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Abstract. **We show that for each $p \in (0,1]$ there exists a separable p-Banach space $\mathbb{G}_p$ of almost universal disposition, that is, having the following extension property: for each $\varepsilon > 0$ and each isometric embedding $g : X \to Y$, where Y is a finite dimensional p-Banach space and X is a subspace of $\mathbb{G}_p$, there is an ε-isometry $f : Y \to \mathbb{G}_p$ such that $x = f(g(x))$ for all $x \in X$.**

Such a space is unique, up to isometries, does contain an isometric copy of each separable p-Banach space and has the remarkable property of being "locally injective" amongst p-Banach spaces.

We also present a nonseparable generalization which is of universal disposition for separable spaces and "separably injective". No separably injective p-Banach space was previously known for $p < 1$.

Contents

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1. Introduction

1.1. Background. In 1965, Gurariĭ constructed a separable Banach space $\mathbb{G}$ of "almost universal disposition for finite dimensional spaces", that is, having the following extension property: for every isometry $g : X \to Y$, where Y is a finite dimensional Banach space and X is a subspace of $\mathbb{G}$, and every $\varepsilon > 0$ there is an ε-isometry $f : Y \to \mathbb{G}$ such that $f(g(x)) = x$ for all $x \in X$. Here, (ε-)*isometry* is a shortcut for *linear* (ε-)*isometric embedding*.

It is almost obvious that if V is any other separable Banach space fitting in the definition, then there is a linear isomorphism $u : \mathbb{G} \to V$ with $\|u\| \cdot \|u^{-1}\|$ arbitrarily close to 1.

Gurariĭ's creature spurred a considerable interest in Banach space theory and is still an object of intense research. Amongst the main hits we find the following. The space $\mathbb{G}$ constructed by Gurariĭ is isometrically unique, in the class of separable Banach spaces. This was proved by Lusky [22]; see [21] for a simpler proof. The space $\mathbb{G}$ is a Lindenstrauss space, that is, its dual space is isometric to an L_1-space. Moreover, every separable Lindenstrauss space is isometric to a subspace of $\mathbb{G}$ which is the range of a nonexpansive projection. This was proved by Wojtaszczyk [28], see also [23, Proposition 8]. The Gurariĭ space is complemented in no space of type $C(K)$ and it is isomorphic to the space of all continuous affine functions on the Poulsen simplex; see [6, Corollary 2] and [23].

We refer the reader to the survey paper [12] for more information and references reporting recent work.

1.2. The plan of the paper. There is no clear intrinsic reason to restrict attention to Banach spaces in studying the extension of isometries. In this paper we push the notion of "universal disposition" and its relatives into the larger class of quasi-Banach spaces.

We shall construct, for each $p \in (0, 1]$, a separable p-Banach space of almost universal disposition for finite dimensional p-Banach spaces, which turns out to be unique, up to isometries, and that we will call $\mathbb{G}_p$. Our main tools are the push-out construction and the notion of a direct limit, whose adaptations to the p-normed setting are presented in Sections 1.4 and 1.5. The construction itself is carried out in Section 2.

In Section 3 we prove that any two separable p-Banach spaces of almost universal disposition for finite dimensional p-Banach spaces are isometric. As a consequence, $\mathbb{G}_p$ contains an isometric copy of each separable p-Banach space, which improves a classical result by Kalton and provides a complete solution to an old problem in the isometric theory of quasi-Banach spaces. Up to this point the paper is rather elementary and self-contained.

In Section 4 we present a nonseparable generalization. We construct a p-Banach space whose density character is the continuum and which is of universal disposition for separable p-Banach spaces. We also mention a result of Ben Yaacov and Henson [5] with a simpler argument provided by Richard Haydon, showing that it is impossible to reduce the size of the space in the preceding result. We prove that these spaces contain isometric copies of all

p-Banach spaces with density character $\aleph_1$ or less and that they are all isometric under the continuum hypothesis.

Section 5 studies the extension of operators with values in the spaces of (almost) universal disposition. Let us pause a moment for some definitions. First, following a long standing tradition, a quasi-Banach space E would be injective amongst p-Banach spaces if there is a constant $\lambda \geq 1$ such that for every p-Banach space X and every subspace Y of X every operator $t : Y \to E$ extends to an operator $T : X \to E$ with $\|T\| \leq \lambda\|t\|$. Also, we say that E is separably injective amongst p-Banach spaces if the preceding condition holds for X separable and we say that it is locally injective if it holds when X is finite dimensional.

After proving that there is no injective p-Banach space, apart from 0, we show that $\mathbb{G}_p$ is locally (1+)-injective and also that any space of universal disposition for separable p-Banach spaces is separably 1-injective. No separably injective p-Banach space had been previously known for $p < 1$.

In Section 6 we show the existence of a nonexpansive projection on $\mathbb{G}_p$ whose kernel is isometric to $\mathbb{G}_p$. Moreover, this projection is universal in the sense that the class of all its restrictions to closed subspaces contains (up to isometry) all possible nonexpansive operators from separable p-Banach spaces into $\mathbb{G}_p$.

Finally, the closing Section 7 contains some miscellaneous remarks and questions which we found interesting.

1.3. Quasi-Banach spaces. We shall denote by $\mathbb{K}$ the field of scalars, which is fixed to be either the field of real or complex numbers.

A quasinorm on a $\mathbb{K}$-linear space X is a function $\|\cdot\| : X \to \mathbb{R}_+$ satisfying the following conditions:

- If $\|x\| = 0$, then $x = 0$.
- $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{K}$ and every $x \in X$.
- There is a constant C such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$.

A quasinorm gives rise to a linear topology on X, namely the least linear topology for which the unit ball $B = \{x \in X : \|x\| \leq 1\}$ is a neighborhood of zero. This topology is locally bounded, that is, it has a bounded neighborhood of zero. Actually, every locally bounded topology arises in this way. We refer the reader to [18, 26] for general information on locally bounded spaces.

A quasinormed space is a linear space X equipped with a quasinorm. If X is complete for the quasinorm topology we say that X is a quasi-Banach space.

Let $p \in (0, 1]$. A p-normed (respectively, p-Banach) space is a quasi-normed (respectively, quasi-Banach) space whose quasinorm is a p-norm, that is, it satisfies the inequality $\|x + y\|^p \leq \|x\|^p + \|y\|^p$. The case $p = 1$ corresponds to the popular class of Banach spaces. Observe that every p-norm is also a q-norm for each $q \leq p$.

It is an important result of Aoki and Rolewicz that every quasinorm is equivalent to a p-norm for some $p \in (0,1]$ in the sense that they induce the same topology; see [18, Theorem 1.3] or [26, Theorem 3.2.1].

Let X and Y be quasinormed spaces. A linear map $f : X \to Y$ is a (bounded) operator if there is a constant K such that $\|f(x)\|_Y \leq K\|x\|_X$ for all $x \in X$. The infimum of the constants K for which the preceding inequality holds is denoted by $\|f\|$, the quasinorm of f.

If besides one has $(1-\varepsilon)\|x\|_X \leq \|f(x)\|_Y \leq (1+\varepsilon)\|x\|_X$ for some $\varepsilon \in [0,1)$ independent of $x \in X$, then f is called an ε-isometry. If $\|f(x)\|_Y = \|x\|_X$ for all $x \in X$, then f is called an isometry. Isometries are not assumed to be surjective. However, we say that X and Y are isometric if there is a surjective isometry $f : X \to Y$.

Note that there is no quasi-Banach space containing, for every $\varepsilon > 0$ and every $p \in (0,1]$, a subspace ε-isometric to the 2-dimensional space ℓ_p^2, the space $\mathbb{K}^2$ with the p-norm $\|(s,t)\|_p = (|s|^p + |t|^p)^{1/p}$. So, strictly speaking, the title of the paper is a bit exaggerated.

1.4. Push-outs. This section is an adaptation of [2, Section 2.1] to the p-normed setting.

Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of p-Banach spaces, where Γ is a set of indices. We set

$$\ell_p(\Gamma, X_\gamma) = \left\{ (x_\gamma) \in \prod_{\gamma\in\Gamma} X_\gamma : \left(\sum_\gamma \|x_\gamma\|^p \right)^{1/p} < \infty \right\}$$

with the obvious p-norm. If the family has two spaces only, say X and Y, we just write $X \oplus_p Y$. It is important to realize that this construction represents the direct sum in the category of p-Banach spaces and nonexpansive operators in the obvious sense.

Let $u : X \to Y$ and $v : X \to Z$ be operators acting between p-normed spaces. The associated push-out diagram is

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ Z & \xrightarrow{u'} & \mathrm{PO} \end{array} \tag{1}$$

Here, $\mathrm{PO} = \mathrm{PO}(u,v)$ is the quotient of the p-sum $Y \oplus_p Z$ by S, the closure of the subspace $\{(u(x), -v(x)) : x \in X\}$. The map $u' : Z \to \mathrm{PO}$ is the inclusion of Z into $Y \oplus_p Z$, followed by the quotient map of $Y \oplus_p Z$ onto $\mathrm{PO} = (Y \oplus_p Z)/S$. The operator v' is defined analogously. The universal property behind this construction is that the preceding diagram is "minimally commutative", in the sense that if $v'' : Y \to E$ and $u'' : Z \to E$ are operators such that $u'' \circ v = v'' \circ u$, then there is a unique operator $w : \mathrm{PO} \to C$ satisfying $u'' = w \circ u'$ and $v'' = w \circ v'$. Clearly, $w((y,z) + S) = v''(y) + u''(z)$.

As for the quasinorm of the operators appearing in (1) it is obvious that both u' and v' are nonexpansive. The proof of the following remark is left to the reader.

Lemma 1.1. *Referring to Diagram 1, if u is an isometry and $\|v\| \leq 1$, then u' is an isometry.*

1.5. Direct limits. Let (X_α) be a family of p-Banach spaces indexed by a directed set Γ whose preorder is denoted by $\leq$. Suppose that, for each $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$ we have an isometry $f_\alpha^\beta : X_\alpha \to X_\beta$ in such a way that f_α^α is the identity on X_α for every $\alpha \in \Gamma$ and $f_\beta^\gamma \circ f_\alpha^\beta = f_\alpha^\gamma$ provided $\alpha \leq \beta \leq \gamma$. Then $(X_\alpha, f_\alpha^\beta)$ is said to be a directed system of p-Banach spaces.

The direct limit of the system is constructed as follows. First we take the disjoint union $\bigsqcup_\alpha X_\alpha$ and we define an equivalence relation $\sim$ by identifying $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$ if there is $\gamma \in \Gamma$ such that $f_\alpha^\gamma(x_\alpha) = f_\beta^\gamma(x_\beta)$.

Then we may use the natural inclusion maps $\imath_\gamma : X_\gamma \to \bigsqcup_\alpha X_\alpha$ to transfer the linear structure and the p-norm from the spaces X_α to $\bigsqcup_\alpha X_\alpha / \sim$ thus obtaining a p-normed space whose completion is called the direct limit of the system and is denoted by $\varinjlim X_\gamma$. The universal property behind this construction is the following: if we are given a system of nonexpansive operators $u_\gamma : X_\gamma \to Y$, where Y is a p-Banach space, which are compatible with the f_α^β in the sense that $u_\alpha = u_\beta \circ f_\alpha^\beta$ for $\alpha \leq \beta$, then there is a unique nonexpansive operator $u : \varinjlim X_\gamma \to Y$ such that $u \circ \imath_\alpha = u_\alpha$ for every $\alpha \in \Gamma$.

2. Construction of p-Banach spaces of almost universal disposition

Let $\mathscr{C}$ be a class of quasi-Banach spaces. Following [13, Definition 2], let us say that a quasi-Banach space U is of almost universal disposition for the class $\mathscr{C}$ if, for every $\varepsilon > 0$ and for every isometry $g : X \to Y$, where Y belongs to $\mathscr{C}$ and X is a subspace of U, there is an ε-isometry $f : Y \to U$ such that $f(g(x)) = x$ for all $x \in X$.

Here is the main result of the paper.

Theorem 2.1. *For every $p \in (0, 1]$ there exists a unique separable p-Banach space of almost universal disposition for finite dimensional p-Banach spaces, up to isometries. This space contains an isometric copy of every separable p-Banach space.*

From now on we fix $p \in (0, 1]$ once and for all. We remark that everything in this paper is well-known for $p = 1$. However, the spaces we shall construct have rather unexpected properties when $p < 1$ and shed some light on a widely ignored paper by Kalton [16]; see Proposition 5.2 below.

Concerning the last statement of Theorem 2.1, it is perhaps worth noticing that, while it is well-known that the separable Banach space $C[0, 1]$ (as well as $\mathbb{G}$) contains an isometric copy of every separable Banach space, there is no available proof of the corresponding fact for p-Banach spaces for $p < 1$. In [15, Theorem 4.1(a)] it is stated without proof that for $0 < p < 1$ there exists a separable p-Banach space which is "universal" for the class of all separable p-Banach spaces. This result also appears in [26, Theorem 3.2.8] but, as far as we can understand, the rather involved proof only gives “universality with respect to ε-isometries”.

Before embarking into the proof of Theorem 2.1, let us record the following remark.

Lemma 2.2. *Let U be a p-Banach space. We assume that for every $\varepsilon > 0$ and every isometry $g : X \to Y$, where Y is a finite dimensional p-Banach space and X is a subspace of U, there is an ε-isometry $f : Y \to U$ such that $\|f(g(x)) - x\| \le \varepsilon\|x\|$ for all $x \in X$.*

Then U is of almost universal disposition for finite dimensional p-Banach spaces.

Proof. This obviously follows from the fact that if B is a basis of Y, then for every $\varepsilon > 0$ there is δ (depending on ε and B) such that if $t : Y \to U$ is linear map with $\|t(b)\| \le \delta$ for every $b \in B$, then $\|t\| \le \varepsilon$. □

The following result, which should be compared to [16, Lemma 4.2] and the construction in [2, Section 3], is the key step in our construction. It is assumed that the families $\mathfrak{J}$ and $\mathfrak{L}$ are actually sets.

Lemma 2.3. *Let E be a p-Banach space, $\mathfrak{J}$ be a family of isometric embeddings between p-Banach spaces and $\mathfrak{L}$ a family of operators with values in E. Then there is a p-Banach space E' and an isometry $\imath : E \to E'$ having the following property: if $u : A \to B$ is in $\mathfrak{J}$ and $f : A \to E$ is in $\mathfrak{L}$, then there is $f' : B \to E'$ such that $f' \circ u = \imath \circ f$, with $\|f'\| = \|f\|$. Moreover, if f is an ε-isometry, then f' is an ε-isometry too.*

Proof. First of all we observe that one may assume that each operator in $\mathfrak{L}$ is of norm one: otherwise we may replace $\mathfrak{L}$ by the new family $\{f/\|f\| : f \in \mathfrak{L}\}$ since f is an ε-isometry if and only if $1 - \varepsilon \le \|f\| \le 1 + \varepsilon$ and $f/\|f\|$ is a δ-isometry, with $\delta = 1 - (1 - \varepsilon)/\|f\|$.

If $f : X \to Y$ is an operator, then we put $\operatorname{dom}(f) := X$ and $\operatorname{cod}(f) := Y$. Note that $\operatorname{cod}(f)$ may be larger than the range of f. Set $\Gamma = \{(u, t) \in \mathfrak{J} \times \mathfrak{L} : \operatorname{dom}(u) = \operatorname{dom}(t)\}$. We consider the spaces of p-summable families $\ell_p(\Gamma, \operatorname{dom}(u))$ and $\ell_p(\Gamma, \operatorname{cod}(u))$. We have an isometry $\oplus\mathfrak{J} : \ell_p(\Gamma, \operatorname{dom}(u)) \to \ell_p(\Gamma, \operatorname{cod}(u))$ given by $\oplus\mathfrak{J}((x_{(u,t)})_{(u,t)\in\Gamma}) = (u(x_{(u,t)}))_{(u,t)\in\Gamma}$. In a similar vein, we can define a nonexpansive operator $\sum\mathfrak{L} : \ell_p(\Gamma, \operatorname{dom}(u)) \to E$ by letting $\sum\mathfrak{L}((x_{(u,t)})_{(u,t)\in\Gamma}) = \sum_{(u,t)\in\Gamma} t(x_{(u,t)})$. The notation is slightly imprecise because both operators depend on Γ.

Now we can consider the push-out diagram

$$\begin{array}{ccc} \ell_p(\Gamma, \operatorname{dom}(u)) & \xrightarrow{\oplus\mathfrak{J}} & \ell_p(\Gamma, \operatorname{cod}(u)) \\ {\scriptstyle \sum\mathfrak{L}}\big\downarrow & & \big\downarrow{\scriptstyle (\sum\mathfrak{L})'} \\ E & \xrightarrow{(\oplus\mathfrak{J})'} & \mathrm{PO} \end{array} \tag{2}$$

Let us see that the lower arrow does the trick so that we may take $E' = \mathrm{PO}$ and $\imath = (\oplus\mathfrak{J})'$. We already know that $(\oplus\mathfrak{J})'$ is an isometry and also that $(\sum\mathfrak{L})'$ is nonexpansive.

Fix (v,s) in Γ. Put $X=\operatorname{dom}(v)=\operatorname{dom}(s)$ and $Y=\operatorname{cod}(v)$. Let s' be the inclusion of Y into the (v,s)-th coordinate of $\ell_p(\Gamma,\operatorname{cod}(u))$ followed by $(\sum\mathfrak{L})'$. As Diagram (2) is commutative, it is clear that $s'\circ v=(\oplus\mathfrak{J})'\circ s$ and also that s' is nonexpansive.

Now suppose s is an ε-isometry, that is, $(1-\varepsilon)\|x\|_X\leq\|s(x)\|_Y\leq\|x\|_X$ (recall that s is nonexpansive). For $y\in Y$ one has

$$\|s'(y)\|_{\mathrm{PO}}=\|(\imath_{(v,s)}(y),0)+S\|_{\ell_p(\Gamma,\operatorname{cod}(u))\oplus_p E},$$

where $S=\{((\oplus\mathfrak{J})((x_{(u,t)})),-(\sum\mathfrak{L})((x_{(u,t)}))) : (x_{(u,t)})_{(u,t)\in\Gamma}\in\ell_p(\Gamma,\operatorname{dom}(u))\}$.

Clearly,

$$\left\|\imath_{(v,s)}(y)-(u(x_{(u,t)}))_{(u,t)}\right\|^p_{\ell_p(\Gamma,\operatorname{cod}(u))}+\left\|\sum_{(u,t)\in\Gamma}t(x_{u,t})\right\|^p_E\geq\|y-v(x)\|^p_Y+\|s(x)\|^p_E,$$

where $x=x_{(v,s)}$. Now, if $\|x\|_X\geq\|y\|_Y$ one has

$$\|y-v(x)\|^p_Y+\|s(x)\|^p_E\geq\|s(x)\|^p_E\geq(1-\varepsilon)^p\|x\|^p_X\geq(1-\varepsilon)^p\|y\|^p_Y.$$

If $\|x\|_X\leq\|y\|_Y$, then

$$\begin{aligned}\|y-v(x)\|^p_Y+\|s(x)\|^p_E&\geq\|y\|^p_Y-\|v(x)\|^p_Y+(1-\varepsilon)^p\|x\|^p_X\\&\geq\|y\|^p_Y-(1-(1-\varepsilon)^p)\|x\|^p_X\\&\geq(1-\varepsilon)^p\|y\|^p_Y.\end{aligned}$$

Thus, $\|s'(y)\|_{\mathrm{PO}}\geq(1-\varepsilon)\|y\|_Y$ and s' is a nonexpansive ε-isometry. □

Lemma 2.4. *Every separable p-Banach space is isometric to a subspace of a separable p-Banach space of almost universal disposition.*

Proof. Let $\mathfrak{F}$ be a countable family of isometries between finite-dimensional p-normed spaces having the following density property: for every isometry of finite-dimensional p-normed spaces $g:A\to B$ and every $\varepsilon\in(0,1)$ there is $f\in\mathfrak{F}$ and surjective ε-isometries $u:A\to\operatorname{dom}(f)$ and $v:B\to\operatorname{cod}(f)$ making commutative the square

$$\begin{array}{ccc}A&\xrightarrow{g}&B\\ {\scriptstyle u}\downarrow&&\downarrow{\scriptstyle v}\\ \operatorname{dom}(f)&\xrightarrow[f]{}&\operatorname{cod}(f)\end{array}$$

Let S be a separable p-Banach space. We shall construct inductively a chain of separable p-Banach spaces based on the nonnegative integers

$$G_0\xrightarrow{\imath_1}\ldots\longrightarrow G_{n-1}\xrightarrow{\imath_n}G_n\xrightarrow{\imath_{n+1}}G_{n+1}\longrightarrow\ldots$$

as follows. We put $G_0 = S$ and, assuming that G_k and $\imath_k$ have been constructed for $k \leq n$, we take a countable set of operators $\mathfrak{L}_n$ such that for every $\varepsilon \in (0,1)$, every $f \in \mathfrak{F}$ and every ε-isometry $u : \mathrm{dom}(f) \to G_n$, there is $v \in \mathfrak{L}_n$ satisfying $\|u - v\| < \varepsilon$.

Then, we apply Lemma 2.3 with $E = G_n, \mathfrak{J} = \mathfrak{F}, \mathfrak{L} = \mathfrak{L}_n$ and we set $G_{n+1} = E'$ and $\imath_{n+1} = \imath$.

Finally, we consider the direct limit

$$\mathbb{G}_p(S) = \varinjlim G_n$$

and we prove that it satisfies the hypothesis of Lemma 2.2.

So suppose we are given an isometry $g : X \to Y$, where Y is a finite dimensional p-Banach space and X is subspace of $\mathbb{G}_p(S)$ and $\varepsilon > 0$. We shall prove that there is an ε-isometry $f : Y \to \mathbb{G}_p(S)$ such that $\|f(g(x)) - x\| \leq \varepsilon\|x\|$ for all $x \in X$.

Let us fix $\delta > 0$. The precise value of δ required here will be announced later.

First, there is an integer n and a linear map $w : X \to G_n$ such that $\|w(x) - x\| \leq \delta\|x\|$. Moreover, we may take $h \in \mathfrak{F}$ and δ-isometries u and v making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{dom}(h) & \xrightarrow{h} & \mathrm{cod}(h) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{g} & Y \end{array}$$

In fact we can clearly assume that $t = w \circ u$ is in $\mathfrak{L}_n$ and also that it is a δ-isometry.

Let $t' : \mathrm{cod}(h) \to G_{n+1}$ be a δ-isometry extending t and set $f = t' \circ v^{-1}$. Obviously $\|f(g(x)) - x\| = \|w(x) - x\| \leq \delta\|x\|$ for all $x \in X$. Moreover,

$$(1-\delta)^2\|y\| \leq \|f(y)\| \leq (1-\delta)^2\|y\| \qquad (y \in Y)$$

and therefore taking $\delta = \sqrt{1+\varepsilon} - 1$ suffices. □

3. Uniqueness

The following result is the first step towards the proof of uniqueness in Theorem 2.1. It is the p-convex analogue of [21, Lemma 2.1]. As the reader can imagine, the proof has to be different here since one needs to avoid the use of linear functionals to work with p-normed spaces.

Lemma 3.1. *Let X and Y be p-normed spaces and $f : X \to Y$ an ε-isometry, with $\varepsilon \in (0,1)$. Let $i : X \to X \oplus Y$ and $j : Y \to X \oplus Y$ be the canonical inclusions. Then there is a p-norm on $X \oplus Y$ such that $\|f \circ j - i\| \leq \varepsilon$ and both i and j are isometries.*

Proof. Put

$$\|(x,y)\|^p = \inf\left\{\|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p : (x,y) = (x_0,0) + (0,y_1) + (x_2, -f(x_2))\right\}.$$

It is easily seen that this formula defines a p-norm on $X \oplus Y$. Let us check that $\|(x,0)\| = \|x\|_X$ for all $x \in X$. The inequality $\|(x,0)\| \leq \|x\|_X$ is obvious. As for the converse, suppose $x = x_0 + x_2$ and $y_1 = f(x_2)$. Then

$$\begin{aligned}\|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p &= \|x_0\|_X^p + \|f(x_2)\|_Y^p + \varepsilon^p\|x_2\|_X^p \\ &\geq \|x_0\|_X^p + (1-\varepsilon)^p\|x_2\|_X^p + \varepsilon^p\|x_2\|_X^p \\ &= \|x_0\|_X^p + \|(1-\varepsilon)x_2\|_X^p + \|\varepsilon x_2\|_X^p \\ &\geq \|x\|_X^p,\end{aligned}$$

as required.

Next we prove that $\|(0,y)\| = \|y\|_Y$ for every $y \in Y$. That $\|(0,y)\| \leq \|y\|_Y$ is again obvious. To prove the reversed inequality assume $x_0 + x_2 = 0$ and $y = y_1 - f(x_2)$. As $t \to t^p$ is subadditive on $\mathbb{R}_+$ for $p \in (0,1]$, we have

$$\begin{aligned}\|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p &= \|x_2\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p \\ &\geq \|y_1\|_Y^p + (1+\varepsilon^p)\|x_2\|_X^p \\ &\geq \|y_1\|_Y^p + (1+\varepsilon)^p\|x_2\|_X^p \\ &\geq \|y_1\|_Y^p + \|f(x_2)\|_Y^p \\ &\geq \|y\|_Y^p.\end{aligned}$$

To end, let us estimate $\|j \circ f - i\|$. We have

$$\|j \circ f - i\| = \sup_{\|x\| \leq 1} \|j(f(x)) - i(x)\| = \sup_{\|x\| \leq 1} \|(-x, f(x))\| \leq \varepsilon$$

and we are done. □

A linear operator $f : X \to Y$ is called a strict ε-isometry if for every $x \in X$,

$$(1-\varepsilon)\|x\|_X < \|f(x)\|_Y < (1+\varepsilon)\|x\|_X,$$

where $\varepsilon \in (0,1)$. Note that when X is finite dimensional, every strict ε-isometry is an η-isometry for some $\eta < \varepsilon$.

Lemma 3.2. *Let U be a p-Banach space of almost universal disposition for finite dimensional p-Banach spaces and let $f : X \to Y$ be a strict ε-isometry, where Y is a finite-dimensional p-Banach space, X is a subspace of U and $\varepsilon \in (0,1)$. Then for each $\delta > 0$ there exists a δ-isometry $g : Y \to U$ such that $\|g(f(x)) - x\| < \varepsilon\|x\|$ for all $x \in X$.*

Proof. Choose $0 < \eta < \varepsilon$ such that f is an η-isometry. Shrinking δ if necessary, we may assume that $\delta^p + (1+\delta)^p\eta^p < \varepsilon^p$. Let Z denote the direct sum $X \oplus Y$ equipped with the p-norm given by Lemma 3.1 and let $i : X \to Z$ and $j : Y \to Z$ denote the canonical inclusions, so that $\|j \circ f - i\| < \eta$. Let $h : Z \to U$ be a δ-isometry such that $\|h(i(x)) - x\| \leq \delta\|x\|$ for $x \in X$. Then $g = h \circ j$ is a δ-isometry from Y into U and we have

$$\begin{aligned}\|x - g(f(x))\|^p &\leq \|x - h(i(x))\|^p + \|h(i(x)) - h(j(f(x))\|^p \\ &\leq \delta^p\|x\|^p + (1+\delta)^p\|i(x) - j(f(x))\|_Z^p \\ &\leq (\delta^p + (1+\delta)^p\eta^p)\|x\|^p < \varepsilon^p\|x\|^p,\end{aligned}$$

as required. □

We are now ready for the proof of the uniqueness. Note that the following result, together with Lemma 2.4, completes the proof of Theorem 2.1.

Theorem 3.3. *Let U and V be separable p-Banach spaces of almost universal disposition for finite dimensional p-Banach spaces. Let $f : X \to V$ be a strict ε-isometry, where X is a finite dimensional subspace of U and $\varepsilon \in (0,1)$. Then there exists a bijective isometry $h : U \to V$ such that $\|h(x) - f(x)\|_V \leq \varepsilon\|x\|_U$ for every $x \in X$. In particular, U and V are isometrically isomorphic.*

Proof. Fix $0 < \eta < \varepsilon$ such that f is an η-isometry and then choose $0 < \lambda < 1$ such that

$$(\star) \qquad \eta^p \frac{1 + 3\lambda^p}{1 - \lambda^p} < \varepsilon.$$

Let $\varepsilon_n = \lambda^n\eta$. We define inductively sequences of linear operators $(f_n), (g_n)$ and finite dimensional subspaces (X_n), (Y_n) of U and V, respectively, so that the following conditions are satisfied:

(0) $X_0 = X$, $Y_0 = f[X]$, and $f_0 = f$;
(1) $f_n : X_n \to Y_n$ is an ε_n-isometry;
(2) $g_n : Y_n \to X_{n+1}$ is an ε_{n+1}-isometry;
(3) $\|g_n f_n(x) - x\| < \varepsilon_n\|x\|$ for $x \in X_n$;
(4) $\|f_{n+1}g_n(y) - y\| < \varepsilon_{n+1}\|y\|$ for $y \in Y_n$;
(5) $X_n \subset X_{n+1}$, $Y_n \subset Y_{n+1}$, $\bigcup_n X_n$ and $\bigcup_n Y_n$ are dense in U and V, respectively.

We use condition (0) to start the inductive construction. Suppose that f_i, X_i, Y_i, for $i \leq n$, and g_i for $i < n$, have been constructed. We easily find g_n, X_{n+1}, f_{n+1} and Y_{n+1} using Lemma 3.2.

To guarantee that Condition (5) holds, we may start by taking sequences (x_n) and (y_n) dense in U and V, respectively and then we require first that X_{n+1} contains both x_n and $g_n[Y_n]$ and then that Y_{n+1} contains both y_n and $f_{n+1}[X_{n+1}]$. Thus, the construction can be carried out.

Fix $n \in \omega$ and $x \in X_n$ with $\|x\| = 1$. Using (4), we get

$$\|f_{n+1}g_n f_n(x) - f_n(x)\|^p < \varepsilon_{n+1}^p \cdot \|f_n(x)\|^p \leq \varepsilon_{n+1}^p \cdot (1 + \varepsilon_n)^p = (\lambda^{n+1}\eta)^p \cdot (1 + \lambda^n\eta)^p.$$

Using (3), we get

$$\|f_{n+1}g_nf_n(x)-f_{n+1}(x)\|^p \le \|f_{n+1}\|^p\cdot\|g_nf_n(x)-x\|^p < (1+\varepsilon_{n+1})^p\cdot\varepsilon_n^p = (\lambda^p\eta)^p\cdot(1+\lambda^{n+1}\eta)^p.$$

These inequalities give

$$\begin{aligned}\|f_n(x)-f_{n+1}(x)\|^p &< (\lambda^n\eta+\lambda^n\eta\lambda^{n+1}\eta)^p+(\lambda^n\eta\lambda^{n+1}\eta+\lambda^{n+1}\eta)^p\\ &< \eta^p(\lambda^{np}+2\lambda^{(n+1)p}+\lambda^{(n+1)p}) = \eta^p(\lambda^{np}+3\lambda^{(n+1)p}).\end{aligned}\tag{$\star\star$}$$

Now it is clear that $(f_n(x))_{n\in\omega}$ is a Cauchy sequence. Given $x\in\bigcup_{n\in\omega}X_n$, define $h(x)=\lim_{n\ge m}f_n(x)$, where m is such that $x\in X_m$. Then h is an ε_n-isometry for every $n\in\omega$, hence it is an isometry. Consequently, it extends to an isometry on $h:U\to V$ that we do not relabel. Furthermore, ($\star$) and ($\star\star$) give

$$\|f(x)-h(x)\| \le \sum_{n=0}^{\infty}\eta^p(\lambda^{np}+3\lambda^{(n+1)p}) = \eta^p\frac{1+3\lambda^p}{1-\lambda^p} < \varepsilon$$

It remains to see that h is a bijection. To this end, we check as before that $(g_n(y))_{n\ge m}$ is a Cauchy sequence for every $y\in Y_m$. Once this is done, we obtain an isometry $g:V\to U$. Conditions (3) and (4) tell us that $g\circ h$ is the identity on U and that $h\circ g$ is the identity on V. This completes the proof. □

4. Nonseparable generalizations

As the reader may expect, we say that a quasi-Banach space U is of universal disposition for a given class of quasi-Banach spaces $\mathscr{C}$ if, whenever $g:X\to Y$ is an isometry, where Y belongs to $\mathscr{C}$ and X is a subspace of U, then there is an isometry $f:Y\to U$ such that $f(g(x))=x$ for all $x\in X$.

Using $\mathbb{G}_p$ as an isometrically universal separable p-Banach space and iterating Lemma 2.3 until the first uncountable ordinal ω_1 we now proceed as in [2, Proposition 3.1(a)] to prove the following.

Theorem 4.1. *There is a p-Banach space of universal disposition for separable p-Banach spaces and whose density character is the continuum.*

Proof. Let ω_1 be the first uncountable ordinal. We may regard ω_1 as the set of all countable ordinals equipped with the obvious order; see [11] for details. We are going to define a transfinite sequence of p-Banach spaces $(G_p^\alpha, f_\alpha^\beta)$ indexed by ω_1 having the following properties:

(a) For each $\alpha\in\omega_1$ the density character of G_p^α is at most the continuum.
(b) If $\beta=\alpha+1$ and $g:X\to Y$ is an isometry, where Y is a separable p-Banach space and X is a subspace of G_p^α, then there is an isometry $f:Y\to G_p^\beta$ such that $f(g(x))=f_\alpha^\beta(x)$ for all $x\in X$.

We proceed by transfinite induction on $\alpha \in \omega_1$. Let us fix an arbitrary p-Banach space C with density $2^{\aleph_0}$. Then, we take $G_p^0 = C$ to start.

The inductive step is as follows. We fix $\gamma \in \omega_1$ and we assume that the directed system $(G_p^\alpha, f_\alpha^\beta)$ has been constructed for $\alpha, \beta < \gamma$ in such a way that (a) and (b) hold for $\alpha, \beta < \gamma$.

We want to see that we can continue the system in such a way that (a) and (b) now hold for $\alpha, \beta < \gamma + 1$. We shall distinguish two cases.

First, assume γ is a limit ordinal. Then we take $G_p^\gamma = \varinjlim_{\alpha<\gamma} G_p^\alpha$ and $f_\alpha^\gamma = \imath_\alpha$. It is clear that $\operatorname{dens}(G_p^\gamma) \le 2^{\aleph_0}$ and there is nothing else to prove since γ cannot arise as $\alpha + 1$ for $\alpha < \gamma$.

Now, suppose γ is a successor ordinal, say $\gamma = \delta + 1$. To construct $G_p^{\delta+1}$ we consider the set of all isometric embeddings between subspaces of $\mathbb{G}_p$ and we call it $\mathfrak{J}$ and the set $\mathfrak{L}$ of all G_p^δ-valued isometries whose domain is a subpace of $\mathbb{G}_p$ – recall that G_p^δ is already defined by the induction hypothesis. Now, we let $E = G_p^\delta$ and we apply Lemma 2.3 with $\varepsilon = 0$ to get the push-out space $G_p^{\delta+1} = E'$ and $f_\delta^{\delta+1} = \imath$. Observe that $G_p^{\delta+1}$ has density character at most $\mathfrak{c}$ since it is a quotient of the direct sum of G_p^δ and $\ell_p(\Gamma, \operatorname{cod}(u))$, where Γ is a subset of $\mathfrak{J} \times \mathfrak{L}$, with $|\mathfrak{J}|, |\mathfrak{L}| \le \mathfrak{c}$ and $\operatorname{cod}(u)$ separable.

Now, for $\alpha < \delta$ we put $f_\alpha^{\delta+1} = f_\delta^{\delta+1} \circ f_\alpha^\delta$ and the Principle of Transfinite Induction goes at work.

The remainder of the proof is rather easy. We define U as the direct limit of the system $(G_p^\alpha)_\alpha$ and we consider the natural isometries $\imath_\alpha : G_p^\alpha \to U$, so that

$$U = \bigcup_{\alpha\in\omega_1} \imath_\alpha[G_p^\alpha].$$

Observe that it is not necessary to take closures here. Obviously, the density character of U is at most the continuum.

Suppose $g : X \to Y$ is an isometry, where Y is a separable p-Banach space and X a subspace of U. Then there is $\alpha \in \omega_1$ so that $X \subset \imath_\alpha[G_p^\alpha]$. It is straightforward from (c) that there is an isometry $f : Y \to G_p^{\alpha+1}$ such that $\imath_{\alpha+1}(f(g(x))) = x$ for every $x \in X$. $\square$

The following result, due to Ben Yaacov and Henson [5], with a more straightforward proof found by Richard Haydon, shows that it is impossible to reduce the size of the space in Theorem 4.1. Formally, this result was stated and proved for $p = 1$, however an easy adaptation gives exactly the same for any $p \in (0, 1]$. Namely, a p-Banach space of universal disposition for the class of p-Banach spaces of dimension three or less must already have density $2^{\aleph_0}$. This was asked in [2, Problem 2] for $p = 1$.

Proposition 4.2 (Ben Yaacov and Henson [5]). *Let $0 < p \le 1$, let H denote the 2-dimensional Hilbert space and suppose $X \supset H$ is a p-Banach space with the following property:*

(SG) *Given an isometric embedding $i\colon H \to F$, where F is a 3-dimensional p-Banach space, there exists an isometric embedding $j\colon F \to X$ such that $j \circ i$ is the inclusion $H \subset X$.*

Then the density of X is at least the continuum.

Proof. (Haydon) Let S be the positive part of the sphere of H. Given $\phi \in S$, we define a norm on $H \oplus \mathbb{K}$ (recall that $\mathbb{K}$ is the scalar field) by the formula

$$\|(x,\lambda)\|_\phi^p = \max\Big\{\|x\|_2^p, |\lambda|^p + |(x|\phi)|^p\Big\},$$

where $(\cdot|\cdot)$ denotes the usual scalar product on H. Note that $\|(0,1)\|_\phi = 1$ and $\|\cdot\|_\phi$ extends the Euclidean norm $\|\cdot\|_2$ of H, where $x \in H$ is identified with $(x,0)$. Using (SG), for each $\phi \in S$ we can find $e_\phi \in X$ such that the map $i_\phi\colon H \oplus \mathbb{K} \to X$, defined by $i_\phi(x,\lambda) = x + \lambda e_\phi$, is an isometric embedding with respect to $\|\cdot\|_\phi$.

Fix $\phi, \psi \in S$ such that $\phi \neq \psi$ and let $\|\cdot\|$ denote the p-norm of X. Fix $\mu > 0$ and let $w = \mu\phi \in H \subset X$. Then

$$\|e_\phi - e_\psi\|^p \geq \|e_\phi + w\|^p - \|e_\psi + w\|^p = \|(\mu\phi,1)\|_\phi^p - \|(\mu\phi,1)\|_\psi^p.$$

Finally, observe that $\|(\mu\phi,1)\|_\phi^p = 1 + \mu^p$ and

$$\|(\mu\phi,1)\|_\psi^p = \max\Big\{\mu^p, 1 + \mu^p|(\phi|\psi)|^p\Big\} = \mu^p,$$

whenever μ is big enough, because $|(\phi|\psi)| < 1$ (recall that ϕ, ψ are distinct vectors from the open unit hemisphere of H). Thus, we conclude that $\|e_\phi - e_\psi\| \geq 1$ whenever $\phi \neq \psi$, which shows that the density of X is at least $|S| = 2^{\aleph_0}$. □

A couple of additional remarks about Theorem 4.1 are in order. First, it is clear that any p-Banach space of universal disposition for the class of all separable p-Banach spaces must contain an isometric copy of every p-Banach space of density $\aleph_1$. This is so because every quasi-Banach space X of density $\aleph_1$ can be written as $X = \bigcup_{\alpha \in \omega_1} X_\alpha$, where each X_α is a separable subspace of X and $X_\alpha \subset X_\beta$ whenever $\alpha \leq \beta$ are countable. For the same reason, if we assume the continuum hypothesis, then we can easily obtain uniqueness up to isometries in Theorem 4.1. See Section 7.3 for more on this.

5. Some forms of injectivity for p-Banach spaces

In this Section we study the extension of operators with values in $\mathbb{G}_p$ and its nonseparable relatives.

Definition 5.1. Let E be a p-Banach space.

(a) We say that E is injective amongst p-Banach spaces if for every p-Banach space X and every subspace Y of X, every operator $t : Y \to E$ can be extended to an operator $T : X \to E$. If this can be achieved with $\|T\| \leq \lambda\|t\|$ for some fixed $\lambda \geq 1$, then E is said to be λ-injective amongst p-Banach spaces.

(b) E is said to separably injective or separably λ-injective amongst p-Banach spaces if the preceding condition holds when X is separable.

(c) Finally, E is said to be locally injective amongst p-Banach spaces if there is a constant λ such that every finite dimensional p-Banach space X and every subspace Y of X, every operator $t : Y \to E$ can be extended to an operator $T : X \to E$ with $\|T\| \leq \lambda\|t\|$.

These notions play a fundamental role in Banach space theory. As it is well-known, a Banach space is injective (amongst Banach spaces) if and only if it is a complemented subspace of $\ell_\infty(I)$ for some set I. Also, a Banach space is locally injective if and only if it is a $\mathscr{L}_\infty$-space and it is locally injective with constant λ for every $\lambda > 1$ if and only if it is a Lindenstrauss space.

As for separable injectivity, Sobczyk theorem asserts that c_0 is separably 2-injective and a deep result by Zippin states that every separable separably injective Banach space has to be isomorphic to c_0. Nevertheless, there is a large variety of (nonseparable) separably injective Banach spaces, see [**29**, **3**].

Let us call a p-Banach space E locally (1+)-injective if it satisfies (c) above with each $\lambda > 1$.

Proposition 5.2. *Let $0 < p < 1$.*

(a) *No nonzero p-Banach space is injective amongst p-Banach spaces.*

(b) *Every space of almost universal disposition for finite dimensional p-Banach spaces, in particular $\mathbb{G}_p$, is locally (1+)-injective amongst p-Banach spaces.*

(c) *All spaces of universal disposition for separable p-Banach spaces, in particular those appearing in Theorem 4.1, are separably 1-injective amongst p-Banach spaces.*

Proof. (a) Let E be a p-Banach space with density character $\aleph$. Let μ denote Haar measure on the product of a family of $2^{\aleph}$ copies of $\mathbb{T}$, the unit circle. Then there is no nonzero operator from $L_p(\mu)$ to E (recall that $p < 1$). This was proved for $\aleph = \aleph_0$ by Kalton (see [**16**, p. 163, at the end of Section 3]) and by Popov in general [**25**, Theorem 1].

Thus, if we fix some nonzero $e \in E$ and we consider the subspace $\mathbb{K}$ of constant functions in $L_p(\mu)$, then the operator $\lambda \in \mathbb{K} \mapsto \lambda e \in E$ cannot be extended and E is not injective.

(b) Assume U is of almost universal disposition for finite dimensional p-Banach spaces. Let X be a finite dimensional p-Banach space, Y a subspace of X and $t : Y \to U$ an operator of norm one. We will prove that, for each $\varepsilon > 0$, there is an extension $T : X \to U$ with

$\|T\| \leq 1+\varepsilon$. Consider the push-out diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t'} \\ t[Y] & \xrightarrow{\imath} & \mathrm{PO} \end{array}$$

where the unlabelled arrow is plain inclusion. As $\imath$ is an isometry, for each $\varepsilon > 0$, there is an ε-isometry $u : \mathrm{PO} \to U$ such that $\imath(t(y)) = u(t(y))$ for all $y \in Y$. Then $u \circ t'$ is an extension of t to X with norm at most $1+\varepsilon$.

(c) Replace "finite dimensional" by "separable", take the closure of $t(Y)$, delete the word "almost", and put $\varepsilon = 0$ in the proof of (b). □

6. Universal operators on p-Gurariĭ spaces

Throughout this section, we again fix $p \in (0,1]$. Our aim is to construct a nonexpansive projection $\mathbb{P}_p \colon \mathbb{G}_p \to \mathbb{G}_p$ whose kernel is isometric to $\mathbb{G}_p$ and the following condition is satisfied:

(P) Given a nonexpansive operator $T \colon X \to \mathbb{G}_p$, where X is a separable p-Banach space, there exists an isometry $i \colon X \to \mathbb{G}_p$ such that $\mathbb{P}_p \circ i = T$.

This will show, in particular, that $\mathbb{G}_p$ has nontrivial projections.

For the remaining part of the section we fix a locally (1+)-injective separable p-Banach space $\mathbb{H}$. Note that, by Proposition 5.2, we may take $\mathbb{H} = \mathbb{G}_p$. In fact, besides obvious variants like the c_0-sum of $\mathbb{G}_p$ and its finite dimensional counterparts, we do not know essentially different examples, unless $p = 1$, where being locally (1+)-injective is the same as being a Lindenstrauss space and a projection satisfying (P) has already been described in [20].

In order to present the announced construction, we shall define a special category involving $\mathbb{H}$, which is actually a particular case of so-called comma categories. These ideas come from a recent work of Pech & Pech [24] as well as from Kubiś [20], where an abstract theory of almost homogeneous structures has been developed. Namely, let $\mathfrak{K}$ be the category whose objects are pairs of the form $\langle S, f \rangle$, where f is a nonexpansive operator into $\mathbb{H}$ and S is a finite-dimensional p-Banach space. A $\mathfrak{K}$-morphism from $f \colon S \to \mathbb{H}$ into $g \colon T \to \mathbb{H}$ is an isometry $i \colon S \to T$ satisfying $g \circ i = f$. Using the properties of push-outs, we easily obtain the following fact.

Lemma 6.1. *$\mathfrak{K}$ has amalgamations. Namely, given two $\mathfrak{K}$-morphisms i, j with the same domain, there exist $\mathfrak{K}$-morphisms i', j' such that $i' \circ i = j' \circ j$.*

We also need the following strengthening of Lemma 3.1.

Lemma 6.2. *Let $f \colon X \to Y$ be an ε-isometry between finite dimensional p-Banach spaces and let $R \colon X \to \mathbb{H}$, $S \colon Y \to \mathbb{H}$ be nonexpansive linear operators such that $S \circ f$ is ε-close to*

R. Let $\|\cdot\|$ be the p-norm on $X \oplus Y$ constructed in the proof of Lemma 3.1 and let i, j be the canonical isometric embeddings of X and Y, respectively. Then the operator $T\colon X \oplus Y \to \mathbb{H}$, defined by $T(x, y) = Rx + Sy$, is nonexpansive and has the property that $T \circ i = R$, $T \circ j = S$. In particular, i, j become $\mathfrak{K}$-morphisms.

Proof. Fix $(x, y) \in X \oplus Y$ and assume $x = x_0 + x_2$, $y = y_1 - f(x_2)$. Using the fact that $\|R(x_2) - S(f(x_2))\| \le \varepsilon\|x_2\|$, we get

$$\|T(x, y)\|^p = \|R(x_0) + R(x_2) + S(y_1) - S(f(x_2))\|^p \le \|x_0\|_X^p + \|y_1\|_Y^p + \varepsilon^p\|x_2\|_X^p.$$

Recall that $\|(x, y)\|^p$ is the infimum of all expressions as above, therefore $\|T(x, y)\|^p \le \|(x, y)\|^p$. $\square$

We now construct a sequence

$$u_0 \to u_1 \to u_2 \to \dots$$

where each $u_n = \langle U_n, T_n\rangle$ is an object of $\mathfrak{K}$, each arrow in the diagram above is a morphism in $\mathfrak{K}$, which we shall regard as inclusion of the corresponding space U_n so, in particular, each T_{n+1} extends T_n, and the following condition is satisfied:

(†) Given $n \in \mathbb{N}$, $\varepsilon > 0$, given an isometric embedding $e\colon U_n \to V$ and a nonexpansive operator $Q\colon V \to \mathbb{H}$ satisfying $Q \circ e = T_n$, there exist $m > n$ and and an ε-isometric embedding $e'\colon V \to U_m$ such that $e' \circ e$ is ε-close to the identity of U_n and $T_m \circ e'$ is ε-close to Q.

The construction can be done either by following the lines of [20] or, simply, by repeating the construction from the proof of Lemma 2.4, at each stage taking into account a fixed nonexpansive linear operator into $\mathbb{H}$ and having in mind the push-out property of p-Banach spaces that provides necessary extensions of nonexpansive operators. Actually, it can be shown that (†) specifies the sequence $\{u_n\}_{n\in\mathbb{N}}$ uniquely, up to an isomorphism in the appropriate category. Denote by U_∞ the completion of the union $\bigcup_{n\in\mathbb{N}} U_n$ and let T_∞ be the unique linear operator extending all T_ns. We claim that T_∞ is a universal nonexpansive operator onto $\mathbb{H}$. The properties of T_∞ are collected below.

Theorem 6.3. *The operator T_∞ has the following properties:*

(1) *It is nonexpansive and right-invertible (in particular, its range is $\mathbb{H}$).*
(2) *Both its domain and kernel are linearly isometric to $\mathbb{G}_p$.*
(3) *For every nonexpansive linear operator $S\colon X \to \mathbb{H}$ such that X is a separable p-Banach space, there exists a linear isometric embedding $e\colon X \to \mathbb{G}_p$ such that*

$$S = T_\infty \circ e.$$

Note that if $\mathbb{H} = \mathbb{G}_p$ then $\mathbb{P}_p := T_\infty$ is the announced universal projection.

Proof. Obviously, T_∞ is nonexpansive, being the pointwise limit of nonexpansive operators. Condition (3) follows from property (†) of the sequence. Namely, given any operator

$S\colon X \to \mathbb{H}$ from a separable p-Banach space X, we can write $S = \bigcup_{n\in\mathbb{N}} S_n$, where each S_n has a finite-dimensional domain and $S_0 = 0$. Using (†), the amalgamation property, Lemma 6.2 and induction, we get an isometric embedding $e\colon X \to U_\infty$ such that $T_\infty \circ e = S$.

More precisely, we set $\varepsilon_n = 2^{-n}$ and at each step we define an ε_n-isometric embedding $e_n\colon X_n \to U_{k_n}$, where X_n is the domain of S_n and $T_{k_n} \circ e_n$ is ε_n-close to S_n. Having defined e_n, Lemma 6.2 followed by the amalgamation property gives us $\mathfrak{K}$-morphisms $i\colon U_{k_n} \to V$, $j\colon X_{n+1} \to V$ with V a finite dimensional p-Banach space, such that $i \circ e_n$ is ε_n-close to j. In particular, we also have a nonexpansive operator $R\colon V \to \mathbb{H}$ so that $\langle V, R\rangle$ is a $\mathfrak{K}$-object and $R \circ i = T_{k_n}$, $R \circ j = S_{n+1}$. Property (†) gives us $k_{n+1} > k_n$ and an ε_{n+1}-isometry $\ell\colon V \to U_{k_{n+1}}$ such that $T_{k_{n+1}} \circ \ell$ is ε_{n+1}-close to R. Setting $e_{n+1} = \ell \circ j$, we obtain an ε_{n+1}-isometry from X_{n+1} to $U_{k_{n+1}}$ such that $T_{k_{n+1}} \circ e_{n+1}$ is ε_{n+1}-close to S_{n+1}. Finally, the sequence $\{e_n\}_{n\in\mathbb{N}}$ converges to an embedding e satisfying $T_\infty \circ e = S$.

This partially shows (3), since we still need to argue that U_∞ is isometric to $\mathbb{G}_p$. Here we use the property that $\mathbb{H}$ is a locally (1+)-injective p-Banach space, therefore, given an isometric embedding $i\colon U_n \to V$ we find an operator $R\colon V \to \mathbb{H}$ that extends T_n and has p-norm $\leq 1 + \varepsilon$ for any fixed $\varepsilon > 0$. Next, we "correct" the operator R so that it becomes nonexpansive and we use (†) to obtain an ε-isometric embedding of V into U_∞ that "almost realizes" V. This shows that U_∞ is the p-Gurariĭ space and completes the proof of (3) and part of the proof of (2). Now consider the identity operator of $\mathbb{H}$. By (3), it can be obtained (up to isometry) as a restriction of T_∞ to a subspace H of U_∞. This shows that T_∞ is right-invertible, completing the proof of (1). It remains to show that $\ker T_\infty$ is isometric to $\mathbb{G}_p$.

We come back to the sequence $\langle U_n, T_n\rangle$ and define $V_n = \ker T_n$. Then $\{V_n\}_{n\in\mathbb{N}}$ is a chain of subspaces of U_∞ such that the completion V_∞ of its union is precisely the kernel of T_∞. In order to conclude that V_∞ is of almost universal disposition for finite dimensional p-Banach spaces, we proceed in exactly the same way as in the proof of Lemma 2.4 above. The only new ingredient is "plugging" zero operators from finite dimensional spaces into $\mathbb{H}$. This completes the proof. □

Finally, note that the universal projection described above says that $\mathbb{G}_p$ is isomorphic to $\mathbb{G}_p \oplus \mathbb{H}$. In particular, $\mathbb{G}_p$ is isomorphic to $\mathbb{G}_p \oplus \mathbb{G}_p$. We also have:

Corollary 6.4. *$\mathbb{G}_p$ is isomorphic to $c_0(\mathbb{G}_p)$, the space of all sequences converging to 0 in $\mathbb{G}_p$, endowed with the maximum p-norm.*

Proof. Note that $c_0(\mathbb{G}_p)$ is locally (1+)-injective, being the completion of the union of a chain of spaces of the form $\mathbb{G}_p \oplus \cdots \oplus \mathbb{G}_p$ endowed with the maximum norm; all these spaces are locally (1+)-injective, because $\mathbb{G}_p$ is so. Finally, $\mathbb{G}_p \oplus c_0(\mathbb{G}_p)$ is isomorphic to $c_0(\mathbb{G}_p)$. □

7. Miscellaneous remarks and questions

7.1. Mazur's "rotations" problem. A quasi-Banach space is said to be almost isotropic if the orbits of the isometry group are dense in the unit sphere: if $\|x\| = \|y\| = 1$, then for every $\varepsilon > 0$ there is a surjective isometry u such that $\|y - u(x)\| \leq \varepsilon$. If this condition holds even for $\varepsilon = 0$, the space is said to be isotropic: the isometry group acts transitively on the sphere.

A notorious problem that Banach attributes to Mazur in his *"Théorie des Opérations Linéaires"* asks whether ℓ_2 is the only separable isotropic Banach space. This is the problem mentioned by Gurariĭ in the title of [13] and, as far as we know, is still open. We may refer the reader to [8, 4] for two complementary surveys on the topic.

The following remark is immediate from Theorem 3.3.

Corollary 7.1. *The space $\mathbb{G}_p$ is almost isotropic.* □

It is well-known that $\mathbb{G}$ ("our" $\mathbb{G}_p$ when $p = 1$) is not isotropic. However the standard argument depends on Mazur's theorem about the existence of smooth points on any separable Banach space and we do not know how to proceed when $p < 1$.

It is worth remarking that the notion of "almost isotropic space" that Gurariĭ manages in [13] is different than ours. Anyway, it is clear from the proof of [13, Theorem 3] that the spaces $\mathbb{G}_p$ are "isotropic" in Gurariĭ 's sense for all $p \in (0, 1]$.

7.2. Ultrapowers of $\mathbb{G}_p$. There is an alternative proof of Theorem 4.1 which is based on the ultraproduct construction; see [17]. Let (X_i) be a family of p-Banach spaces indexed by I and let $\mathscr{U}$ be a countably incomplete ultrafilter on I. Then the space of bounded families $\ell_\infty(I, X_i)$ with the quasi-norm $\|(x_i)\| = \sup_i \|x_i\|$ is a p-Banach space and $c_0^{\mathscr{U}}(X_i) = \{(x_i) : \lim_{\mathscr{U}} \|x_i\| = 0\}$ is a closed subspace of $\ell_\infty(I, X_i)$. The ultraproduct of the family (X_i) with respect to $\mathscr{U}$, denoted by $[X_i]_{\mathscr{U}}$, is the quotient space $\ell_\infty(I, X_i)/c_0^{\mathscr{U}}(X_i)$ with the quotient quasinorm. The class of the family (x_i) in $(X_i)_{\mathscr{U}}$ is denoted by $[(x_i)]$.

The quasinorm in $[X_i]_{\mathscr{U}}$ can be computed as $\|[(x_i)]\| = \lim_{\mathscr{U}} \|x_i\|$.

When all the spaces X_i are the same, say X, the ultraproduct is called the ultrapower of X following $\mathscr{U}$. One has the following generalization of [2, Proposition 5.7] for which we provide a simpler proof.

Proposition 7.2. *If $\mathscr{U}$ is a countably incomplete ultrafilter on the integers, then $[\mathbb{G}_p]_{\mathscr{U}}$ is a p-Banach space of universal disposition for separable p-Banach spaces whose density character is the continuum.*

Proof. We denote by I the index set supporting $\mathscr{U}$. Let X be a separable subspace of $[\mathbb{G}_p]_{\mathscr{U}}$ and $g : X \to Y$ an isometry, where Y is any separable p-Banach space. We will prove that there is an isometry $f : Y \to [\mathbb{G}_p]_{\mathscr{U}}$ such that $f(g(x)) = x$ for every $x \in X$. Clearly, we may and do assume that Y/X has dimension one.

So, let (x^n) be a normalized, linearly independent sequence whose linear span is dense in X and $y^0 \in Y\backslash X$. Let X^n be the subspace spanned by $(x^1, \dots, x^n)$ in X^n and Y^n the subspace spanned by $g[X^n]$ and y^0 in Y.

Also, let us fix representatives (x_i^n) so that $x^n = [(x_i^n)]$ for every n. We may assume $\|x_i^n\| = 1$ for every n and every i. For $i \in I$ and $n \in \mathbb{N}$, let us denote by X_i^n the subspace of $\mathbb{G}_p$ spanned by $(x_i^1, \dots, x_i^n)$. We define a linear map $\jmath_{n,i} : X^n \to X_i^n$ by letting $\jmath_{n,i}(x^k) = x_i^k$ for $1 \leq k \leq n$ and linearly on the rest.

To proceed, we observe that the sets

$$I_\varepsilon^n = \{i \in I \text{ such that } \jmath_{n,i} : X^n \to X_i^n \text{ is a strict } \varepsilon\text{-isometry}\}$$

are in $\mathscr{U}$ for every n and every $\varepsilon > 0$. Let (J_n) be a sequence of subsets of $\mathscr{U}$ with $\bigcap_n J_n = \varnothing$. For each $i \in I$, set $n(i) = \max\{n \in \mathbb{N} : i \in J_n \cap I_{1/n}^n\}$ and observe that $n(i) \to \infty$ along $\mathscr{U}$.

Let us form the ultraproducts $[X^{n(i)}]_\mathscr{U}$, $[X_i^{n(i)}]_\mathscr{U}$ and $[Y_i^{n(i)}]_\mathscr{U}$. It is obvious that $[X^{n(i)}]_\mathscr{U}$ and $[X_i^{n(i)}]_\mathscr{U}$ are isometric through the ultraproduct operator $[(\jmath_{n(i),i})]$. Moreover, there is a linear isometry $\kappa : X \to [X^{n(i)}]_\mathscr{U}$ that we may define taking $\kappa(x) = [(x_i)]$, where $x_i \in X^{n(i)}$ is any point minimizing the "distance" from x to $X^{n(i)}$ and the same applies to Y and $[Y_i^{n(i)}]_\mathscr{U}$.

For each $i \in I$ consider the composition $g \circ \jmath_{n(i),i}^{-1}$ which is a strict $(1/n(i))$-isometry from $X_i^{n(i)}$ into $Y^{n(i)}$. On account of Lemma 3.2 we may find an $(1/n(i))$-isometry $f_i : Y^{n(i)} \to \mathbb{G}_p$ such that $\|f_i(g(\jmath_{n(i),i}^{-1}(x))) - x\| \leq \|x\|/n(i)$ for every $x \in X_i^{n(i)}$. It is now obvious that if $f : Y \to [\mathbb{G}_p]_\mathscr{U}$ denotes the composition of the embedding $Y \to [Y_i^{n(i)}]_\mathscr{U}$ with the ultraproduct operator $[(f_i)]$ one obtains an isometry with $f(g(x)) = x$ for every $x \in X$. $\square$

Notice that, while it is unclear whether the spaces arising in the proof of Theorem 4.1 are isotropic or not, it follows from Corollary 7.1 and rather standard ultraproduct techniques that every ultrapower of $\mathbb{G}_p$ built over a countably incomplete ultrafilter is isotropic.

7.3. Universal spaces. As we have already mentioned, under the continuum hypothesis, all the spaces having the properties appearing in Theorem 4.1 are isometric. It was observed in [2, Proposition 4.7] that, in the Banach space setting, the uniqueness cannot be proved in ZFC, the usual setting of set theory, with the axiom of choice. This depends on the fact that it is consistent with ZFC that there is no Banach space of density $2^{\aleph_0}$ containing an isometric copy of all Banach spaces of density $2^{\aleph_0}$, a recent result by Brech and Koszmider [7]. Whether or not the same happens to p-Banach spaces is left open to reflection.

7.4. Vector-valued Sobczyk's theorem without local convexity. Sobczyk's theorems states that c_0, the Banach space of all sequences converging to zero with the sup norm is separably injective – amongst Banach spaces, of course. More interesting for us is that if E is a separably injective Banach space, then so is $c_0(E)$ – the space of sequences converging to 0 in E. Several proofs of this fact are available. Some of them made strong use of local convexity. For instance, Johnson-Oikhberg's argument in [14] is based on M-ideal theory, while Castillo-Moreno proof in [10] uses the bounded approximation property, a very rare property outside the Banach space setting. It seems, however, that Rosenthal proof in [27] would survive for p-Banach spaces and in any case the proof in [9] applies verbatim to p-Banach spaces. So we have the following.

Proposition 7.3. *If E is separably injective amongst p-Banach spaces, then so is $c_0(E)$.*

We do not know whether there is a nontrivial separable space, separably injective amongst p-Banach spaces when $p < 1$, but our guess is no. In any case, such a space would necessarily be a complemented subspace of $\mathbb{G}_p$.

7.5. Operators on $\mathbb{G}_p$ when $p < 1$. It is a classical result in quasi-Banach space theory that every operator from L_p to a q-Banach space for $p < q \leq 1$ is zero. It follows easily that the same is true replacing L_p by $\mathbb{G}_p$. In particular, the dual of $\mathbb{G}_p$ is trivial. In a similar vein, there is no nonzero operator from $\mathbb{G}_p$ into any L_q (here q can be 0) and there is no compact operator on $\mathbb{G}_p$; the first statement follows from the fact that there is no nonzero operator from L_p/H_p to L_0, see [1] and the second one from the fact that every operator defined on L_p is either zero or an isomorphism on a copy of ℓ_2, see [18, Theorem 7.20] for which is perhaps the simplest proof.

We do not know whether $\mathbb{G}_p$ is isomorphic to all its quotients or complemented subspaces. In particular we don't know whether $\mathbb{G}_p$ is isomorphic to its quotient by a line.

This is clearly connected to the notion of a K-space. Recall that a quasi-Banach space X is said to be a K-space if whenever Z is a quasi-Banach space with a subspace L of dimension one such that Z/L is isomorphic to X, then L is complemented in Z and so Z is isomorphic to $\mathbb{K} \oplus X$.

It would be interesting to know whether the spaces $\mathbb{G}_p$ are K-spaces or not. The case $p = 1$ is solved in the affirmative by a deep result of Kalton and Roberts [19, Theorem 6.3], who proved that every $\mathscr{L}_\infty$-space, and in particular the Gurariĭ space, is a K-space.

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