of L by reals, then we get a lattice ordered group; we denote it by $G(L)$.

The set of all reals will be denoted by $\mathbb{R}$. The symbol 0 denotes both the real number zero and the neutral element of L; the meaning of this symbol will be clear from the context. For $(a_n) \in \mathbb{R}^{\mathbb{N}}$ and $a \in \mathbb{R}$ the symbol $a_n \to a$ has the usual meaning.

1.1. Definition. (Cf., e.g., [15].) A nonempty subset α of $L^{\mathbb{N}} \times L$ will be said to be a *convergence in* $\ell(L)$ if it satisfies the following conditions:

(i) If $x_n \to_\alpha x$ and if (y_n) is a subsequence of (x_n), then $y_n \to_\alpha x$.
(ii) If $x_n \to_\alpha x$ and $x_n \to_\alpha y$, then $x = y$.
(iii) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n + y_n \to_\alpha x + y$.
(iv) If $x_n \to_\alpha x$ and $a \in \mathbb{R}$, then $ax_n \to_\alpha ax$.
(v) If $x \in L$, $(a_n) \in \mathbb{R}^{\mathbb{N}}$, $a \in \mathbb{R}$ and $a_n \to a$, then $a_n x \to_\alpha ax$.

The system of all convergences in $\ell(L)$ will be denoted by $\operatorname{Conv}_\ell L$.

1.2. Definition. (Cf. [3].) A nonempty subset α of $L^{\mathbb{N}} \times L$ will be said to be a *convergence in* $G(L)$ if it satisfies the conditions (i), (ii), (iii) from 1.1, and if also the following conditions are fulfilled:

(i_1) If $((x_n), x) \in L^{\mathbb{N}} \times L$ and if each subsequence (y_n) of (x_n) has a subsequence (z_n) such that $z_n \to_\alpha x$, then $x_n \to_\alpha x$.
(ii_1) If $x \in L$ and $(x_n) = \operatorname{const} x$, then $x_n \to_\alpha x$.
(iii_1) If $x_n \to_\alpha x$, then $-x_n \to_\alpha -x$.
(iv_1) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n \wedge y_n \to_\alpha x \wedge y$ and $x_n \vee y_n \to_\alpha x \vee y$.
(v_1) If $x_n \to_\alpha x$, $y_n \to_\alpha x$, $(z_n) \in L^{\mathbb{N}}$ and $x_n \leqslant z_n \leqslant y_n$ for each $n \in \mathbb{N}$, then $z_n \to_\alpha x$.

The system of all convergences in $G(L)$ will be denoted by $\operatorname{Conv}_g L$.

Let us remark that in the paper [14] the Urysohn property (i_1) (which will be systematically applied below) was not assumed to be valid when investigating a sequential convergence in a lattice ordered group.

We denote by d the system of all elements $((x_n), x) \in L^{\mathbb{N}} \times L$ having the property that there is $m \in \mathbb{N}$ such that $x_n = x$ for each $n \geqslant m$. It is easy to verify that d belongs to $\operatorname{Conv}_g L$, hence $\operatorname{Conv}_g L$ is nonempty. The system $\operatorname{Conv}_g L$ will be considered to be partially ordered by inclusion. It is obvious that d is the least element of $\operatorname{Conv}_g L$.

Let us remark that the conditions (i), (ii), (iii), (i_1), (ii_1) and (iii_1) define a convergence group in the sense of [18] or a FLUSH convergence on the corresponding group (cf. [17]).

1.3. Definition. A nonempty subset α of $L^{\mathbb{N}} \times L$ will be said to be a *convergence in* L if $\alpha \in \operatorname{Conv}_\ell L \cap \operatorname{Conv}_g L$. The system of all convergences in L will be denoted by $\operatorname{Conv} L$. If $\operatorname{Conv} L \neq \emptyset$, then the set $\operatorname{Conv} L$ will be partially ordered by inclusion.

The vector lattice L is said to be *archimedean* if, whenever $x, y \in L$ and $0 \leqslant nx \leqslant y$ for each $n \in \mathbb{N}$, then $x = 0$.

1.4. Lemma. *Let L be non-archimedean. Then* $\operatorname{Conv} L = \emptyset$.

P r o o f. There exist $x, y \in L$ such that $0 < nx \leqslant y$ for each $n \in \mathbb{N}$. By way of contradiction, assume that $\alpha \in \operatorname{Conv} L$. Because $\frac{1}{n} \to 0$ in $\mathbb{R}$, in view of 1.1, (v) we infer that $\frac{1}{n} y \to_\alpha 0$. Since $0 < x \leqslant \frac{1}{n} y$ for each $n \in \mathbb{N}$, according to (ii_1) and (v_1) of 1.2 the relation $x_n \to_\alpha x$ is valid, where $(x_n) = \operatorname{const} x$. Thus in view of ($\mathrm{ii}_1$) and (ii) we have arrived at a contradiction. □

1.5. Lemma. *Let $\alpha \in \operatorname{Conv}_g L$. Then α satisfies the condition* (iv) *from* 1.1.

P r o o f. Let $x_n \to_\alpha x$ and let $a \in \mathbb{R}$. There is $m \in \mathbb{N}$ with $|a| \leqslant m$. We have

$$x_n \to_\alpha x \Rightarrow |x_n - x| \to_\alpha 0,$$

whence in view of (iii) and by induction we get $m|x_n - x| \to_\alpha 0$. Since

$$0 \leqslant |ax_n - ax| = |a|\ |x_n - x| \leqslant m|x_n - x|,$$

according to (v_1) we obtain $|ax_n - ax| \to_\alpha 0$, thus $ax_n \to_\alpha ax$. □

1.6. Corollary. *Let $\alpha \in \operatorname{Conv}_g L$. Then $\alpha \in \operatorname{Conv} L$ if and only if α satisfies the condition* (v) *from* 1.1.

If $L \neq \{0\}$, then the element d of $\operatorname{Conv}_g L$ does not satisfy the condition (v) of 1.1. Hence if $L \neq \{0\}$, then $\operatorname{Conv}_g L$ fails to be a subset of $\operatorname{Conv} L$.

The positive cone $\{x \in L \colon\ x \geqslant 0\}$ of L will be denoted by L^+. Under the inherited partial order and the operation $+$, L^+ is a lattice ordered semigroup.

1.7. Definition. A convex subsemigroup β of $(L^+)^{\mathbb{N}}$ will be said to be a 0-*convergence in* $G(L)$ if the following conditions are satisfied:

(I) If $(g_n) \in \beta$, then each subsequence of (g_n) belongs to β.

(II) If $(g_n) \in (L^+)^{\mathbb{N}}$ and if each subsequence of (g_n) has a subsequence belonging to (β), then (g_n) belongs to β.

(III) Let $x \in L^+$. Then $\operatorname{const} x$ belongs to β if and only if $x = 0$.

The system of all 0-convergences in $G(L)$ will be denoted by 0-$\operatorname{Conv}_g L$. Let d_0 be the set of all $(x_n) \in (L^+)^{\mathbb{N}}$ such that $((x_n), 0) \in d$. Then $d_0 \in$ 0-$\operatorname{Conv}_g L$. Hence 0-$\operatorname{Conv}_g G \neq \emptyset$. The system 0-$\operatorname{Conv}_g L$ is partially ordered by inclusion.

Let $\alpha \in \operatorname{Conv}_g L$. Put

$$\varphi_1(\alpha) = \{(|x_n - x|)\colon\ x_n \to_\alpha x\}. \tag{1}$$

Conversely, let $\beta \in$ 0-$\operatorname{Conv}_g L$. Denote

$$\varphi_2(\beta) = \{((x_n), x)\colon\ (|x_n - x|) \in \beta\}. \tag{2}$$

1.8. Lemma. (Cf. [4], Lemma 1.4 and Theorem 1.6.) *φ_1 and φ_2 are inverse isomorphisms of* $\operatorname{Conv}_g L$ *onto* 0-$\operatorname{Conv}_g L$, *or of* 0-$\operatorname{Conv}_g L$ *onto* $\operatorname{Conv}_g L$, *respectively.*

1.9. Definition. A nonempty subset β of $(L^+)^{\mathbb{N}}$ will be said to be a 0-*convergence in* L if $\beta \in$ 0-$\operatorname{Conv}_g L$ and if, moreover, the following condition is satisfied:

(IV) If $x \in L$ and $a_n \to 0$ in $\mathbb{R}$, then $(a_n x) \in \beta$.

Let 0-$\operatorname{Conv} L$ be the set of all 0-convergences in L. If this set is nonempty, then it will be considered to be partially ordered by inclusion.

Now let α and β run over the set $\operatorname{Conv} L$ or 0-$\operatorname{Conv} L$, respectively, and let φ_1 and φ_2 be defined as in (1) and (2). Then by a routine proof and by using 1.5 we obtain the following result which is analogous to 1.8:

1.10. Lemma. (i) $\operatorname{Conv} L = \emptyset \Leftrightarrow$ 0-$\operatorname{Conv} L = \emptyset$. (ii) *If* $\operatorname{Conv} L \neq \emptyset$, *then φ_1 and φ_2 are inverse isomorphisms of* $\operatorname{Conv} L$ *onto* 0-$\operatorname{Conv} L$, *or of* 0-$\operatorname{Conv} L$ *onto* $\operatorname{Conv} L$, *respectively.*

As we remarked in the introduction, we are interested in studying the partially ordered system $\operatorname{Conv} L$. Now, in view of 1.10, it suffices to investigate the system 0-$\operatorname{Conv} L$. Next, according to 1.4, it suffices to consider the case when L is archimedean.

In what follows we assume that L is an archimedean vector lattice.

Let $\emptyset \neq A \subseteq (L^+)^{\mathbb{N}}$. The set A will be said to be regular with respect to $G(L)$ (or L, respectively) if there is $\alpha \in 0\text{-Conv}_g L$ (or $\alpha \in 0\text{-Conv}\, L$) such that $A \subseteq \alpha$.

2.1. Lemma. *Let $\emptyset \neq A \subseteq (L^+)^{\mathbb{N}}$. Then the following conditions are equivalent:*

(i) *A fails to be regular with respect to $G(L)$.*

(ii) *There exist $0 < z \in L$, positive integers m, k, elements $(y_n^1), \ldots, (y_n^k)$ of A and subsequences (x_n^1) of $(y_n^1), \ldots, (x_n^k)$ of (y_n^k) such that*

$$z \leqslant m(x_n^1 \vee x_n^2 \vee \ldots \vee x_n^k) \quad \textit{for each } n \in \mathbb{N}.$$

P r o o f. The implication (ii)⇒(i) is obvious. Let (i) be valid. In view of the results of [4] (cf. also [10], Proposition 2.1) there exist $0 < z \in L$, positive integers m_1, k, elements $(y_n^1), \ldots, (y_n^k)$ of A and subsequences (x_n^1) of $(y_n^1), \ldots, (x_n^k)$ of (y_n^k) such that

$$z \leqslant m_1(x_n^1 + x_n^2 + \ldots + x_n^k) \quad \text{for each } n \in \mathbb{N}.$$

Hence according to Lemma 2.4, [10] there is $m \in \mathbb{N}$ with

$$z \leqslant m(x_n^1 \vee x_n^2 \vee \ldots \vee x_n^k) \quad \text{for each } n \in \mathbb{N}.$$

□

Let A_0 be the set of all sequences (x_n) in L having the property that there are $0 \leqslant x \in L$ and $(a_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n \to 0$ in $\mathbb{R}$ and $x_n = a_n x$ for each $n \in \mathbb{N}$.

2.2. Lemma. *The set A_0 is regular with respect to $G(L)$ and also with respect to L.*

P r o o f. By way of contradiction, assume that A_0 fails to be regular with respect to $G(L)$. Then the condition (ii) from 2.1. holds for A_0.

For each $i \in \{1, 2, \ldots, k\}$ there are $0 < x^i \in L$ and $(a_n^i) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n^i \to 0$ in $\mathbb{R}$ and

$$x_n^i = a_n^i x^i \quad \text{for each } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ we put $a_n = \max\{a_n^1, a_n^2, \ldots, a_n^k\}$. Then $a_n \to 0$ in $\mathbb{R}$ and

$$\begin{aligned} 0 < z \leqslant m(x_n^1 \vee x_n^2 \vee \ldots \vee x_n^k) &= m(a_n^1 x^1 \vee \ldots \vee a_n^k x^k) \\ &\leqslant m a_n (x^1 \vee \ldots \vee x^k) \quad \text{for each } n \in \mathbb{N}. \end{aligned}$$

Next, for each $n \in \mathbb{N}$ there is $n(1) \in \mathbb{N}$ such that $ma_{n(1)} < \frac{1}{n}$, hence

$$0 < z < \tfrac{1}{n}(x^1 \vee \ldots \vee x^k) \quad \text{for each } n \in \mathbb{N}.$$

Thus $nz < x^1 \vee \ldots \vee x^k$ for each $n \in \mathbb{N}$, which is impossible, because L is archimedean. Thus there is $\alpha \in 0\text{-Conv}_g\, L$ with $A_0 \subseteq \alpha$. Then α fulfils the condition (IV), hence $\alpha \in 0\text{-Conv}\, L$. □

2.3. Theorem. *Let L be an archimedean vector lattice. Then* $\text{Conv}\, L \neq \emptyset$.

Proof. In view of 2.2 there is $\alpha \in 0\text{-Conv}\, L$ with $A_0 \subseteq \alpha$. Hence $0\text{-Conv}\, L \neq \emptyset$. Thus according to 1.10 we have $\text{Conv}\, L \neq \emptyset$. □

2.4. Lemma. *Let $\alpha \in 0\text{-Conv}\, L$. Then $A_0 \subseteq \alpha$.*

Proof. This follows immediately from the fact that α satisfies the condition (IV) of 1.9. □

2.5. Corollary. *Let I be a nonempty set and for each $i \in I$ let $\alpha_i \in 0\text{-Conv}\, L$. Then $\emptyset \neq \bigcap_{i \in I} \alpha_i \in 0\text{-Conv}\, L$.*

Let us denote by d^0 the intersection of all $\alpha_i \in 0\text{-Conv}\, L$ with $A_0 \subseteq \alpha_i$ (such α_i do exist in view of 2.2). According to 2.4 and 2.5 we obtain:

2.6. Corollary. *d^0 is the least element of $0\text{-Conv}\, L$. If $\alpha \in 0\text{-Conv}\, L$, then the interval $[d^0, \alpha]$ of the partially ordered set $0\text{-Conv}\, L$ is a complete lattice.*

2.7. Proposition. $d^0 = A_0$.

Proof. In view of the definition of d^0 we have $A_0 \subseteq d^0$. Let $(z_n) \in d^0$. Then in view of [10], Proposition 2.1, and according to 2.4 there are $m, k \in \mathbb{N}$, elements $(y_n^1), \ldots, (y_n^k)$ of A_0 and subsequences (x_n^1) of $(y_n^1), \ldots, (x_n^k)$ of (y_n^k) such that

$$z_n \leqslant m(x_n^1 \vee \ldots \vee x_n^k).$$

For each $i \in \{1, 2, \ldots, k\}$ there are $x^i \in L^+$ and $(a_n^i) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n^i \to 0$ in $\mathbb{R}$ and $x_n^i = a_n^i x^i$ for each $n \in \mathbb{N}$. Put $a_n = \max\{a_n^1, \ldots, a_n^k\}$. Hence $a_n \to 0$ in $\mathbb{R}$ and

$$z_n \leqslant a_n(mx^1 \vee \ldots \vee mx^n).$$

Thus $(z_n) \in A_0$ and therefore $d^0 \subseteq A_0$. □

For each $X \subseteq (L^+)^{\mathbb{N}}$ let us denote by X^* the set of all $(x_n) \in (L^+)^{\mathbb{N}}$ such that each subsequence of (x_n) has a subsequence which belongs to X.

Let A_1 be the set of all $(x_n) \in (L^+)^{\mathbb{N}}$ which have the following property: there exist $0 \leqslant x \in L$ and $m \in \mathbb{N}$ such that $x_n \leqslant \frac{1}{n}x$ for each $n \geqslant m$.

Another constructive characterization of d^0 is given by the following lemma.

2.8. Lemma. $d^0 = A_1^*$.

Proof. Since $A_1 \subseteq A_0$, we clearly have $A_1^* \subseteq d^0$. Let $(x_n) \in d^0$. In view of 2.7 there are $x \in L^+$ and $(a_n) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $x_n = a_n x$ for each $n \in \mathbb{N}$. Let (y_n) be a subsequence of (x_n) and let (b_n) be the corresponding subsequence of (a_n); hence $y_n = b_n x$ for each $n \in \mathbb{N}$. There exists a subsequence (c_n) of (b_n) such that $c_n \leqslant \frac{1}{n}$ for each $n \in \mathbb{N}$. Put $z_n = c_n x$ for each $n \in \mathbb{N}$. Then $(c_n x)$ is a subsequence of (y_n) and $(c_n x) \in A_1$. Hence $(x_n) \in A_1^*$ and thus $d^0 \subseteq A_1^*$. □

2.9. Proposition. *There exists an archimedean vector lattice L such that* 0-Conv L *has no greatest element.*

Proof. It suffices to apply an analogous example as in [3], Section 5 (with the distinction that the real functions under consideration in the example are not assumed to be integer valued). □

2.10. Theorem. *Let L be an archimedean vector lattice. Suppose that L is $(\aleph_0, 2)$-distributive. Then* 0-Conv L *possesses a greatest element.*

Proof. This is a consequence of 2.6 and of the fact that 0-Conv$_g$ L has a greatest element (cf. [12]). □

Lemma 1.10 and Lemma 2.6 yield that each interval of the partially ordered set 0-Conv L is, at the same time, an interval of 0-Conv$_g$ L. Hence in view of [5], Theorem 2.5 we obtain:

2.11. Proposition. *Each interval of* 0-Conv L *is a Brouwerian lattice.*

3. The sets of the form $\alpha \cup A_0$

Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$ be such that α is regular with respect to $G(L)$. We shall investigate the problem whether the set $\alpha \cup A_0$ is regular with respect to L.

First we shall deal with the case when L is a projectable vector lattice. (Projectable lattice ordered groups and vector lattices were studied by several authors; cf. e.g., [2] and [16].)

For the sake of completeness we recall the following notions.

Let L be a vector lattice and $X \subseteq L$. We put

$$X^d = \{y \in L\colon\ |y| \wedge |x| = 0 \quad \text{for each } x \in X\}.$$

Then X^d is said to be a polar of L. The vector lattice L is called projectable if for each $x \in L$, the set $\{x\}^d$ is a direct factor of L.

An element $e \in L$ is called a strong unit of L if for each $x \in L$ there is $n \in \mathbb{N}$ such that $x \leqslant ne$.

Since each strong unit of an archimedean vector lattice L_1 is, at the same time, a strong unit of the Dedekind completion of L_1, we have

3.1. Proposition. (Cf., e.g., [19], Theorem V.3.1.) *Let L_1 be an archimedean vector lattice having a strong unit. Then there is a set I such that there exists an isomorphism of L_1 into the vector lattice $\prod\limits_{i \in I} R_i$, where $R_i = \mathbb{R}$ for each $i \in I$.*

3.2. Lemma. *Let $\alpha \in \operatorname{Conv}_g L$. Then the following conditions are equivalent:*

(i) *The set $\alpha \cup A_0$ fails to be regular with respect to $G(L)$.*

(ii) *There are $t, z \in L$ and $(z_n) \in \alpha$ such that $0 < z \leqslant t$ and*

$$z = z_n \vee (z \wedge \tfrac{1}{n} t) \quad \textit{for each } n \in \mathbb{N}.$$

P r o o f. According to 2.1, (ii)$\Rightarrow$(i). Suppose that (i) is valid. Thus in view of 2.7 and 2.8, the set $\alpha \cup A_1$ fails to be regular with respect to $G(L)$. Hence the condition (ii) from 2.1 holds, where $A = \alpha \cup A_1$.

If $(x_n^1), \ldots, (x_n^k) \in \alpha$, then α would not be regular with respect to $G(L)$, which is a contradiction. If $(x_n^1), \ldots, (x_n^k) \in A_1$, then we obtain a contradiction with respect to 2.2. Hence without loss of generality we can suppose that there is $k(1) \in \mathbb{N}$ with $1 < k(1) < k$ such that

$$(x_n^1), \ldots, (x_n^{k(1)}) \in \alpha \quad \text{and } (x_n^{k(1)+1}), \ldots, (x_n^k) \in A_1.$$

Put $z_n = m(x_n^1 \vee \ldots \vee x_n^{k(1)})$ for each $n \in \mathbb{N}$. Then $(z_n) \in \alpha$.

For each $j \in \{k(1)+1, \ldots, k\}$ there are $0 < y^j \in L$ and $(a_n^j) \in (\mathbb{R}^+)^{\mathbb{N}}$ such that $a_n^j \to 0$ in $\mathbb{R}$ and $y_n^j = a_n^j y^j$ for each $n \in \mathbb{N}$. Denote

$$a_n = \max\{a_n^{k(1)+1}, \ldots, a_n^k\}, \quad t = y^{k(1)+1} \vee \ldots \vee y^k.$$

There is a subsequence $(n(1))$ of the sequence (n) such that

$$m a_{n(1)} < \tfrac{1}{n} \quad \text{for each } n \in \mathbb{N}.$$

Hence we have

$$m(x_{n(1)}^{k(1)+1} \vee \ldots \vee x_{n(1)}^{k}) \leqslant \tfrac{1}{n}t \quad \text{for each } n \in \mathbb{N}.$$

Therefore

$$0 < z \leqslant z_{n(1)} \vee \tfrac{1}{n}t \quad \text{for each } n \in \mathbb{N}.$$

Becuase $(z_{n(1)}) \in \alpha$, it suffices to write z_n instead of $z_{n(1)}$. Thus

$$z = z \wedge (z_n \vee \tfrac{1}{n}t) = (z \wedge z_n) \vee (z \wedge \tfrac{1}{n}t) \quad \text{for each } n \in \mathbb{N}. \tag{1}$$

If $z \wedge t = 0$, then $z \wedge \frac{1}{n}t = 0$ for each $n \in \mathbb{N}$, whence $z \leqslant z_n$ for each $n \in \mathbb{N}$ and thus α fails to be regular, which is a contradiction. Therefore $z \wedge t > 0$ and then, without loss of generality, we can take $z \wedge t$ instead of z; hence we have $z \leqslant t$. Next, $(z \wedge z_n) \in \alpha$, thus without loss of generality we can take $(z \wedge z_n)$ instead of (z_n). Hence in view of (1) we infer that (ii) is valid. □

3.3. Proposition. *Assume that L is projectable. Let $\alpha \in 0\text{-}\mathrm{Conv}_g\, L$. Then $\alpha \cup A_0$ is regular with respect to L.*

Proof. In view of 2.7 it suffices to verify that $\alpha \cup A_0$ is regular with respect to $G(L)$.

By way of contradiction, suppose that $\alpha \cup A_0$ fails to be regular with respect to $G(L)$. Then the condition (ii) from 3.2 is valid. There exists $m \in \mathbb{N}$ such that $z \nleqq \frac{1}{m}t$. Thus

$$z^0 = (z - \tfrac{1}{m}t)^+ > 0. \tag{1$'$}$$

Let us denote by P the polar of L generated by z^0; i.e., $P = \{z^0\}^{dd}$. Since L is projectable, P is a direct factor in L. For each $g \in L$ let $g(P)$ be the component of g in P. In view of the condition (ii) of 3.2 we have

$$z(P) = z_n(P) \vee (z(P) \wedge \tfrac{1}{n}t(P)) \quad \text{for each } n \in \mathbb{N}. \tag{2}$$

If $z(P) = 0$, then $z^0 = z^0(P) = 0$, which is a contradiction. Thus $z(P) > 0$. Next, from $z \leqslant t$ we infer that $z(P) \leqslant t(P)$.

Let L_1 be the convex ℓ-subgroup of $G(P)$ generated by the element $t(P)$. Then $t(P)$ is a strong unit of L_1 and L_1 is a linear subspace of L. Let I and φ be as in 3.2. For each $i \in I$ we have $\varphi(z(P))(i) \geqslant 0$. According to the definition of P we obtain

$$(z - \tfrac{1}{m}t)^- \in P^d$$

whence $(z - \frac{1}{m}t)(P) = z_0(P)$. In view of $(1')$,

$$0 < z^0 = z^0(P) = z(P) - \tfrac{1}{m}t(P), \tag{3}$$

hence the set $I_1 = \{i \in I\colon\ \varphi(z(P))(i) > 0\}$ is nonempty.

Let $i \in I_1$ and $n > m$. According to (3),

$$\varphi(z(P))(i) \geqslant \tfrac{1}{n}\varphi(t(P))(i). \tag{4}$$

Also, in view of (2),

$$\begin{aligned}\varphi(z(P))(i) &= \varphi(z_n(P))(i) \vee (\varphi(z(P))(i) \wedge \tfrac{1}{n}\varphi(t(P))(i)) \\ &= \max\{\varphi(z_n(P))(i), \min\{\varphi(z(P))(i), \tfrac{1}{n}\varphi(t(P))(i))\}\}.\end{aligned}$$

Thus according to (4),

$$\varphi(z(P))(i) = \max\{\varphi(z_n(P)(i), \tfrac{1}{n}\varphi(t(P))(i)\}.$$

By applying (4) again we get

$$\varphi(z(P))(i) = \varphi(z_n(P))(i).$$

Therefore $\varphi(z(P))(i) = \varphi(z_n(P))(i)$ for each $i \in I$. Hence

$$0 < z(P) = z_n(P) \quad \text{for each } n > m. \tag{5}$$

Since $z_n(P) \leqslant z_n$ for each $n \in \mathbb{N}$ and since (z_n) is regular with respect to L, we infer that $(z_n(P))$ is regular with respect to L. Thus in view of (5) we have arrived at a contradiction. □

Now let us drop the assumption that L is projectable. We denote by L' the Dedekind completion of L. It is well-known that L' is projectable.

3.4. Lemma. *Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$. Assume that α is regular with respect to $G(L)$. Then α is regular with respect to $G(L')$.*

P r o o f. By way of contradiction, assume that α fails to be regular with respect to $G(L')$. Then the condition (ii) from 2.1 holds (with the distinction that $z \in L'$ and A is replaced by α). There exists $0 < z_1 \in L$ with $z_1 \leqslant z$. But by applying 2.1 again we infer that α fails to be regular with respect to L, which is a contradiction. □

3.5. Lemma. *Let $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$. Assume that α is regular with respect to $G(L)$. Then α is regular with respect to $G(L)$.*

P r o o f. This is an immediate consequence of 2.1. □

3.6. Theorem. *Let* $\emptyset \neq \alpha \subseteq (L^+)^{\mathbb{N}}$. *Assume that* α *is regular with respect to* $G(L)$. *Then* $\alpha \cup A_0$ *is regular with respect to* $G(L)$ *and with respect to* L.

Proof. In view of 3.4, α is regular with respect to $G(L')$. Because $G(L')$ is projectable, according to 3.3 we obtain that $\alpha \cup A_0$ is regular with respect to $G(L')$. Thus 3.5 yields that $\alpha \cup A_0$ is regular with respect to $G(L)$. Now it follows from 2.7 that $\alpha \cup A_0$ is regular with respect to L. □

3.7. Corollary. *Let* $\alpha \in 0\text{-}\mathrm{Conv}_g L$. *Then* $\alpha \vee d^0$ *does exist in* $0\text{-}\mathrm{Conv}_g L$ *and in* $0\text{-}\mathrm{Conv}\, L$.

3.8. Proposition. *The following conditions are equivalent:*

(i) $0\text{-}\mathrm{Conv}\, L$ *has the greatest element.*
(ii) $0\text{-}\mathrm{Conv}_g L$ *has the greatest element.*

Proof. We obviously have (ii)⇒(i). Let (i) hold and let β be the greatest element of $0\text{-}\mathrm{Conv}\, L$. Let $\alpha \in 0\text{-}\mathrm{Conv}_g L$. According to 3.7, the element $\alpha \vee d^0$ does exist in $0\text{-}\mathrm{Conv}\, L$. Thus $\alpha \leqslant \alpha \vee d^0 \leqslant \beta$. Hence β is the greatest element of $0\text{-}\mathrm{Conv}_g L$. □

3.9. Corollary. *Let* $0\text{-}\mathrm{Conv}\, L$ *have the greatest element. Then* $0\text{-}\mathrm{Conv}\, L$ *is a complete lattice and* $0\text{-}\mathrm{Conv}\, L$ *is a principal dual ideal of* $0\text{-}\mathrm{Conv}_g L$ *generated by the element* d^0.

Let us remark that if L_1 is a convex ℓ-subgroup of $G(L)$, then it is a linear subspace of L.

3.10. Theorem. *There exists a convex* ℓ*-subgroup* L_1 *of* $G(L)$ *such that the following conditions are satisfied:*

(i) $\mathrm{Conv}\, L_1$ *is a complete lattice.*
(ii) *If* L_2 *is a convex* ℓ*-subgroup of* $G(L)$ *such that* $\mathrm{Conv}\, L_2$ *is a complete lattice, then* $L_2 \leqslant L_1$.

Proof. This follows from 3.8 and from [10], Theorem 5.5. □

Let L_1 be a vector lattice. If neither the operation $+$ nor the multiplication of elements of L_1 by reals is taken into account, then we obtain a lattice which will be denoted by L_1^0.

3.11. Theorem. *Let* L_i $(i = 1, 2)$ *be archimedean vector lattices. Assume that the lattices* L_1^0 *and* L_2^0 *are isomorphic and that* $\mathrm{Conv}\, L_1$ *possesses a greatest element. Then* $\mathrm{Conv}\, L_2$ *possesses a greatest element as well.*

P r o o f. According to 1.10, 0-Conv L_1 possesses a greatest element. Then in view of 3.8, 0-Conv$_g$ L has a greatest element. Since L_1^0 is isomorphic to L_2^0, by applying [10], Theorem 3.5 we conclude that 0-Conv$_g$ L_2 has a greatest element as well. Now according to 3.8 and 1.10, Conv L_2 possesses a greatest element. □

4. Disjoint sequences

A sequence (x_n) in L will be said to be disjoint (or orthogonal) if $x_n \wedge x_m = 0$ whenever n and m are distinct positive integers.

The following assertion follows from the results proved in [4].

(A) Assume that L possesses a disjoint sequence all members of which are strictly positive. Then there exist infinitely many elements α_i of 0-Conv$_g$ L such that each α_i is generated by a disjoint sequence.

4.1. Lemma. (Cf. [4].) *Let (x_n) be a disjoint sequence in L. Then the set (x_n) is regular with respect to $G(L)$.*

4.2. Lemma. *Let (x_n) be a disjoint sequence in L. Then the set $\{(x_n)\} \cup A_0$ is regular with respect to $G(L)$ and with respect to L.*

P r o o f. This is a consequence of 4.1 and 3.6. □

If $(x_n) \in (L^+)^{\mathbb{N}}$ and the set $\{(x_n)\}$ is regular in $G(L)$ then the least element α of 0-Conv$_g$ L satisfying the relation $\{(x_n)\} \cup A_0 \subseteq \alpha$ will be denoted by $\alpha(x_n)$.

Let (x_n) be a disjoint sequence in L such that $x_n > 0$ for each $n \in \mathbb{N}$. Then $(x_n) \notin d_0$. On the other hand, (x_n) can belong to d^0 (cf. Proposition 4.6 below).

4.3. Lemma. *Let (x_n) and (y_n) be disjoint sequences in L such that $x_n \wedge y_m = 0$ for each $m, n \in \mathbb{N}$. Let $y_n > 0$ for each $n \in \mathbb{N}$ and $(y_n) \notin d^0$. Then $(y_n) \notin \alpha(x_n)$.*

P r o o f. By way of contradiction, assume that $y_n \in \alpha(x_n)$. Then in view of [10], Lemma 2.3 there are $m, k \in \mathbb{N}$ and $(z_n^1), \ldots, (z_n^k) \in (L^+)^{\mathbb{N}}$ such that each (z_n^i) $(i = 1, 2, \ldots, k)$ is a subsequence of a sequence belonging to $\{(x_n)\} \cup A_0$ and

$$0 < y_n \leqslant m(z_n^1 \vee \ldots \vee z_n^k) \quad \text{for each } n \in \mathbb{N}.$$

Since $(y_n) \notin A_0$, without loss of generality we can assume that $(z_n^1), \ldots, (z_n^{k-1})$ are subsequences of (x_n) and that (z_n^k) is a subsequence of $(\frac{1}{n}x)$ for some $0 < x \in L$. Thus

$$0 < y_n \leqslant (mz_n^1 \vee \ldots \vee mz_n^{k-1}) \vee \tfrac{1}{n}x' \quad \text{for each } n \in \mathbb{N},$$

where $x' = mx$. But $y_n \wedge (mz_n^1 \vee \ldots \vee mz_n^{k-1}) = 0$, whence $y_n \leqslant \frac{1}{n}x'$ for each $n \in \mathbb{N}$. Since $(y_n) \notin d^0$, we have arrived at a contradiction. □

4.4. Theorem. *Assume that L possesses an infinite orthogonal subset. Next, suppose that no disjoint sequence (x_n) in L with $x_n > 0$ for each $n \in \mathbb{N}$ belongs to d^0. Then $0\text{-Conv}\, L$ is infinite.*

P r o o f. In view of the assumption there are disjoint sequences (x_n^i) $(i \in \mathbb{N})$ in L such that $x_n^i > 0$ for each $n, i \in \mathbb{N}$, and $x_n^i \wedge x_m^j = 0$ whenever $m, n, i, j \in \mathbb{N}$ and $i \neq j$. In view of 4.2 we have $\alpha(x_n^i) \in 0\text{-Conv}_g\, L$ for each $i \in \mathbb{N}$. Let i, j be distinct elements of $\mathbb{N}$. According to 4.3, $\alpha(x_n^i) \neq \alpha(x_n^j)$. □

For a relevant result concerning convergences in a lattice ordered group cf. [4].

4.5. Theorem. *Assume that L possesses no infinite orthogonal subset. Then $0\text{-Conv}\, L$ is a one-element set.*

P r o o f. The case $L = \{0\}$ is trivial; let $L \neq \{0\}$. The system $0\text{-Conv}_g\, L$ was described in [4], Section 6. According to [4], if $\alpha \in 0\text{-Conv}_g\, L$ and $(\frac{1}{n}x) \in \alpha$ for each $0 < x \in L$, then α is the greatest element of $0\text{-Conv}_g\, L$; hence only this greatest element of $0\text{-Conv}_g\, L$ can belong to $0\text{-Conv}\, L$. □

4.6. Proposition. *Assume that L is orthogonally complete. Then each disjoint sequence in L belongs to d^0.*

P r o o f. Let (x_n) be a disjoint sequence in L. Then (nx_n) is disjoint as well. Since L is orthogonally complete, there exists $x = \bigvee_{n \in \mathbb{N}} nx_n$ in L. For each $n \in \mathbb{N}$ we have $0 \leqslant x_n \leqslant \frac{1}{n}x$, whence $(x_n) \in d^0$. □

4.7. Corollary. *The assertion* (A) *does not hold in general if* $0\text{-Conv}_g\, L$ *is replaced by* $0\text{-Conv}\, L$.

4.8. Proposition. *Assume that $L \neq \{0\}$ has a strong unit and that (x_n) is a disjoint sequence in L such that $x_n > 0$ for each $n \in \mathbb{N}$. Then there is a sequence (a_n) with $a_n \in \mathbb{N}$ for each $n \in \mathbb{N}$ having the property that $(a_n x_n) \notin d^0$.*

P r o o f. Let e be a strong unit in L. Since L is archimedean, for each $n \in \mathbb{N}$ there is $a_n \in \mathbb{N}$ such that

$$a_n x_n \nleqslant e. \tag{1}$$

By way of contradiction, assume that $(a_n x_n) \in d^0$. Hence in view of 2.8 there is a subsequence $(b_n y_n)$ of $(a_n x_n)$ such that $(b_n y_n) \in A_1$. Thus there are $m \in \mathbb{N}$ and $0 < x \in L$ such that $b_n y_n \leqslant \frac{1}{n}x$ for each $n \geqslant m$. Next, since e is a strong unit in L, there is $k \in \mathbb{N}$ with $x \leqslant ke$. Thus

$$b_n y_n \leqslant \frac{k}{n} e \quad \text{for each } n \geqslant m.$$

Hence for $n > \max\{m, k\}$ we have $b_n y_n \leqslant e$. But in view of (1) the relation $b_n y_n \nleqq e$ is valid for each $n \in \mathbb{N}$, which is a contradiction. □

4.9. Proposition. *Assume that L has a strong unit. Then* (A) *is valid with* $\operatorname{Conv}_g L$ *replaced by* $\operatorname{Conv} L$.

Proof. This is a consequence of 4.3 and 4.8. □

5. Direct sums of linearly ordered vector lattices

Let us denote by $\mathcal{S}$ the class of all archimedean vector lattices which can be expressed as the direct sum of linearly ordered vector lattices. Next, let $\mathcal{L}$ be the class of all linearly ordered vector lattices.

In this section it will be shown that if $L \in \mathcal{S}$, then $0\text{-}\operatorname{Conv} L$ is a complete lattice which has no atom.

The case $L = \{0\}$ being trivial, we assume in the present section that L is a nonzero archimedean vector lattice which can be represented as

$$L = \sum_{i \in I} L_i, \quad \text{where } L_i \in \mathcal{L} \text{ for each } i \in I. \tag{1}$$

Also, without loss of generality we can suppose that $L_i \neq \{0\}$ for each $i \in I$.

5.1. Proposition. $0\text{-}\operatorname{Conv} L$ *is a complete lattice.*

Proof. From (1) it follows that L is completely distributive. Hence in view of 2.10, $0\text{-}\operatorname{Conv} L$ possesses a greatest element. Thus $0\text{-}\operatorname{Conv} L$ is a complete lattice. □

5.2. Lemma. *Let (x_n) be a disjoint sequence in L such that $x_n > 0$ for each $n \in \mathbb{N}$. Then (x_n) is not upper-bounded in L.*

Proof. This is an immediate consequence of (1). □

In view of 5.2 and 2.8 we obtain

5.3. Corollary. *Let (x_n) be as in 5.2. Then (x_n) does not belong to d^0.*

5.4. Proposition. *Let I be finite. Then* $0\text{-}\operatorname{Conv} L$ *is a one-element set.*

Proof. From (1) we infer that L has no infinite orthogonal subset. Hence in view of 4.5, $0\text{-}\operatorname{Conv} L$ is a one-element set. □

5.5. Proposition. *Let I be infinite. Then* 0-Conv L *is infinite.*

Proof. According to (1), L possesses an infinite orthogonal subset. Then 4.4 and 5.3 yield that 0-Conv L is infinite. □

5.6. Lemma. *Let $\alpha \in$ 0-Conv L. Assume that $(x_n) \in \alpha$, $x_n > 0$ for each $n \in \mathbb{N}$, and that the sequence (x_n) is disjoint. Then α fails to be an atom of* 0-Conv L.

Proof. Consider the sequences (x_{2n}) and (x_{2n+1}). In view of 5.3, $(x_{2n}) \notin d^0$ and $(x_{2n+1}) \notin d^0$. Hence by applying the notation from Section 4 we have

$$d^0 < \alpha(x_{2n}) \leqslant \alpha, \quad d^0 < \alpha(x_{2n+1}) \leqslant \alpha.$$

Next, according to 4.3, $\alpha(x_{2n}) \neq \alpha(x_{2n+1})$. Hence α cannot be an atom of 0-Conv L. □

For $x \in L$ and $i \in I$, let $x(i)$ be the component of x in L_i. We put $\operatorname{Sup} x = \{i \in I : x(i) \neq 0\}$. If (x_n) is a sequence in L, then we denote

$$\operatorname{Sup}(x_n) = \bigcup_{n \in \mathbb{N}} \operatorname{Sup} x_n.$$

5.7. Lemma. *Let $(x_n) \in (L^+)^{\mathbb{N}}$ be such that $\{(x_n)\}$ is regular and suppose that $\operatorname{Sup}(x_n)$ if finite. Then $\alpha(x_n) = d^0$.*

Proof. In view of the assumption there is a finite subset $I(1)$ of I such that $x_n \in L(1) = \sum\limits_{i \in I(1)} L_i$ for each $n \in \mathbb{N}$. Then according to 4.5, (x_n) belongs to the least element of 0-Conv $L(1)$. Next, in view of 2.8, (x_n) belongs to d^0. Hence $\alpha(x_n) = d^0$. □

5.8. Lemma. *Let $(x_n) \in (L^+)^{\mathbb{N}}$ be such that $\{(x_n)\}$ is regular and suppose that $\operatorname{Sup}(x_n)$ is infinite. Then $\alpha(x_n)$ contains a disjoint sequence with strictly positive elements.*

Proof. Since $\operatorname{Sup}(x_n)$ is infinite and (1) holds, there is a subsequence (y_n) of (x_n) such that for each $n \in \mathbb{N}$, $\operatorname{Sup} y_n$ is not a subset of the set

$$\operatorname{Sup} y_1 \cup \ldots \cup \operatorname{Sup} y_{n-1}.$$

Therefore the sequence (y_n) is disjoint and belongs to $\alpha(x_n)$. □

5.9. Theorem. *Let $L \in \mathcal{S}$. Then* 0-Conv L *has no atom.*

Proof. By way of contradiction, assume that α is an atom of 0-Conv L. Then there is $(x_n) \in (L^+)^{\mathbb{N}}$ such that $\alpha = \alpha(x_n)$. If $\operatorname{Sup}(x_n)$ is finite, then 5.7 yields a contradiction. If $\operatorname{Sup}(x_n)$ is infinite, then by means of 5.8 and 5.6 we arrive at a contradiction. □

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