ON FORBIDDEN CONFIGURATION OF 0-DISTRIBUTIVE LATTICES

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Dedicated to my Mother

Abstract. In this paper we obtain the forbidden configuration for 0-distributive lattices.

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Introduction

Grillet and Varlet [2] introduced the concept of 0-distributive lattices as a generalization of distributive lattices.

A lattice L with 0 is called 0-*distributive* if, for every triplet $\langle a, b, c\rangle$ of elements of L, $a \wedge b = a \wedge c = 0$ implies $a \wedge (b \vee c) = 0$. Dually, one can define the 1-distributive lattice.

Grillet (see Varlet [4]) has given the forbidden configuration of modular 0-distributive lattices as follows:

Theorem 1. *A modular lattice L with* 0 *is* 0*-distributive if and only if it contains no sublattice isomorphic to the lattice of Figure* 1(a) *or to the lattice of Figure* 1(b) *(next page).*

In Stern [3], it is mentioned that, till now no such forbidden configuration is known for 0-distributivity.

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In this paper we obtain forbidden configuration for 0-distributive lattices. The idea of the proof is taken from the paper of Davey, Poguntke and Rival [1].

A subset A of a lattice L is an antichain if x is incomparable with y for each pair of distinct elements x, y in A. For example, if a triplet $\langle a, b, c\rangle$ of elements of L violates 0-distributivity, then $\{a, b, c\}$ must be a three-element antichain.

For antichains A and B in L we define $A \ll B$ if, for every $a \in A$ there is $b \in B$ such that $a \leqslant b$; this defines a partial order on the set of antichains in L.

Throughout this paper, let L be a lattice of finite length with 0 which does not satisfy 0-distributivity and let $\{a, b, c\}$ be a maximal antichain in L such that $\langle a, b, c\rangle$ violates 0-distributivity. Since L is of finite length we may, without loss of generality, assume that every proper sub-interval of L satisfies 0-distributivity and $a \vee b \vee c = 1$.

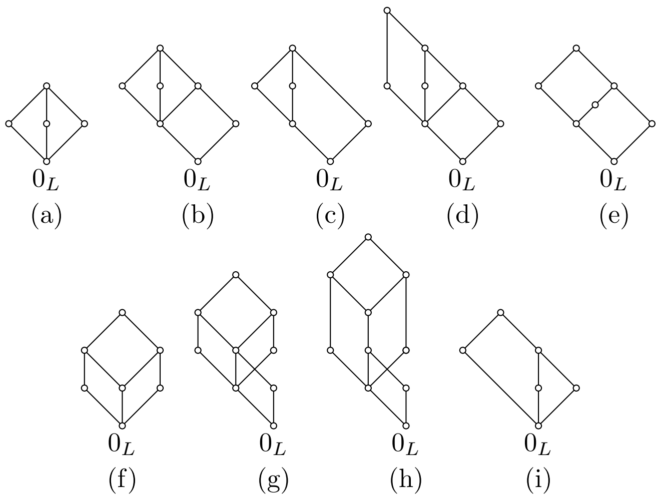

Figure 1

Lemma 1. (i) *If* $b < s < 1$, *then* $b \wedge c = s \wedge c$. (ii) *If* $c < t < 1$, *then* $b \wedge c = b \wedge t$.

P r o o f. Assume $b \wedge c < s \wedge c$. Hence $b < [b \vee (s \wedge c)]$. Clearly, $a \wedge [b \vee (s \wedge c)] \geqslant 0$. We claim that $a \wedge [b \vee (s \wedge c)] = 0$. If $a \wedge [b \vee (s \wedge c)] > 0$, then $\langle a \wedge [b \vee (s \wedge c)], b, s \wedge c\rangle$ is a triplet which violates 0-distributivity in the interval $[0, s]$, a contradiction to the assumption that every proper sub-interval satisfies 0-distributivity.

Therefore $a \wedge [b \vee (s \wedge c)] = 0$. But then $\langle a, [b \vee (s \wedge c)], c\rangle$ violates 0-distributivity, a contradiction to maximality of $\{a, b, c\}$. Thus, $b \wedge c = s \wedge c$. Similarly (ii) can be proved. $\square$

Lemma 2. (i) *If* $b \wedge c < s < b$, *then* $s \vee c = 1$. (ii) *If* $b \wedge c < t < c$, *then* $b \vee t = 1$.

P r o o f. Suppose $s \vee c < 1$. Clearly $c < s \vee c < 1$. By Lemma 1, $b \wedge c = b \wedge (s \vee c)$. But then $b \wedge c = b \wedge (s \vee c) \geqslant s > b \wedge c$, a contradiction. Hence $s \vee c = 1$. Similarly (ii) can be proved. □

Theorem 2. *A finite lattice* L *with* 0 *is* 0*-distributive if and only if it contains no sublattice isomorphic to one of the lattices of Figure* 1(a), (b), (c), (d), (e), (f), (g), (h), (i).

P r o o f. Suppose $a \wedge b = a \wedge c = 0$ but $a \wedge (b \vee c) \neq 0$. Let p be an atom of L such that $p \leqslant a \wedge (b \vee c)$. Then $p \leqslant b \vee c$ implies $b \vee c = (b \vee p) \vee (c \vee p)$.

We have the following three main cases:

[A] $p \vee b = p \vee c$;

[B] $p \vee b \leqslant p \vee c$;

[C] $p \vee b \parallel p \vee c$.

C a s e [A]: Suppose $b \vee p = c \vee p$. Then $b \vee c = b \vee p = c \vee p$. This case has the following two subcases:

[A_1] $(b \wedge c) \vee p = b \vee p$;

[A_2] $(b \wedge c) \vee p < b \vee p$.

S u b c a s e [A_1]: If $(b \wedge c) \vee p = b \vee p$, then $b \wedge c \neq 0$, otherwise $p = b$, a contradiction to the fact that $p \nleqslant b$. Therefore, $L_1 = \{0, b, c, b \wedge c, p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(c).

S u b c a s e [A_2]: Let $(b \wedge c) \vee p < b \vee p$. This subcase has the following three subcases:

[A_{21}] $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[A_{22}] $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[A_{23}] $[(b \wedge c) \vee p] \wedge b > b \wedge c < [(b \wedge c) \vee p] \wedge c$.

S u b c a s e [A_{21}]: If $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$, then $L_2 = \{0, p, b, c, b \wedge c, (b \wedge c) \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(b) when $b \wedge c > 0$ and to the lattice of Figure 1(a) if $b \wedge c = 0$.

S u b c a s e [A_{22}]: Suppose $b \wedge c < [(b \wedge c) \vee p] \wedge b$ and $b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds.

Clearly, $b \wedge c < [(b \wedge c) \vee p] \wedge b \leqslant b$. If $[(b \wedge c) \vee p] \wedge b = b$, then $[(b \wedge c) \vee p] = b \vee p$, a contradiction to $[(b \wedge c) \vee p] < b \vee p$ (Subcase [A_2]). Thus, $b \wedge c < [(b \wedge c) \vee p] \wedge b < b$. By Lemma 2, $\{[(b \wedge c) \vee p] \wedge b\} \vee c = 1$. Then $L_3 = \{0, b \wedge c, p, (b \wedge c) \vee p, [(b \wedge c) \vee p] \wedge b, c, 1\}$ forms a sublattice isomorphic to the lattice of Figure 1(e). Note that in this case $b \wedge c \neq 0$, otherwise $0 = b \wedge c < [(b \wedge c) \vee p] \wedge b = p \wedge b = 0$.

From the symmetry of b, c, the subcases $b \wedge c < [(b \wedge c) \vee p] \wedge c$ and $b \wedge c = [(b \wedge c) \vee p] \wedge b$ follow.

S u b c a s e [A_{23}]: Suppose $b \wedge c < [(b \wedge c) \vee p] \wedge b$ and $b \wedge c < [(b \wedge c) \vee p] \wedge c$ hold.

Put $[(b \wedge c) \vee p] \wedge b = x$ and $[(b \wedge c) \vee p] \wedge c = y$. As shown in case $[A_{22}]$, we have $b \wedge c < [(b \wedge c) \vee p] \wedge b < b$ and $b \wedge c < [(b \wedge c) \vee p] \wedge c < c$. By Lemma 2, $\{[(b \wedge c) \vee p] \wedge b\} \vee c = 1$ and $\{[(b \wedge c) \vee p] \wedge c\} \vee b = 1$, that is, $x \vee c = y \vee b = 1$. Clearly, $p \wedge x = p \wedge y = 0$ and every proper sub-interval $[0, (b \wedge c) \vee p]$ of L is 0-distributive, hence we have $p \wedge (x \vee y) = 0$. Clearly, $x \vee y \leqslant (b \wedge c) \vee p$. If $x \vee y < (b \wedge c) \vee p$, then $L_4 = \{0, p, c, y, x \vee y, [(b \wedge c) \vee p], 1\}$ forms a sublattice isomorphic to the lattice of Figure 1(e) and if $x \vee y = (b \wedge c) \vee p$, then $L_4 = \{0, p, x, y, x \vee y, x \wedge y = b \wedge c\}$ forms a sublattice isomorphic to the lattice of Figure 1(c).

Case [B]: Without loss of generality, suppose $b \vee p < c \vee p$. Then $b \vee c = c \vee p$. Note that $(b \wedge c) \vee p \neq b \vee c$, otherwise $b \vee p = c \vee p$, a contradiction to $b \vee p < c \vee p$. This case has the following two subcases:

[B$_1$] $(b \wedge c) \vee p = b \vee p$;

[B$_2$] $(b \wedge c) \vee p < b \vee p$.

Subcase [B$_1$]: Suppose $(b \wedge c) \vee p = b \vee p$ holds. Clearly, $b < b \vee p < 1$, and by Lemma 1, $b \wedge c = (b \vee p) \wedge c$. Then $L_5 = \{0, b, c, b \wedge c, p, b \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(e) when $b \wedge c > 0$ and is isomorphic to the lattice of Figure 1(i) when $b \wedge c = 0$.

Subcase [B$_2$]: Suppose $(b \wedge c) \vee p < b \vee p$. We have the following three subcases in this case.

[B$_{21}$] $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[B$_{22}$] $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[B$_{23}$] $[(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c$;

[B$_{24}$] $[(b \wedge c) \vee p] \wedge b > b \wedge c < [(b \wedge c) \vee p] \wedge c$.

Subcase [B$_{21}$]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. It is clear that $b < b \vee p < 1$, hence by Lemma 1, $b \wedge c = (b \vee p) \wedge c$. Thus, $L_6 = \{0, b, c, b \wedge c, p, (b \wedge c) \vee p, b \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(d) when $b \wedge c > 0$ and is isomorphic to the lattice of Figure 1(i) when $b \wedge c = 0$.

Subcase [B$_{22}$]: Suppose $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. Further, we claim that $[(b \wedge c) \vee p] \wedge b \neq b$. If possible, then $b \leqslant [(b \wedge c) \vee p]$. Taking join with p, we get $b \vee p \leqslant (b \wedge c) \vee p$, a contradiction to $(b \wedge c) \vee p < b \vee p$. Hence $(b \wedge c) < [(b \wedge c) \vee p] \wedge b < b$. By Lemma 2, $\{[(b \wedge c) \vee p] \wedge b\} \vee c = 1$. Thus, $L_7 = \{0, c, b \wedge c, p, [(b \wedge c) \vee p] \wedge b, (b \wedge c) \vee p, 1\}$ forms a sublattice isomorphic to the lattice of Figure 1(e).

Subcase [B$_{23}$]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c$ holds. Along similar lines as in Subcase [B$_{22}$], one can show that $\{[(b \wedge c) \vee p] \wedge c\} \vee b = 1$, which implies that $b \vee p = 1$, a contradiction to $b \vee p < 1$. Hence this case can not occur. Similarly, Subcase [B$_{24}$] also.

C a s e [C]: Suppose $b \vee p \parallel c \vee p$. Then $b \vee p$, $c \vee p < b \vee c$. Note that $(b \wedge c) \vee p \leqslant (b \vee p) \wedge (c \vee p)$ is always true. Hence, we have the following two subcases:

[C$_1$] $(b \wedge c) \vee p = (b \vee p) \wedge (c \vee p)$;

[C$_2$] $(b \wedge c) \vee p < (b \vee p) \wedge (c \vee p)$.

S u b c a s e [C$_1$]: Suppose $(b \wedge c) \vee p = (b \vee p) \wedge (c \vee p)$ holds. For this subcase, we have the following three subcases:

[C$_{11}$] $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[C$_{12}$] $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[C$_{13}$] $[(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c$.

S u b c a s e [C$_{11}$]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. This together with $(b \wedge c) \vee p = (b \vee p) \wedge (c \vee p)$ gives $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c = b \wedge (c \vee p) = c \wedge (b \vee p)$. Thus, $L_8 = \{0, p, b, c, (b \wedge c) \vee p, b \vee p, c \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(g) when $b \wedge c \neq 0$ and L_8 forms a sublattice isomorphic to the lattice of Figure 1(f) when $b \wedge c = 0$.

S u b c a s e [C$_{12}$]: Suppose $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. We claim that $b \wedge (c \vee p) \neq b$. Otherwise $b \leqslant c \vee p$, which gives $b \vee c \leqslant c \vee p$, a contradiction to $c \vee p < b \vee c$. Hence $b \wedge c < b \wedge (c \vee p) < b$. By Lemma 2, $[b \wedge (c \vee p)] \vee c = 1$, which implies $c \vee p = 1$, a contradiction. Thus, this case can not occur. Similarly, one can show that the Subcase [C$_{13}$] can not occur.

S u b c a s e [C$_2$]: Suppose $(b \wedge c) \vee p < (b \vee p) \wedge (c \vee p) < 1$ holds. For this subcase, we have the following three subcases.

[C$_{21}$] $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[C$_{22}$] $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$;

[C$_{23}$] $[(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c$.

S u b c a s e [C$_{21}$]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. This subcase has again the following three subcases.

[C$_{211}$] $(b \vee p) \wedge c = b \wedge c = (c \vee p) \wedge b$;

[C$_{212}$] $(b \vee p) \wedge c = b \wedge c < (c \vee p) \wedge b$;

[C$_{213}$] $(b \vee p) \wedge c > b \wedge c < (c \vee p) \wedge b$.

S u b c a s e [C$_{211}$]: Suppose $(b \vee p) \wedge c = b \wedge c = (c \vee p) \wedge b$ holds. Then $L_9 = \{0, p, b, c, b \wedge c, (b \wedge c) \vee p, (b \vee p) \wedge (c \vee p), b \vee p, c \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(h) when $b \wedge c > 0$ and $L_9 = \{0, b, c, (b \vee p) \wedge (c \vee p), b \vee p, c \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(f) when $b \wedge c = 0$.

From Subcase C$_{12}$, we can show that the Subcases [C$_{212}$], [C$_{213}$], [C$_{22}$] and [C$_{23}$] can not occur.

The converse is obvious.

This completes the proof. □

R e m a r k. Note that the lattice of Figure 2 violates 0-distributivity, but has no sublattice isomorphic to Figure 1 (a), (b), (c), (d), (e), (f), (g), (e), (h) or (i). Hence, Theorem 2 fails in a lattice of infinite chains.

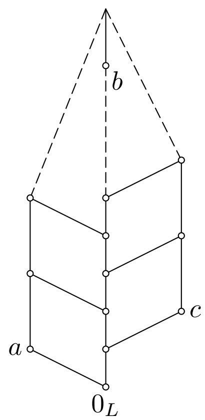

Figure 2

From the proof of Theorem 2 and Lemma 2, the next theorem can be easily proved.

Theorem 3. *A finite modular lattice L with 0 is 0-distributive if and only if it contains no sublattice isomorphic to the lattice of Figure* 3(a) *or to the lattice of Figure* 3(b) *(where covers indicated by double lines are preserved).*

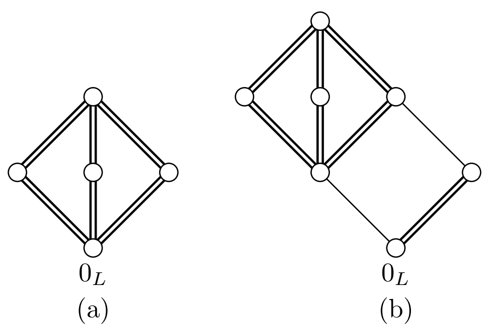

Figure 3

A c k n o w l e d g e m e n t. The author is grateful to Professor B. N. Waphare and the learned referee for their fruitful suggestions.

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