DOMINATION NUMBERS ON THE COMPLEMENT OF THE BOOLEAN FUNCTION GRAPH OF A GRAPH

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Abstract. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. The Boolean function graph $B(G, L(G), \mathrm{NINC})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \mathrm{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_1(G)$. In this paper, we determine domination number, independent, connected, total, point-set, restrained, split and non-split domination numbers in the complement $\overline{B}_1(G)$ of $B_1(G)$ and obtain bounds for the above numbers.

Keywords: domination number, eccentricity, radius, diameter, neighborhood, perfect matching, Boolean function graph

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1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. For a connected graph G, the *eccentricity* $e_G(v)$ of a vertex v in G is the distance to a vertex farthest from v. Thus, $e_G(v) = \max\{d_G(u, v) \colon u \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of a vertex v in G as $e(v)$ and use $d(u, v)$ to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted $r(G)$ and $\mathrm{diam}(G)$ respectively. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set

$N_G[v] = N_G(v) \cup \{v\}$ is called the closed neighborhood of v. A set S of edges in a graph G is said to be *independent* if no two of the edges in S are adjacent. A set of independent edges covering all the vertices of a graph G is called a *perfect matching*. An edge $e = (u, v)$ is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore [12]. A set $D \subseteq V(G)$ is said to be a *dominating set* of G if every vertex in $V(G) - D$ is adjacent to some vertex in D. D is said to be a minimal dominating set if $D - \{u\}$ is not a dominating set for any $u \in D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called a *connected (independent) dominating set* if the induced subgraph $\langle D \rangle$ is connected (independent) [14], [2]. D is called a *total dominating set* if every vertex in $V(G)$ is adjacent to some vertex in D [5].

A dominating set D is called a *cycle dominating set* if the subgraph $\langle D \rangle$ has a Hamiltonian cycle and is called a *perfect dominating set* if every vertex in $V(G) - D$ is adjacent to exactly one vertex in D [6]. D is called a *restrained dominating set* if every vertex in $V(G) - D$ is adjacent to another vertex in $V(G) - D$ [7]. By γ_{c}, γ_{i}, γ_{o}, γ_{t}, γ_{p} and γ_{r}, we mean the minimum cardinality of a connected dominating set, independent dominating set, cycle dominating set, total dominating set, perfect dominating set and restrained dominating set respectively.

Sampathkumar and Pushpalatha [13] introduced the concept of point-set domination number of a graph. A set $D \subseteq V(G)$ is called a *point-set dominating set* (psd-set) if for every set $T \subseteq V(G) - D$, there exists a vertex $v \in D$ such that the subgraph $\langle T \cup \{v\} \rangle$ induced by $T \cup \{v\}$ is connected. The point-set domination number $\gamma_{\mathrm{ps}}(G)$ is the minimum cardinality of a psd-set of G, where G is a connected graph. Kulli and Janakiram [11] introduced the concept of split and non-split domination in graphs. A dominating set D of a connected graph G is a *split (non-split) dominating set* if the induced subgraph $\langle V(G) - D \rangle$ is disconnected (connected). The *split (non-split) domination number* $\gamma_{\mathrm{s}}(G)(\gamma_{\mathrm{ns}}(G))$ of G is the minimum cardinality of a split (non-split) dominating set.

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. As for every graph (undirected, uniformly weighted), there exists an adjacency $(0, 1)$ matrix, we call the general operation a Boolean operation. Boolean operation on a given graph uses the adjacency relation between two vertices or two edges and incidence relationship between vertices and edges to define a new structure from the given graph. This extracts information from the original graph and encodes it into a new structure. If it is possible to decode the given graph from the encoded graph in polynomial time, such operation may be used to analyze various structural properties of the original graph in terms of the Boolean graph. If it is not possible to decode the original graph in polynomial time, then that operation can be used in graph coding or coding of certain grouped signals.

Whitney [16] introduced the concept of the line graph $L(G)$ of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The Middle graph $M(G)$ of a graph G was introduced by Hamada and Yoshimura [8]. Chikkodimath and Sampathkumar [4] also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad [3] in 1966. Sastry and Raju [15] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. This motivates us to define and study other graph operations. Using $L(G)$, G, incidence and non-incidence, complementary operations, complete and totally disconnected structures, one can get thirty-two graph operations. As total graphs, semi-total edge graphs, semi-total vertex graphs and quasi-total graphs and their complements (8 graphs) have already been defined and studied, we study all other similar remaining graph operations. This is illustrated below.

$$\boxed{G/\overline{G}/K_p/\overline{K}_p} \text{---} \boxed{\text{incident (INC), not incident (NINC)}} \text{---} \boxed{L(G)/\overline{L}(G)/K_q/\overline{K}_q}.$$

Here, $\overline{G}$ and $L(G)$ denote the complement and the line graph of G respectively. K_p is the complete graph on p vertices.

The Boolean function graph $B(G, L(G), \text{NINC})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \text{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G [10]. For brevity, this graph is denoted by $B_1(G)$. In other words, $V(B_1(G)) = V(G) \cup V(L(G))$; and $E(B_1(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L}(G)))] \cup (E(G) \cup E(L(G)))$, where $\overline{G}$, $L(G)$ and $T(G)$ denote the complement, the line graph and the total graph of G respectively. The vertices of G and $L(G)$ are referred to as point and line vertices respectively.

The Boolean function graph $B(G, L(G), \text{NINC})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \text{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G [10]. For brevity, this graph is denoted by $B_1(G)$. In other words, $V(B_1(G)) = V(G) \cup V(L(G))$; and $E(B_1(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L}(G)))] \cup (E(G) \cup E(L(G)))$, where $\overline{G}$, $L(G)$ and $T(G)$ denote the complement,

the line graph and the total graph of G respectively. The vertices of G and $L(G)$ are referred to as point and line vertices respectively.

In this paper, we determine the domination numbers mentioned above for the complement $\overline{B}_1(G)$ of the graph $B_1(G)$. The definitions and details not furnished in this paper may be found in [9].

2. Prior results

Theorem 2.1 [13]. *Let $G=(V,E)$ be a connected graph. A set $S \subseteq V$ is a point-set dominating set of G if and only if for every independent set W in $V-S$, there exists a vertex u in S such that $W \subseteq N_G(u) \cap (V-S)$.*

Observation [10].

2.2. *$\overline{G}$ and $\overline{L}(G)$ are induced subgraphs of $\overline{B}_1(G)$.*

2.3. *The degree of a point vertex in $\overline{B}_1(G)$ is $p-1$; and the degree of a line vertex e' in $\overline{B}_1(G)$ is $q+1-\deg_{L(G)}(e')$.*

2.4. *$\overline{B}_1(G)$ is a connected graph, for any graph G.*

3. Main results

3.1. Domination number in $\overline{B}_1(G)$.

Proposition 3.1. $\gamma(\overline{B}_1(G))=1$ *if and only if* $G \cong nK_1$, $n \geqslant 1$, *or* $G \cong K_2$.

Theorem 3.1. *Let G be any graph not totally disconnected. If there exists an edge in G not lying in any triangle, then $\gamma(\overline{B}_1(G)) \leqslant 3$.*

Proof. Let $e=(u,v)$ be an edge in G not lying in any triangle and e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge e. Then $D=\{u,v,e'\} \subseteq V(\overline{B}_1(G))$. Since $e \in E(G)$ does not lie on any triangle, there exists no vertex in G adjacent to both u and v. Therefore, all the point vertices of $\overline{B}_1(G)$ are adjacent to at least one of u and v. Let x' be a line vertex in $\overline{B}_1(G)$ and x be the corresponding edge in G. If x is adjacent to e in G, then x' is adjacent to either u or v. If x is not adjacent to e in G, then x' is adjacent to e' in $\overline{B}_1(G)$, since $\overline{L}(G)$ is an induced subgraph of $\overline{B}_1(G)$. Hence, D is a dominating set of $\overline{B}_1(G)$. Thus, $\gamma(\overline{B}_1(G)) \leqslant 3$. □

Remark 3.1. D is also a connected (total) dominating set of $\overline{B}_1(G)$ and hence $\gamma_c(\overline{B}_1(G)) \leqslant 3$, or $\gamma_t(\overline{B}_1(G)) \leqslant 3$. This bound is attained if $G \cong C_n$, for $n \geqslant 4$.

Theorem 3.2. *Let C be the set of all complete subgraphs of G. If $k = \min\{|C_i|\colon C_i \in C$ and $C_i \nsubseteq C_j$, for any $C_j \in C\}$, then $\gamma(\overline{B}_1(G)) \leqslant k+1$.*

Proof. Let $C_i \in C$ be such that $|C_i| = k$ and e be an edge in C_i. Let e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge e. Then $D = V(C_i) \cup \{e'\}$ is a dominating set of $\overline{B}_1(G)$ and hence $\gamma(\overline{B}_1(G)) \leqslant k+1$.

The next theorem relates $\gamma(\overline{B}_1(G))$ to the point covering number γ_o of G. □

Theorem 3.3. *Let G be a graph not totally disconnected. Then $\gamma(\overline{B}_1(G))) \leqslant \gamma_o(G)+1$.*

Proof. Let D be a minimal point cover for G such that $|D| = \gamma_o(G)$. Then $D \subseteq V(\overline{B}_1(G))$ dominates all the line vertices of $\overline{B}_1(G)$. Also since D is a point cover for G, $V(G)-D$ is independent and hence $\langle V(G)-D\rangle$ is a complete subgraph of $V(\overline{B}_1(G)) - D$. Therefore, $\gamma(\langle V(G)-D\rangle) = 1$. Thus, $\gamma(\overline{B}_1(G)) \leqslant \gamma_o(G)+1$.

This bound is attained if $G \cong K_{1,n}$, $n \geqslant 2$. □

The following theorems give the upper bounds of $\gamma(\overline{B}_1(G))$ in terms of the domination number γ of $\overline{G}$.

Theorem 3.4. *Let G be any graph having no isolated vertices and $G \neq K_n$, for $n \geqslant 3$. Then $\gamma(\overline{B}_1(G)) \leqslant \gamma(\overline{G})+2$.*

Proof. Let D be a minimal dominating set of $\overline{G}$. Then D dominates all the point vertices of $\overline{B}_1(G)$ and the line vertices corresponding to the edges incident with the vertices in D. Let $u \in D$. Then there exists an edge $e = (u,v)$ in $E(G)$ incident with u. If e' is the line vertex in $\overline{B}_1(G)$ corresponding to the edge e, then $D' = D \cup \{v, e'\}$ is a dominating set of $\overline{B}_1(G)$. Hence, $\gamma(\overline{B}_1(G)) \leqslant \gamma(\overline{G})+2$. □

Theorem 3.5. *Let G be any graph. Then $\gamma\overline{B}_1(G)) \leqslant \gamma(\overline{G})$, if $\gamma(\overline{G}) = \alpha_0(G)$.*

Proof. Let D be a minimal dominating set of $\overline{G}$ with $|D| = \gamma(\overline{G})$. Then D is also a point cover of G and hence D dominates both point and line vertices in $\overline{B}_1(G)$. Therefore, $\gamma(\overline{B}_1(G)) \leqslant \gamma(\overline{G})$. □

Theorem 3.6. *Let G be any graph with $\gamma(\overline{G}) \neq \alpha_0(G)$. Then $\gamma(\overline{B}_1(G)) \leqslant \gamma(\overline{G})+1$, if one of the following holds.*

(i) *There exists a minimal dominating set D of $\overline{G}$ such that $D \subseteq V(G)$ is not independent in G; and*

(ii) *For all minimal dominating sets D of $\overline{G}$, where D is independent in G,*

  (a) *There exists a vertex $u \in V(G)-D$ such that all the edges in $\langle V(G)-D\rangle_G$ are incident with u. That is, $\langle V(G)-D\rangle_G$ is a star.*

(b) *There exists an edge $e \in E(G)$ such that e is nonadjacent to all the edges in $\langle V(G) - D\rangle_G$ or*

(c) *$\langle V(G) - D\rangle_G$ contains K_2 as one of its components.*

Proof. Let G be any graph with $\gamma(\overline{G}) \neq \alpha_0(G)$.

(i) Let there exist a minimal dominating set D of $\overline{G}$ such that $D \subseteq V(G)$ is independent in G. Since $\overline{G}$ is an induced subgraph of $\overline{B}_1(G)$, D dominates all the point vertices in $\overline{B}_1(G)$ and the line vertices corresponding to the edges incident with the vertices in D. Since D is an independent set in G, there exists at least one edge e in $\langle D\rangle$, the subgraph of G induced by D. Let e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge e. Since $\overline{L}(G)$ is an induced subgraph of $\overline{B}_1(G)$, e' is adjacent to all the line vertices corresponding to the edges in $\langle V(G)-D\rangle_G$. Thus, $D' = D \cup \{e'\}$ is a dominating set of $\overline{B}_1(G)$.

(ii) Assume all the minimal dominating sets D of $\overline{G}$ are independent in G.

(a) Let there exist a vertex $u \in V(G) - D$ such that all the edges in $\langle V(G) - D\rangle$ are incident with u. Then $D' = D \cup \{u\}$ is a minimal dominating set of $\overline{B}_1(G)$.

(b) Let there exist an edge $e \in E(G)$ such that e is not adjacent to all the edges in $\langle V(G) - D\rangle$ and let e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge e. Then $D \cup \{e'\}$ is a dominating set of $\overline{B}_1(G)$.

(c) Let $\langle V(G) - D\rangle_G$ contain K_2 as one of its components and let e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge in K_2. Then $D \cup \{e'\}$ is a dominating set of $\overline{B}_1(G)$. From the above it follows that $\gamma(\overline{B}_1(G)) \leqslant \gamma(\overline{G}) + 1$.

This bound is attained if G is a 3-regular graph on 6 vertices. □

Remark 3.2.

1. Any proper subset D of $V(G)$ is a dominating set of $\overline{B}_1(G)$ if and only if D is a dominating set of $\overline{G}$ and is a point cover for G.

2. For any graph G having no isolated vertices, $\gamma(\overline{B}_1(G)) \leqslant \alpha_1(G)$, if $\gamma(\overline{L}(G)) = \alpha_1(G)$.

3. Any proper subset D of $V(L(G))$ is a dominating set of $\overline{B}_1(G)$ if and only if D is a dominating set of $\overline{L}(G)$ and is a line cover for G.

4. For any (p, q) graph G having no isolated vertices with $p \geqslant 6$ if there exists a perfect matching in G, then $\gamma(\overline{B}_1(G)) \leqslant p/2$.

Theorem 3.7. *Let G be any disconnected graph not totally disconnected. If G contains K_2, P_3 or C_3 as one of its components, then $\gamma(\overline{B}_1(G)) = 2$.*

Proof. Let G contain K_2 as one of its components and $u \in V(K_2)$ and e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge in K_2. Then $\{u, e'\}$ is a dominating set of $\overline{B}_1(G)$. Hence, $\gamma(\overline{B}_1(G)) = 2$. Assume G contains P_3 or C_3 as one of its components. Let v be a vertex in $V(P_3)$ or $V(C_3)$ and e' be the line vertex in

$\overline{B}_1(G)$ corresponding to an edge in P_3 or C_3 not incident with v. Then $\{v, e'\}$ is a dominating set of $\overline{B}_1(G)$. Hence, $\gamma(\overline{B}_1(G)) = 2$. $\square$

Remark 3.3. Let G be disconnected such that $G \neq nK_1$ and $G_1, G_2, \ldots, G_t$ $(t \geqslant 2)$ be the components of G. If at least one of the graphs $\overline{B}_1(G_i)$ has a dominating set containing at least one point and one line vertex, then it is a dominating set of $\overline{B}_1(G)$.

Example 3.1.

(i) $\gamma(\overline{B}_1(P_n)) = 2$, if $n = 3, 4$; and
$= 3$, if $n \geqslant 5$.

(ii) $\gamma(\overline{B}_1(C_n)) = 2$, if $n = 3$; and
$= 3$, if $n \geqslant 4$.

(iii) $\gamma(\overline{B}_1(K_n)) = n - 1$, if $n \geqslant 3$.

(iv) $\gamma(\overline{B}_1(K_{1,n})) = 2$, if $n \geqslant 2$.

(v) $\gamma(\overline{B}_1(W_n)) = 3$, if $n = 3, 4, 5$; and
$= 4$, if $n \geqslant 6$, where W_n is a wheel on n vertices.

3.2. Independent, connected and cycle domination numbers.

Theorem 3.8. *Let G be any graph not totally disconnected. Then $\gamma_{\rm i}(\overline{B}_1(G)) = 2$ if and only if G is one of the following.*

(i) $G \cong$ *(double star)* $\cup\, mK_1$, $m \geqslant 0$;
(ii) $G \cong K_{1,n} \cup mK_1$, $n \geqslant 2$ *and* $m \geqslant 0$; *and*
(iii) G *contains* $K_2 \cup K_1$, P_3 *or* C_3 *as one of its components.*

Proof. Let $\gamma_{\rm i}(\overline{B}_1(G)) = 2$. Then there exists an independent dominating set D of two vertices in $\overline{B}_1(G)$.

Case (i): D contains two point vertices of $\overline{B}_1(G)$. Let $D = \{u, v\}$, where u, v are adjacent vertices in G. If u and v are not adjacent, then D cannot be independent in $\overline{B}_1(G)$. Since D is a dominating set of $\overline{B}_1(G)$, no vertex in $V(G) - D$ is adjacent to both u and v and each edge in G adjacent to the edge $(u, v) \in E(G)$. Hence, $G \cong$ (double star) $\bigcup mK_1$, $m \geqslant 0$, or $G \cong K_{1,n}$, $\bigcup mK_1$, for $n \geqslant 2$ and $m \geqslant 0$.

Case (ii): D contains one point and one line vertex. Let $D = \{v, e'\}$, where $v \in V(G)$ and e' is the line vertex in $\overline{B}_1(G)$ corresponding to an edge e in G not incident with v. Since D is a dominating set of $\overline{B}_1(G)$, edges in G adjacent to e must be incident with v in G or there can exist vertices and edges in G not adjacent to v and e respectively. Hence, G contains $K_2 \cup K_1$, P_3 or C_3 as one of its components.

Case (iii): D contains two line vertices. Let $D = \{e_1', e_2'\}$, where e_1' and e_2' are the line vertices in $\overline{B}_1(G)$ corresponding to two adjacent edges e_1 and e_2 in G. Then the fact that D is a dominating set of $\overline{B}_1(G)$ implies that $G \cong P_3$.

Conversely, if G is one of the graphs given as in (i), (ii) and (iii), then there exists a minimal independent dominating set for $\overline{B}_1(G)$ containing two vertices. □

Theorem 3.9. *Let C_3 be a maximal clique in G. If all the edges of G are incident with at least one of the vertices in C_3, then $\gamma_{\rm i}(\overline{B}_1(G)) \leqslant 3$.*

Proof. Let $D = V(C_3)$. By hypothesis, D dominates both point and line vertices in $\overline{B}_1(G)$ and is an independent subset of $\overline{B}_1(G)$. Hence, $\gamma_{\rm i}(\overline{B}_1(G)) \leqslant 3$. □

Theorem 3.10. *Let G be any graph not totally disconnected and D a minimal point cover for G. Then $\gamma_{\rm i}(\overline{B}_1(G)) \leqslant \alpha_0(G)$ if and only if $\langle D\rangle$ is a maximal clique in G.*

Proof. Let D be a minimal point cover for G such that $\langle D\rangle$ is a maximal clique in G with $|D| = \alpha_0(G)$. Then D is an independent dominating set of $\overline{B}_1(G)$ and hence $\gamma_{\rm i}(\overline{B}_1(G)) \leqslant \alpha_0(G)$. Conversely, assume $\gamma_{\rm i}(\overline{B}_1(G)) \leqslant \alpha_0(G)$. If $\langle D\rangle$ is not a maximal clique in G, then there exists a vertex in G adjacent to all the vertices of D and D does not dominate the corresponding point vertices in $\overline{B}_1(G)$, which is a contradiction.

This bound is attained if $G \cong K_1 + K_2 + K_2 + K_1$. □

Corollary 3.10.1. *Let G be any graph not totally disconnected and D a maximal clique in G with $|D| = \varphi(G)$, where $\varphi(G)$ is the clique number of G. Then $\gamma_{\rm i}(\overline{B}_1(G)) \leqslant \varphi(G)$ if and only if D is a point cover for G.*

This bound is attained if $G \cong K_{1,n} + e$, $n \geqslant 3$.

In the following, the graphs G for which $\gamma_{\rm c}(\overline{B}_1(G)) = 2$ are found.

Theorem 3.11. *Let G be any graph not totally disconnected. Then $\gamma_{\rm c}(\overline{B}_1(G)) = \gamma_{\rm t}(\overline{B}_1(G)) = 2$ if and only if G is one of the following graphs.*

(a) *$K_{1,n} \cup K_{1,m} \cup tK_1$, $m, n \geqslant 2$ and $t \geqslant 0$; and*

(b) *G contains K_2 as one of its components.*

Proof. Assume $\gamma_{\rm c}(\overline{B}_1(G)) = 2$. Then there exists a minimal connected dominating set D of $\overline{B}_1(G)$ containing two vertices.

Case (i): D contains two point vertices. Let $D = \{u, v\} \subseteq V(\overline{B}_1(G))$, where u, $v \in V(G)$. Since D is a connected dominating set of $\overline{B}_1(G)$, D must be a connected dominating set of $\overline{G}$ and is a point cover for G. In other words, D is an independent point cover for G and is a dominating set of $\overline{G}$. Hence, there exists no vertex in G adjacent to both u and v and all the edges in G are incident with exactly one of u and v. Therefore, $K_{1,n} \cup K_{1,m} \cup tK_1$, $m, n \geqslant 1$ and $t \geqslant 0$, with u and v as center vertices.

Case (ii): D contains one point and one line vertex. Let $D = \{u, e'\}$, where $u \in V(G)$ and e' is the line vertex in $\overline{B}_1(G)$ corresponding to an edge in G incident with u. Since D is a dominating set of $\overline{B}_1(G)$, vertices adjacent to $u \in V(G)$ are incident with $e \in E(G)$ and hence G contains K_2 as one of its components.

Case (iii): D contains two line vertices. Let $D = \{e_1', e_2'\} \subseteq V(\overline{B}_1(G))$, where e_1' and e_2' are the line vertices in $\overline{B}_1(G)$ corresponding to any two independent edges e_1 and e_2 in G respectively. If there exists an edge e in G adjacent to both e_1 and e_2 then D does not dominate the corresponding line vertex. Similarly, if there exists a vertex u in G not incident with at least one of e_1 and e_2, then D does not dominate u in $\overline{B}_1(G)$. Hence $G \cong 2K_2$.

Conversely, if G is one of the graphs as in (a) or (b), then there exists a minimal connected dominating set in $\overline{B}_1(G)$ containing two vertices. □

Proposition 3.2. *If G contains three independent vertices such that each edge in G is incident with exactly one of the vertices and no vertex in G is adjacent to all the three vertices, then $\gamma_c(\overline{B}_1(G)) \leqslant 3$.*

Proof. Let D be a subset of $V(G)$ containing three independent vertices satisfying the given condition. Then D is a connected dominating set of $\overline{B}_1(G)$ and hence $\gamma_c(\overline{B}_1(G))) \leqslant 3$.

This bound is attained if $G \cong C_6$.

The following propositions are stated without proof. □

Proposition 3.3. *Let $D \subseteq V(G)$ be an independent set of G and a dominating set of $\overline{B}_1(G)$ with $|D| = 3$. Then D is a cycle dominating set of $\overline{B}_1(G)$ if and only if G is a bipartite graph with the bipartition $[D, V(G) - D]$.*

Proposition 3.4. *Let M be a perfect matching in G with $|M| = 3$. Then the set of line vertices corresponding to the edges in M is a cycle dominating set of $\overline{B}_1(G)$.*

Remark 3.4. There exists no 3-cycle dominating set in $\overline{B}_1(G)$ containing one point vertex and two line vertices or one line vertex and two point vertices.

Remark 3.5. From Theorem 3.8. and Theorem 3.11., it follows that $\gamma(\overline{B}_1(G)) = 2$ if and only if G is one of the following graphs.

(i) $G \cong$ (double star) $\cup\, mK_1$, for $m \geqslant 0$;

(ii) $G \cong K_{1,n} \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 0$;
(iii) $G \cong K_{1,n} \cup K_{1,m} \cup tK_1$, for $n, m \geqslant 2$ and $t \geqslant 0$; and
(iv) G contains K_2, P_3 or C_3 as one of its components.

In the following, the total domination number $\gamma_{\rm t}$ of $\overline{B}_1(G)$ is obtained.

Proposition 3.5. *If G contains a triangle, which is not contained in any K_n ($n \geqslant 4$), then $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant 5$.*

P r o o f. Let there exist a C_3 in G not contained in any K_n ($n \geqslant 4$) and $D = V(C_3) \cup \{$the line vertices corresponding to any two edges in $C_3\}$. Then D is a total dominating set of $\overline{B}_1(G)$, since $\langle D\rangle \cong C_5$ in $\overline{B}_1(G)$. Thus, $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant 5$. □

N o t e 3.1. By Remark 3.1., if there exists an edge in G not lying in any triangle, then $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant 3$.

Proposition 3.6. *Let C be the set of all cliques in G and $k = \min\{|C_i|\colon\ C_i \in C$ and $C_i \subsetneqq C_j$ for any $C_j \in C\}$. Then $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant k + \{k/2\}$.*

P r o o f. Let D' be a line cover for C_i, where $C_i \in C$ and $|C_i| = k$. Then $V(C_i) \cup$ $\{$the line vertices in $\overline{B}_1(G)$ corresponding to the edges in $D'\}$ is a total dominating set of $\overline{B}_1(G)$. Hence, $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant k + \{k/2\}$. □

Proposition 3.7. *If G has no perfect matching, then $\gamma_{\rm t}(\overline{B}_1(G)) = \beta_1(G) + 2$.*

P r o o f. Let D be the set of independent edges in G with $|D| = \beta_1(G)$. Since G has no perfect matching, there exists a vertex $v \in V(G)$ not incident with any of the edges in D. Let $e \in E(G)$ be an edge incident with v and e' the corresponding line vertex in $\overline{B}_1(G)$. If $D' = \{e'\} \cup \{$the line vertices corresponding to the edges in $D\}$, then $D' \cup \{v\}$ is a total dominating set of $\overline{B}_1(G)$. Hence, $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant \beta_1(G) + 2$.

Similarly, the following propositions can be proved. □

Proposition 3.8. *$\gamma_{\rm t}(\overline{B}_1(G)) \leqslant \alpha_0(G) + 1$ if there exists a point cover D of G and a vertex $v \in V(G) - D$ such that $|D| = \alpha_0(G)$ and the radius of the subgraph $\langle D \cup \{v\}\rangle$ of G is at least two.*

Proposition 3.9. *$\gamma_{\rm t}(\overline{B}_1(G)) \leqslant \gamma(\overline{G}) + 1$ if there exists a dominating set D of $\overline{G}$ with $|D| = \gamma(\overline{G})$ such that $\langle D\rangle$ is not totally disconnected in G and $r(\langle D\rangle) \geqslant 2$ in G.*

Proposition 3.10. *$\gamma_{\rm t}(\overline{B}_1(G)) \leqslant \delta(G) + 2$ if there exists a vertex v of minimum degree in G with $r(\langle N(v)\rangle) \geqslant 2$.*

Proposition 3.11. *If G has pendant vertices, then $\gamma_{\rm t}(\overline{B}_1(G)) \leqslant 3$.*

3.3. Perfect, point-set domination numbers. In the following, the perfect domination number $\gamma_{\rm p}$ of $\overline{B}_1(G)$ is determined.

Theorem 3.12. *Let G be any graph having no isolated vertices. If there exists at least one edge $e = (u, v)$ in G such that each vertex in $V(G) - \{u, v\}$ is adjacent to exactly one of u and v, then $\gamma_{\rm p}(\overline{B}_1(G)) \leqslant 3$.*

Proof. Let there exist an edge $e = (u, v)$ in G such that each vertex in $V(G) - \{u, v\}$ is adjacent to exactly one of u and v. Let e' be the line vertex in $\overline{B}_1(G)$ corresponding to the edge e. Then $D = \{u, v, e'\}$ is a dominating set of $\overline{B}_1(G)$. By our hypothesis, each point vertex in $V(\overline{B}_1(G)) - D$ is adjacent to exactly one of u, v, e'. Let $e_1 \in E(G)$. If e_1 is adjacent to e, then the corresponding line vertex in $\overline{B}_1(G)$ is adjacent to exactly one of u and v. If e_1 is not adjacent to e, then the line vertex is adjacent to e only. Hence, D is a perfect dominating set of $\overline{B}_1(G)$ and $\gamma_{\rm p}(\overline{B}_1(G)) \leqslant 3$. □

Theorem 3.13. *Let $D \subseteq V(G)$ be a dominating set of $\overline{B}_1(G)$ with $|D| \geqslant 3$. Then D is a perfect dominating set of $\overline{B}_1(G)$ if and only if D is independent in G and $|N_G(v) \cap D| = |D| - 1$, for all $v \in V(G) - D$.*

Proof. Let $D \subseteq V(G)$ be a dominating set of $\overline{B}_1(G)$ with $|D| \geqslant 3$. Assume the given conditions. Since D is independent in G, each line vertex in $V(\overline{B}_1(G)) - D$ is adjacent to exactly one vertex in D. Since $|N_G(v) \cap D| = |D| - 1$, for all $v \in V(G) - D$, each vertex in $V(G) - D$ is adjacent to $|D| - 1$ vertices in D and hence each point vertex in $V(\overline{B}_1(G)) - D$ is adjacent to exactly one vertex in D. Therefore, D is a perfect dominating set of $\overline{B}_1(G)$. Conversely, assume D is a perfect dominating set of $\overline{B}_1(G)$. If D is not independent in G, then there exists an edge $e = (u, v)$ in $\langle D \rangle_G$, where $u, v \in D$ and the corresponding line vertex is adjacent to both vertices u and v in D, which is a contradiction. If $|N_G(v) \cap D| = |D|$, for at least one $v \in V(G) - D$, then $v \in V(\overline{B}_1(G)) - D$ is adjacent to none of the vertices in D. Similarly, if $|N_G(v) \cap D| < |D| - 1$ for at least one $v \in V(G) - D$, then $v \in V(\overline{B}_1(G)) - D$ is adjacent to at least two vertices in D, which is a contradiction. Hence, $|N_G(v) \cap D| = |D| - 1$, for all $v \in V(G) - D$. □

Remark 3.6. Theorem 3.13. can be restated as follows. Let G be a graph having no isolated vertices. Then $D \subseteq V(G)$ is a perfect dominating set of $\overline{B}_1(G)$ if and only if G is bipartite with the bipartition $[D, V - D]$ such that $\deg_G(v) = |D| - 1$, for all $v \in V(G) - D$.

Theorem 3.14. *Let G be a graph having no isolated vertices and D a perfect matching in G. Then the set of line vertices in $\overline{B}_1(G)$ corresponding to the edges in D is a perfect dominating set of $\overline{B}_1(G)$ if and only if either $|D| = 3$ or $G \cong nK_2$, $n \geqslant 2$.*

P r o o f. Assume D is a perfect matching in G and $|D| = 3$. Let D' be the set of line vertices in $\overline{B}_1(G)$ corresponding to the edges in D. Since $|D'| = 3$ and $\overline{L}(G)$ is an induced subgraph of $\overline{B}_1(G)$, each line vertex in $V(\overline{B}_1(G)) - D'$ is adjacent to exactly one vertex in D'. Also since D is a perfect matching in G, each point vertex in $V(\overline{B}_1(G)) - D'$ is adjacent to exactly one vertex in D'. Hence, D' is a perfect dominating set of $\overline{B}_1(G)$. If $G \cong nK_2$, $n \geqslant 2$, then the line vertices corresponding to the edges in the perfect matching of G form a perfect dominating set of $\overline{B}_1(G)$.

Conversely, assume D' is a perfect dominating set of $\overline{B}_1(G)$. If $|D| > 3$, then each line vertex in $V(\overline{B}_1(G)) - D'$ is adjacent to at least two vertices in D'. If $|D| = 2$, then any line vertex (if any) in $V(\overline{B}_1(G)) - D'$ is adjacent to none of the vertices in D'. Hence, $|D| = 3$ or $G \cong nK_2$, $n \geqslant 2$. $\square$

R e m a r k 3.7. Let G be any graph not totally disconnected. Then the set $V(G)$ is not a perfect dominating set of $\overline{B}_1(G)$ and the set $V(L(G))$ is a perfect dominating set of $\overline{B}_1(G)$ if and only if $G \cong nK_2$, $n \geqslant 2$.

In the following, the point-set domination number $\gamma_{\rm ps}$ of $\overline{B}_1(G)$ is determined. We use Theorem 2.1.

Theorem 3.15. *Let G be any graph not totally disconnected. Then $\gamma_{\rm ps}(\overline{B}_1(G)) = 2$ if and only if G is one of the following graphs.*

(i) *(double star) $\bigcup mK_1$, for $m \geqslant 0$;*
(ii) *$K_{1,n} \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 0$;*
(iii) *$K_{1,n} \cup K_{1,m} \cup tK_1$, for $m, n \geqslant 2$ and $t \geqslant 0$; and*
(iv) *G contains K_2, P_3 or C_3 as one of its components.*

P r o o f. By Remark 3.5, $\gamma(\overline{B}_1(G)) = 2$ if and only if G is one of the graphs given above. Let D be a dominating set of $\overline{B}_1(G)$ with $|D| = 2$ and $W \subseteq V(\overline{B}_1(G)) - D$ be independent. Then there exists a point vertex or a line vertex in D adjacent to all the vertices in W. Hence, D is a point-set dominating set of $\overline{B}_1(G)$ by Theorem 2.1. and hence $\gamma_{\rm ps}(\overline{B}_1(G)) = 2$. $\square$

Theorem 3.16. *Let G be any graph containing at least one edge. If there exists an edge $e = (u, v)$ in G not lying in any triangle, then $D = \{u, v, e'\}$, where e' is the line vertex corresponding to e, is a point-set dominating set (psd-set) of $\overline{B}_1(G)$ if and only if G is one of the following graphs: $K_{1,2} \cup mK_1$, $2K_2 \cup mK_1$ and $P_4 \cup mK_1$, where $m \geqslant 0$.*

Proof. If G contains C_4 or $K_{1,n}$ $(n \geqslant 3)$ as a subgraph, then D is not a psd-set of $\overline{B}_1(G)$. Since $e = (u, v)$ does not lie in any triangle, G is either $K_{1,2}$ or P_4, if G is connected. If G is disconnected and if G contains $K_{1,2} \cup K_2$ or $P_4 \cup K_2$ as a subgraph, then also D is not a psd-set of $\overline{B}_1(G)$. Hence, G is one of the above graphs. The converse is obvious. $\square$

Theorem 3.17. *Let G be any graph having no isolated vertices and $D \subseteq V(G)$ a dominating set of $\overline{B}_1(G)$. Then D is a psd-set of $\overline{B}_1(G)$ if and only if*

(a) $\deg_G(v) = 1$, *for all* $v \in V(G) - D$;

(b) $|N_G(u) \cap (V(G) - D)| = 1$, *for all* $u \in D$; *and*

(c) $|N_G(u) \cap D| \leqslant 1$, *for all* $u \in D$.

Proof. Let $D \subseteq V(G)$ be a dominating set of $\overline{B}_1(G)$. Then D is a dominating set of $\overline{G}$ and is a point cover for G and hence $V(G) - D$ is independent.

(i) If there exists a vertex $v \in V(G) - D$ with $\deg_G(v) \geqslant 2$, then the set of line vertices corresponding to any two edges incident with v is an independent set in $V(\overline{B}_1(G)) - D$ and there exists no vertex in D adjacent to both the line vertices, which is a contradiction. Hence, $\deg_G(v) = 1$, for all $v \in V(G) - D$.

(ii) Let there exist a vertex $u \in D$ with $|N_G(u) \cap (V(G) - D)| \geqslant 2$ and $N(u) \cap (V(G) - D) = \{u_1, u_2\}$ and let $(u, u_2) = e_2$. Then $W = \{u_1, e'_2\}$, where e'_2 is the line vertex corresponding to e_2, is an independent set of $V(\overline{B}_1(G)) - D$ and there exists no vertex in D adjacent to both the vertices in W. Hence, $|N_G(u) \cap (V(G) - D)| \leqslant 1$. But D is a point cover for G and therefore $|N_G(u) \cap (V(G) - D)| = 1$.

(iii) Let there exist a vertex $u \in D$ with $|N_G(u) \cap D| \geqslant 2$ and let e'_1 and e'_2 be the line vertices in $\overline{B}_1(G)$ corresponding to the edges in $\langle D \rangle_G$ adjacent to u. Since D is a point cover for G, there exists at least one vertex, say, $v \in V(G) - D$, adjacent to u. Then $W = \{v, e'_1, e'_2\} \subseteq V(\overline{B}_1(G)) - D$ is an independent set in $V(\overline{B}_1(G)) - D$ and no vertex in D is adjacent to all the vertices in W. Hence, $|N_G(u) \cap D| \leqslant 1$, for all $u \in D$. Conversely, let $D \subseteq V(G)$ be a dominating set of $\overline{B}_1(G)$ satisfying (a), (b) and (c). Let W be an independent set in $V(\overline{B}_1(G)) - D$. Since D is a point cover for G, $V(G) - D$ is independent and hence $\langle V(G) - D \rangle$ is a clique in $V(\overline{B}_1(G)) - D$. Therefore, the number of point vertices in W is at most 1. If W contains only line vertices then by conditions (a), (b) and (c), it follows that $|W| \leqslant 2$ and there exists a point vertex in D adjacent to all vertices in W. Let W contain both point and line vertices. Then W contains one point vertex and at most two line vertices and there exists a vertex in D adjacent to all the vertices in W. Hence, D is a psd-set of $\overline{B}_1(G)$. $\square$

Corollary 3.17.1. *Let G be any graph having no isolated vertices and $D \subseteq V(G)$ a dominating set of $\overline{G}$ and $e \in E(\langle D\rangle)$ in G. Then $D \cup \{e'\}$, where e' is the line vertex corresponding to e, is a psd-set of $\overline{B}_1(G)$ if and only if*

(i) *$V(G) - D$ is independent;*

(ii) *$\deg_G(v) = 1$, for all $v \in V(G) - D$;*

(iii) *$|N_G(u) \cap (V(G) - D)| = 1$, for all $u \in D$; and*

(iv) *$|N_G(u) \cap D| \leqslant 1$, for all $u \in D$.*

Theorem 3.18. *Let G be any graph having no isolated vertices and $D \subseteq V(G)$. Then D is a psd-set of $\overline{B}_1(G)$ if and only if G contains no triangles.*

Proof. Assume $D \subseteq V(G)$ is a psd-set of $\overline{B}_1(G)$ and G contains triangles. Let W be the set of line vertices in $\overline{B}_1(G)$ corresponding to the edges in any triangle in G. Then W is an independent set in $V(\overline{B}_1(G)) - D$ and there is no vertex in D adjacent to all the vertices in W, which is a contradiction. Hence, G is a triangle-free. Conversely, if G is triangle-free, then for every independent set W in $V(\overline{B}_1(G)) - D$ there exists at least one vertex in D adjacent to all the vertices in W. Hence, D is a psd-set of $\overline{B}_1(G)$. □

Remark 3.8. Similarly, $D \subseteq V(L(G))$ is a psd-set of $\overline{B}_1(G)$ if and only if G is triangle-free.

Remark 3.9.

(i) Let G be any (p,q) graph having triangles. Then $\gamma_{\rm ps}(\overline{B}_1(G)) \leqslant p+k$, where k is the minimum number of edges to be deleted in G such that G becomes triangle-free.

(ii) If there exists a perfect matching M containing at least three edges in G, then the line vertices corresponding to the edges in M form a psd-set of $\overline{B}_1(G)$ if and only if $G \cong nK_2$, for $n \geqslant 2$.

Proposition 3.12. *Let $D' \subseteq V(L(G))$ be a dominating set of $\overline{B}_1(G)$ and let D be the set of edges in G corresponding to the vertices in D'. Then D' is a psd-set of $\overline{B}_1(G)$ if and only if either* (i) *there is no edge in G joining the end vertices of any two edges in D or* (ii) *if there exists such an edge e in G, then e is an edge in D.*

Proof. Assume $D' \subseteq V(L(G))$ is a dominating set of $\overline{B}_1(G)$. Let e_1 and e_2 be any two edges in G such that e_1', $e_2' \in D'$ and let v_1 and v_2 be the distinct end vertices of e_1 and e_2 respectively. If v_1 and v_2 are adjacent in G, then $W = \{v_1, v_2\} \subseteq V(\overline{B}_1(G))$ is independent in $V(\overline{B}_1(G)) - D$ and there exists no vertex in D adjacent to both v_1 and v_2, which is a contradiction. Hence, (i) or (ii) is true. The converse part can be proved easily. □

3.4. Restrained, split and non-split domination numbers. In the following, the restrained domination number $\gamma_{\rm r}$ of $\overline{B}_1(G)$ is determined. The following propositions are stated without proof.

Proposition 3.13. *Let G be any graph not totally disconnected and $G \ncong K_2 \cup K_1$. Then $\gamma_{\rm r}(\overline{B}_1(G)) = 2$ if and only if G is one of the following graphs.*

(i) *(double star) $\bigcup mK_1$, for $m \geqslant 0$;*

(ii) *$K_{1,n} \cup K_{1,m} \cup tK_1$, for $m, n \geqslant 2$ and $t \geqslant 0$;*

(iii) *$K_{1,n} \cup mK_1$, for $n \geqslant 3$ and $m \geqslant 0$; and*

(iv) *G contains K_2, P_3 or C_3 as one of its components.*

Proposition 3.14. *Let G be any graph having at least two edges. If there exists at least one edge in G not lying in any triangle, then $\gamma_{\rm r}(\overline{B}_1(G)) \leqslant 3$. If $G \ncong C_3 \cup mK_1$, for $m \geqslant 0$, and each edge in G lies in exactly one triangle, then $\gamma_{\rm r}(\overline{B}_1(G)) \leqslant 4$.*

Proposition 3.15. *$\gamma_{\rm r}(\overline{B}_1(G)) \leqslant \alpha_0(G) + 1$, where $\alpha_0(G) \geqslant 3$.*

Remark 3.10. If $\alpha_0(G) = 2$, then the point cover D of G must be independent. If D is not independent, then the line vertices corresponding to the edges in D are not adjacent to any vertex in $V(\overline{B}_1(G)) - D$. If $\alpha_0(G) = 1$, then $G \cong K_{1,n}$, for $n \geqslant 2$ and $\gamma_{\rm r}(\overline{B}_1(G)) = 3 = \alpha_0(G) + 2$.

Proposition 3.16. *$\gamma_{\rm r}(\overline{B}_1(G)) \leqslant \alpha_1(G)$, if $\gamma(\overline{L}(G)) = \alpha_1(G)$.*

Proposition 3.17. *$\gamma_{\rm r}(\overline{B}_1(G)) \leqslant \gamma(\overline{G}) + 1$, if $\gamma(\overline{G}) \neq \alpha_0(G)$ and there exists a dominating set D of $\overline{G}$ with $|D| = \gamma(\overline{G})$ such that D is not independent in G. If every dominating set of $\overline{G}$ is independent in G, then $\gamma_{\rm r}(\overline{B}_1(G)) \leqslant \gamma(\overline{G}) + 2$.*

Proposition 3.18. *Let $G \ncong K_n$, $n = 2, 3, 4$, and let C denote the set of all complete subgraphs of G. Let $C' = \{C_i \in C\colon\ C_i \nsubseteqq C_j$ for any $j\}$ and $k = \min\{|C_i|\colon C_i \in C'\}$. Then $\gamma_{\rm r}(\overline{B}_1(G)) \leqslant k + 1$.*

Proposition 3.19. *If there exists a perfect matching in G containing at least 3 edges, then $\gamma_{\rm r}(\overline{B}_1(G)) \leqslant p/2$.*

Proposition 3.20. *The set of all point vertices in $\overline{B}_1(G)$ is a restrained dominating set of $\overline{B}_1(G)$ if and only if $\Delta(L(G)) \leqslant q-2$. If $\Delta(L(G)) = q-1$, then the set $V(G) \cup \{$a center of $L(G)\}$ is a restrained dominating set of $\overline{B}_1(G)$. Similarly, the set of all line vertices in $\overline{B}_1(G)$ is a restrained dominating set of $\overline{B}_1(G)$ if and only if $\Delta(G) \leqslant p-2$. If $\Delta(G) = p-1$, then the set $V(L(G)) \cup \{$a center of $G\}$ is a restrained dominating set of $\overline{B}_1(G)$.*

In the following, the split domination number $\gamma_{\rm s}$ of $\overline{B}_1(G)$ occurs. We give upper bounds for $\gamma_{\rm s}(\overline{B}_1(G))$.

Theorem 3.19. *Let G be any graph with $\Delta(G) \geqslant 2$. Then $\gamma_{\rm s}(\overline{B}_1(G)) \leqslant p-1$.*

Proof. Let $v \in V(G)$ be such that $\deg_G(v) = \Delta(G) \geqslant 2$ and let D' be the set of line vertices in $\overline{B}_1(G)$ corresponding to the edges in G incident with v. Then $D = D' \cup \{V(G) - N[v]\} \subseteq V(\overline{B}_1(G))$ is a dominating set of $\overline{B}_1(G)$ and v is isolated in $\langle V(\overline{B}_1(G)) - D\rangle$ and is disconnected. Hence, D is a split dominating set of $\overline{B}_1(G)$. Thus, $\gamma_{\rm s}(\overline{B}_1(G)) \leqslant \Delta(G) + p - (\Delta(G)+1)$ and $\gamma_{\rm s}(\overline{B}_1(G)) \leqslant p-1$.

This bound is attained if $G \cong C_3$. $\square$

Theorem 3.20. *Let G be any graph not totally disconnected. Then $\gamma_{\rm s}(\overline{B}_1(G)) \leqslant q+2-\Delta(G)$.*

Proof. Let $v \in V(G)$ be such that $\deg_G(v) = \Delta(G)$ and let D' be the set of line vertices in $\overline{B}_1(G)$ such that the corresponding edges in G are not incident with v. Let w be any vertex in G adjacent to v. Then $D = D' \cup \{v, w\}$ is a dominating set of $\overline{B}_1(G)$ and the line vertex corresponding to the edge (v,w) in G is isolated in $\langle V(\overline{B}_1(G)) - D\rangle$. Hence, D is a split dominating set of $\overline{B}_1(G)$. Thus, $\gamma_{\rm s}(\overline{B}_1(G)) \leqslant q+2-\Delta(G)$.

This bound is attained when $G \cong K_{1,n}$, $n \geqslant 2$ or $G \cong nK_2$, $n \geqslant 1$. $\square$

Remark 3.11. If $\overline{G}$ is disconnected, then $\gamma_{\rm s}(\overline{B}_1(G)) \leqslant q$.

Proposition 3.21. *The non-split domination number $\gamma_{\rm ns}(\overline{B}_1(G))$ satisfies $\gamma_{\rm ns}(\overline{B}_1(G)) \leqslant \min\{\delta(G)+2, \alpha_0(G)+1, \beta_1(G)+1, \gamma(\overline{G})+2, k+1\}$, where $k = \min\{|C_i|$, C_i is a clique in G not contained in any other clique in $G\}$.*

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