ASYMPTOTIC PROPERTIES FOR HALF-LINEAR DIFFERENCE EQUATIONS

Mariella Cecchi, Firenze, Zuzana Došlá, Brno,
Mauro Marini, Firenze, Ivo Vrkoč, Praha

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Dedicated to Prof. Jaroslav Kurzweil on the occasion of his 80th birthday

Abstract. Asymptotic properties of the half-linear difference equation

$$\Delta(a_n|\Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n) = b_n|x_{n+1}|^{\alpha} \operatorname{sgn} x_{n+1} \tag{$*$}$$

are investigated by means of some summation criteria. Recessive solutions and the Riccati difference equation associated to $(*)$ are considered too. Our approach is based on a classification of solutions of $(*)$ and on some summation inequalities for double series, which can be used also in other different contexts.

Keywords: half-linear second order difference equation, nonoscillatory solutions, Riccati difference equation, summation inequalities

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1. Introduction

Consider the half-linear difference equation

$$\Delta\big(a_n|\Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n\big) = b_n|x_{n+1}|^{\alpha} \operatorname{sgn} x_{n+1}, \tag{1}$$

where $a = \{a_n\}, b = \{b_n\}$ are positive real sequences for $n \geqslant 1$ and $\alpha > 0$.

The qualitative behavior of solutions of (1) has been investigated, from different point of view, in several recent papers: see, e.g., [3], [4], [10], [12] and references

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therein. Clearly, if $x = \{x_n\}$ is a solution of (1), then $-x$ is a solution too. Hence, for the sake of simplicity, we restrict our study to solutions x for which $x_n > 0$ for large n. It is easy to show that any nontrivial solution of (1) is nonoscillatory and for large n monotone, see, e.g., [2, Lemma 1]. More precisely, any nontrivial solution x of (1) belongs to one of the two classes listed below:

$$\begin{aligned}\mathbb{M}^+ &= \{x \text{ solution of } (1)\colon\ x_k > 0,\ \Delta x_k > 0 \text{ for large } k\},\\ \mathbb{M}^- &= \{x \text{ solution of } (1)\colon\ x_k > 0,\ \Delta x_k < 0 \text{ for } k \geqslant 1\}.\end{aligned}$$

Clearly, solutions with initial conditions $x_1 > 0$, $\Delta x_1 > 0$ are in the class $\mathbb{M}^+$; also $\mathbb{M}^- \neq \emptyset$ as follows, e.g., from [2, Theorem 1] or from [1, Th. 6.10.4], [11], with minor changes. Since both the classes are nonempty, the set of solutions of (1) presents a dichotomy.

The aim of this paper is to continue the study started in [4], by characterizing this dichotomy by means of some summation criteria and by examining the role of the so-called asymptotically constant solutions. These results can be interpreted also in the context of recessive and dominant solutions of (1), because in many cases, the classes $\mathbb{M}^-$ and $\mathbb{M}^+$ coincide with these solutions, respectively.

Special attention is given to the corresponding Riccati difference equation

$$\Delta w_n - b_n + \bigl(1 - S(a_n, w_n)\bigr) w_n = 0, \tag{2}$$

where

$$S(a_n, w_n) = \frac{a_n}{\bigl|(a_n)^{1/\alpha} + |w_n|^{1/\alpha} \operatorname{sgn} w_n\bigr|^\alpha} \operatorname{sgn}\bigl((a_n)^{1/\alpha} + |w_n|^{1/\alpha} \operatorname{sgn} w_n\bigr).$$

Equation (2) is closely related to (1). Indeed, when (1) is nonoscillatory, for any solution x of (1) the sequence $w = \{w_n\}$, where

$$w_n = \frac{a_n |\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n}{|x_n|^\alpha \operatorname{sgn} x_n}, \tag{3}$$

is a solution of (2) for large n.

The paper is organized as follows. Section 2 is devoted to summation inequalities. They originate from analogous ones involving double integrals and are presented in an independent form, because they can be applied also in other different contexts, as it is shown in [6]. In Section 3 a brief review on qualitative behavior of solutions of (1) is given. Using these results, in Section 4 some summation characterizations of classes $\mathbb{M}^-$ and $\mathbb{M}^+$ are presented jointly with applications to recessive solutions of (1). Finally, in Section 5 asymptotic properties of solutions of the generalized Riccati equation (2) are obtained. Several illustrative examples complete the paper.

We close this section by introducing some notation. For any solution x of (1) denote by $x^{[1]} = \{x_n^{[1]}\}$ its quasi-difference, where

$$x_n^{[1]} = a_n|\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n.$$

Put

$$W_1 = \lim_{N\to\infty} \sum_{n=1}^{N} b_n \biggl(\sum_{k=n}^{N} \Bigl(\frac{1}{a_{k+1}} \Bigr)^{1/\alpha} \biggr)^{\alpha}, \; W_2 = \lim_{N\to\infty} \sum_{n=1}^{N} \biggl(\frac{1}{a_n} \sum_{k=n}^{N} b_k \biggr)^{1/\alpha}$$

$$Z_1 = \lim_{N\to\infty} \sum_{n=2}^{N} \biggl(\frac{1}{a_n} \sum_{k=1}^{n-1} b_k \biggr)^{1/\alpha}, \; Z_2 = \lim_{N\to\infty} \sum_{n=2}^{N} b_n \biggl(\sum_{k=1}^{n-1} \Bigl(\frac{1}{a_{k+1}} \Bigr)^{1/\alpha} \biggr)^{\alpha}$$

and

$$Y_a = \sum_{n=1}^{\infty} \Bigl(\frac{1}{a_n} \Bigr)^{1/\alpha}, \quad Y_b = \sum_{n=1}^{\infty} b_n, \quad Y_{ab} = \sum_{n=1}^{\infty} \Bigl(\frac{1}{a_n} \Bigr)^{1/\alpha} \biggl(\sum_{k=n}^{\infty} b_k \biggr)^{(1-\alpha)/\alpha}.$$

R e m a r k 1. It is easy to verify that the following relations hold:
(i_1) If $Z_1 < \infty$, then $Y_a < \infty$.
(i_2) If $W_2 < \infty$, then $Y_b < \infty$.
(i_3) $Z_1 < \infty$ and $W_2 < \infty$ if and only if $Y_a < \infty$ and $Y_b < \infty$.

2. Series relations

As we have shown in [4], [5], [7], an important tool in the asymptotic theory of half-linear differential and difference equations is the change of integration and summation for certain double integrals and series, respectively. This section contributes to this problem by giving two new summation inequalities which will be useful later.

Let $A = \{A_n\}$, $B = \{B_n\}$ be two sequences of nonnegative numbers and let λ, μ be two positive numbers. Denote

$$(4)\quad S_\lambda(A,B) = \lim_{N\to\infty} \sum_{n=1}^{N} B_n \biggl(\sum_{k=1}^{n} A_k \biggr)^{\lambda}, \quad T_\mu(A,B) = \lim_{N\to\infty} \sum_{n=1}^{N} A_n \biggl(\sum_{k=n}^{N} B_k \biggr)^{1/\mu}.$$

When $\lambda = \mu$, it is proved in [4, Theorem 1] that if $\lambda = \mu \geqslant 1$, then for any $N > 1$

$$\biggl(\sum_{n=1}^{N} A_n \biggl(\sum_{k=n}^{N} B_k \biggr)^{1/\mu} \biggr)^{\mu} \geqslant \sum_{n=1}^{N} B_n \biggl(\sum_{k=1}^{n} A_k \biggr)^{\mu}.$$

Similarly, if $\lambda = \mu \leqslant 1$, then for any $N > 1$

$$\sum_{n=1}^{N} B_n \left(\sum_{k=1}^{n} A_k\right)^{\mu} \geqslant \left(\sum_{n=1}^{N} A_n \left(\sum_{k=n}^{N} B_k\right)^{1/\mu}\right)^{\mu}.$$

Hence

$$\begin{aligned} &T_\mu(A,B) < \infty \Longrightarrow S_\mu(A,B) < \infty \quad \text{when } \mu \geqslant 1,\\ &S_\mu(A,B) < \infty \Longrightarrow T_\mu(A,B) < \infty \quad \text{when } \mu \leqslant 1. \end{aligned} \tag{5}$$

To extend relations (5) for $\mu \neq \lambda$, let f, g be two nonnegative functions and $f, g \in L^1_{\mathrm{loc}}[1,\infty)$. Define

$$\begin{aligned} I_\lambda(f,g) &= \lim_{T\to\infty} \int_1^T g(t) \left(\int_1^t f(s)\,\mathrm{d}s\right)^{\lambda} \mathrm{d}t,\\ J_\mu(f,g) &= \lim_{T\to\infty} \int_1^T f(t) \left(\int_t^T g(s)\,\mathrm{d}s\right)^{1/\mu} \mathrm{d}t. \end{aligned}$$

If $\lambda = \mu = 1$, in view of the Fubini theorem we have $I_1(f,g) = J_1(f,g)$. Further, if $\int^\infty g(t)\,\mathrm{d}t = \infty$, then $I_\lambda(f,g) = J_\mu(f,g) = \infty$. In general, the following holds.

Lemma 1.
(i_1) *If $\mu < \lambda$, then*

$$I_\lambda(f,g) < \infty \quad \Longrightarrow \quad J_\mu(f,g) < \infty.$$

(i_2) *If $\mu > \lambda$, then*

$$J_\mu(f,g) < \infty \quad \Longrightarrow \quad I_\lambda(f,g) < \infty.$$

P r o o f. The assertion has been proved in [5, Lemmas 1, 2], under the stronger assumption that f, g are positive and continuous on $[0,\infty)$. In the more general case considered here, the assertion follows again from [5, Lemmas 1, 2], with minor changes. □

A similar result for series holds.

Lemma 2.

(i_1) *If* $\mu < \lambda$, *then*

$$S_\lambda(A,B) < \infty \Longrightarrow T_\mu(A,B) < \infty.$$

(i_2) *If* $\mu > \lambda$, *then*

$$T_\mu(A,B) < \infty \Longrightarrow S_\lambda(A,B) < \infty.$$

Proof. The assertions easily follow from Lemma 1. Claim (i_1). Define

$$f(t) = A_n,\ g(t) = B_n,\ \text{if } t \in [n, n+1).$$

We have

$$\begin{aligned}
\int_1^{N+1} g(t)\Big(\int_1^t f(s)\,\mathrm{d}s\Big)^\lambda \mathrm{d}t &= \sum_{k=1}^N \int_k^{k+1} g(t)\Big(\int_1^t f(s)\,\mathrm{d}s\Big)^\lambda \mathrm{d}t \\
&= \sum_{k=1}^N B_k \int_k^{k+1} \Big(\sum_{i=1}^{k-1}\int_i^{i+1} f(s)\,\mathrm{d}s + \int_k^t f(s)\,\mathrm{d}s\Big)^\lambda \mathrm{d}t \\
&\leqslant \sum_{k=1}^N B_k \int_k^{k+1} \Big(\sum_{i=1}^{k-1} A_i + A_k\Big)^\lambda \mathrm{d}t = \sum_{k=1}^N B_k \Big(\sum_{i=1}^k A_i\Big)^\lambda.
\end{aligned}$$

Since $S_\lambda(A,B) < \infty$, we have $I_\lambda(f,g) < \infty$. Applying Lemma 1, we obtain $J_\mu(f,g) < \infty$. Therefore

$$\begin{aligned}
J_\mu^N(f,g) &= \int_1^{N+1} f(t)\Big(\int_t^{N+1} g(s)\,\mathrm{d}s\Big)^{1/\mu} \mathrm{d}t = \sum_{k=1}^N \int_k^{k+1} f(t)\Big(\int_t^{N+1} g(s)\,\mathrm{d}s\Big)^{1/\mu} \mathrm{d}t \\
&= \sum_{k=1}^N A_k \int_k^{k+1} \Big(\int_t^{k+1} g(s)\,\mathrm{d}s + \int_{k+1}^{N+1} g(s)\,\mathrm{d}s\Big)^{1/\mu} \mathrm{d}t \\
&= \sum_{k=1}^N A_k \int_k^{k+1} \Big(B_k(k+1-t) + \sum_{i=k+1}^N B_i\Big)^{1/\mu} \mathrm{d}t.
\end{aligned}$$

Since $t \in [k, k+1]$, we have $1 \geqslant (k+1-t)$. Hence

$$\begin{aligned}
J_\mu^N(f,g) &\geqslant \sum_{k=1}^N A_k \int_k^{k+1} \Big(B_k(k+1-t) + (k+1-t)\sum_{i=k+1}^N B_i\Big)^{1/\mu} \mathrm{d}t \\
&= \sum_{k=1}^N A_k \int_k^{k+1} (k+1-t)^{1/\mu} \Big(\sum_{i=k}^N B_i\Big)^{1/\mu} \mathrm{d}t \\
&= \sum_{k=1}^N A_k \Big(\sum_{i=k}^N B_i\Big)^{1/\mu} \int_k^{k+1} (k+1-t)^{1/\mu}\,\mathrm{d}t = \frac{\mu}{\mu+1}\sum_{k=1}^N A_k\Big(\sum_{i=k}^N B_i\Big)^{1/\mu},
\end{aligned}$$

which yields $T_\mu(A,B) < \infty$. The claim (i_2) follows in a similar way. □

Notice that the vice-versa of Lemma 2 can fail. To this end, consider the sequences $A = \{1\}$, $B = \{(n^2+n)^{-1}\}$. Then $S_3(A,B) = \infty$, $T_{1/2}(A,B) < \infty$ and so the converse of Lemma 2(i_1) is not true. Similarly, $S_{1/2}(A,B) < \infty$, $T_3(A,B) = \infty$ and so the converse of Lemma 2(i_2) is not true, either.

3. Classification of solutions

According to the asymptotic behavior of a solution x of (1) and its quasi-difference $x^{[1]}$, both classes can be *a-priori* divided into the following subclasses:

$$
\begin{aligned}
\mathbb{M}_l^+ &= \{x \in \mathbb{M}^+ : \lim_n x_n = l_x,\ 0 < l_x < \infty\},\\
\mathbb{M}_{\infty,l}^+ &= \{x \in \mathbb{M}^+ : \lim_n x_n = \infty,\ \lim_n x_n^{[1]} = l_x,\ 0 < l_x < \infty\},\\
\mathbb{M}_{\infty,\infty}^+ &= \{x \in \mathbb{M}^+ : \lim_n x_n = \lim_n x_n^{[1]} = \infty\},\\
\mathbb{M}_l^- &= \{x \in \mathbb{M}^- : \lim_n x_n = l_x,\ 0 < l_x < \infty\},\\
\mathbb{M}_{0,l}^- &= \{x \in \mathbb{M}^- : \lim_n x_n = 0,\ \lim_n x_n^{[1]} = -l_x,\ 0 < l_x < \infty\},\\
\mathbb{M}_{0,0}^- &= \{x \in \mathbb{M}^- : \lim_n x_n = \lim_n x_n^{[1]} = 0\}.
\end{aligned}
$$

In [4], solutions in the subclasses of $\mathbb{M}^+$ and $\mathbb{M}^-$ have been described in terms of the convergence or divergence of the series W_i, Z_i $(i = 1,2)$. More precisely, by using certain summation inequalities, in [4] it is shown that the possible cases concerning the mutual behavior of these series are the following:

$$
\begin{aligned}
&C_1\colon \quad Z_1 = W_1 = Z_2 = W_2 = \infty;\\
&C_2\colon \quad Z_1 = W_1 = \infty,\ Z_2 < \infty,\ W_2 < \infty;\\
&C_3\colon \quad Z_1 < \infty,\ W_1 < \infty,\ Z_2 = W_2 = \infty;\\
&C_4\colon \quad Z_1 < \infty,\ W_1 < \infty,\ Z_2 < \infty,\ W_2 < \infty;\\
&C_5\colon \quad Z_1 = W_1 = \infty,\ Z_2 < \infty,\ W_2 = \infty \text{ (only if } \alpha > 1);\\
&C_6\colon \quad Z_1 = \infty,\ W_1 < \infty,\ Z_2 = W_2 = \infty \text{ (only if } \alpha > 1);\\
&C_7\colon \quad Z_1 = W_1 = Z_2 = \infty,\ W_2 < \infty \text{ (only if } \alpha < 1);\\
&C_8\colon \quad Z_1 < \infty,\ W_1 = Z_2 = W_2 = \infty \text{ (only if } \alpha < 1).
\end{aligned}
$$

Notice that for $\alpha = 1$, i.e. for the linear equation, only the cases C_1–C_4 are possible. Thus cases C_5–C_8 illustrate the difference in passing from the linear equation to the half-linear one.

The following holds, see [4, Proposition 2, Theorems 2,3].

Theorem A. *For solutions of* (1) *we have:*

if C_1 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty}$, $\mathbb{M}^- = \mathbb{M}^-_{0,0}$;
if C_2 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,l}$, $\mathbb{M}^- = \mathbb{M}^-_{l}$;
if C_3 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{l}$, $\mathbb{M}^- = \mathbb{M}^-_{0,l}$;
if C_4 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{l}$, $\mathbb{M}^-_{0,0} = \emptyset$, $\mathbb{M}^-_{0,l} \neq \emptyset$, $\mathbb{M}^-_{l} \neq \emptyset$.

In addition, when $\alpha > 1$,

if C_5 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,l}$, $\mathbb{M}^- = \mathbb{M}^-_{0,0}$;
if C_6 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty}$, $\mathbb{M}^- = \mathbb{M}^-_{0,l}$;

and, when $\alpha < 1$,

if C_7 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty}$, $\mathbb{M}^- = \mathbb{M}^-_{l}$;
if C_8 *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{l}$, $\mathbb{M}^- = \mathbb{M}^-_{0,0}$.

The asymptotic behavior of $x^{[1]}$, where $x \in \mathbb{M}^+_l \cup \mathbb{M}^-_l$, is given by the following result, which will be useful in the sequel.

Lemma 3. *If*

$$Y_a + Y_b = \infty \tag{6}$$

then every solution $x \in \mathbb{M}^-_l$ *satisfies* $\lim_n x^{[1]}_n = 0$ *and every solution* $x \in \mathbb{M}^+_l$ *satisfies* $\lim_n x^{[1]}_n = \infty$.

In the opposite case, i.e. $Y_a + Y_b < \infty$, *every solution* $x \in \mathbb{M}^-_l \cup \mathbb{M}^+_l$ *satisfies* $\lim_n x^{[1]}_n = c_x$, *where* $|c_x| < \infty$.

Proof. Assume (6). In virtue of Remark 1, the case C_4 does not occur. Let $x \in \mathbb{M}^-_l$ and suppose $\lim_n x^{[1]}_n = -c_x < 0$. From Theorem A, the possible cases are C_2 or C_7 and so, from Remark 1 and (6), $Y_b < \infty$, $Y_a = \infty$. Since $x^{[1]}$ is negative increasing, we have $x^{[1]}_n < -c_x$ and, by summation from n to ∞, we obtain a contradiction with the positiveness of x.

Now let $x \in \mathbb{M}^+_l$ and assume $x_n > 0, \Delta x_n > 0$ for $n \geqslant n_0 \geqslant 1$. Again from Theorem A, the possible cases are C_3 or C_8 and we obtain $Y_a < \infty$, $Y_b = \infty$. Summarizing (1) from n_0 to n we have

$$x^{[1]}_{n+1} - x^{[1]}_{n_0} = \sum_{k=n_0}^{n} b_k (x_{k+1})^\alpha \geqslant (x_{n_0+1})^\alpha \sum_{k=n_0}^{n} b_k$$

and the second statement follows.

Now assume $Y_a + Y_b < \infty$. Clearly, if $x \in \mathbb{M}_l^-$, then $\lim_n x_n^{[1]} = c_x$, $|c_x| < \infty$. Let $x \in \mathbb{M}_l^+$: summarizing (1) from n_0 to n (n_0 large) and taking into account the boundedness of x, we have for some $h > 0$

$$x_{n+1}^{[1]} - x_{n_0}^{[1]} = \sum_{k=n_0}^{n} b_k (x_{k+1})^\alpha \leqslant h \sum_{k=n_0}^{n} b_k$$

from where the last statement follows. □

As already mentioned, in [10] the concept of recessive solutions for (1) has been defined using certain asymptotic properties of solutions of (2). It reads for (1) with $b_n > 0$ as follows: there exists a unique solution v of (2) with the property

$$v_n < w_n \quad \text{for large } n$$

for any other solution w of (2) defined in some neighbourhood of ∞. Solution v is said to be *eventually minimal*. The sequence u, where

$$\Delta u_n = \frac{|v_n|^{1/\alpha} \operatorname{sgn} v_n}{(a_n)^{1/\alpha}}\, u_n, \tag{7}$$

is a solution of (1) and is called a *recessive solution* of (1). Any nontrivial solution of (1), which is not recessive, is called a *dominant solution.* Obviously, recessive solutions of (1) are determined up to a constant factor.

From Theorem A and Lemma 3 (see also [4]) the following asymptotic characterization of recessive solutions of (1) completes that in [4, page 12].

Corollary 1. *Let u be an eventually positive solution of* (1). *Except for the case C_4, the solution u is recessive if and only if $u \in \mathbb{M}^-$. In addition, recessive solutions satisfy*

$$\begin{aligned}
&\lim_n u_n = \lim_n u_n^{[1]} = 0 \text{ in cases } C_1,\ C_5,\ C_8;\\
&\lim_n u_n = l_u > 0, \quad \lim_n u_n^{[1]} = 0 \text{ in cases } C_2,\ C_7;\\
&\lim_n u_n = 0, \quad \lim_n u_n^{[1]} = l_u < 0 \text{ in cases } C_3,\ C_6.
\end{aligned}$$

In the case C_4 u is a recessive solution if and only if $\lim_n u_n = 0$ *and* $\lim_n u_n^{[1]} = l_u < 0$.

As already claimed, the set of solutions of (1) exhibits a kind of dichotomy for classes $\mathbb{M}^-$ and $\mathbb{M}^+$. In this section we characterize this dichotomy in a unified way based on summation criteria, jointly with a discussion about the above given classification C_1–C_8.

Put

$$\Gamma_u = \sum^{\infty} \frac{1}{a_n^{1/\alpha} u_n u_{n+1}};$$
$$\Lambda_u = \sum^{\infty} \frac{b_n}{u_n^{[1]} u_{n+1}^{[1]}};$$
$$\Omega_u = \sum^{\infty} \frac{1}{a_n |\Delta u_n|^{\alpha-1} u_n u_{n+1}} = \sum^{\infty} \frac{|\Delta u_n|}{|u_n^{[1]}| u_n u_{n+1}}.$$

Theorem 1. *Let u be a nontrivial solution of* (1) *and assume*

$$Z_2 + W_2 = \infty. \tag{8}$$

Then the following holds:

$$u \in \mathbb{M}^- \iff \Gamma_u = \infty \iff \Lambda_u = \infty \iff \Omega_u = \infty. \tag{9}$$

Proof. First observe that in view of (8) the only possible cases are C_1, C_3, C_5–C_8.

Step 1: $u \in \mathbb{M}^- \iff \Gamma_u = \infty$. In view of Corollary 1, the assertion follows from [3, Th. 4].

Step 2: $u \in \mathbb{M}^- \iff \Lambda_u = \infty$. It is easy to verify that the sequence $y = u^{[1]}$ satisfies the difference equation

$$\Delta\Big(\Big(\frac{1}{b_n}\Big)^{1/\alpha} |\Delta y_n|^{1/\alpha} \operatorname{sgn} \Delta y_n\Big) = \Big(\frac{1}{a_{n+1}}\Big)^{1/\alpha} |y_{n+1}|^{1/\alpha} \operatorname{sgn} y_{n+1}. \tag{10}$$

Obviously, $u \in \mathbb{M}^-$ if and only if $y_n y_n^{[1]} < 0$. Then, by applying Step 1 to (10), the assertion follows.

Step 3: $u \in \mathbb{M}^- \Longrightarrow \Omega_u = \infty$. Let $u \in \mathbb{M}^-$. Taking into account

$$\frac{a_n^{1/\alpha} u_n u_{n+1}}{a_n |\Delta u_n|^{\alpha-1} u_n u_{n+1}} = a_n^{(1-\alpha)/\alpha} |\Delta u_n|^{1-\alpha} = |u_n^{[1]}|^{(1-\alpha)/\alpha} \operatorname{sgn} u_n^{[1]}, \tag{11}$$

if $\lim_n u_n^{[1]} = l_u, -\infty < l_u < 0$, by the comparison criterion, the series Γ_u and Ω_u have the same behavior. Then, by applying Step 1, the assertion follows. If $\lim_n u_n^{[1]} = 0$ and $\alpha \geqslant 1$, again, by using the comparison criterion, the assertion follows. If $\lim_n u_n^{[1]} = 0$ and $\alpha < 1$, then, in virtue of (8) and Theorem A, the possible cases are C_1, C_7, C_8. If C_1 or C_8 occurs, taking into account that $u^{[1]}$ is negative increasing, we have for large n_0

$$\frac{1}{|u_{n_0}^{[1]}|} \sum_{n=n_0}^{\infty} \frac{|\Delta u_n|}{u_n u_{n+1}} = \frac{1}{|u_{n_0}^{[1]}|} \sum_{n=n_0}^{\infty} \Delta\Big(\frac{1}{u_n}\Big) \tag{12}$$

and so $\Omega_u = \infty$. Now assume the case C_7. By Corollary 1, $\lim_n u_n^{[1]} = 0$. Summing (1) from n to ∞, we obtain

$$-u_n^{[1]} = \sum_{k=n}^{\infty} b_k (u_{k+1})^{\alpha}.$$

Since $u \in \mathbb{M}_l^-$, we have

$$-u_n^{[1]} \sim \sum_{k=n}^{\infty} b_k,$$

and therefore

$$|\Delta u_n|^{\alpha-1} \sim \Big(\frac{1}{a_n}\Big)^{(\alpha-1)/\alpha} \Big(\sum_{k=n}^{\infty} b_k\Big)^{(\alpha-1)/\alpha},$$

where the symbol $c_n \sim d_n$ means that $\lim_n c_n/d_n$ is finite and different from zero. Then

$$\frac{|\Delta u_n|}{|u_n^{[1]}|} = \frac{1}{a_n} \frac{1}{|\Delta u_n|^{\alpha-1}} \sim \Big(\frac{1}{a_n}\Big)^{1/\alpha} \Big(\sum_{k=n}^{\infty} b_k\Big)^{(1-\alpha)/\alpha}. \tag{13}$$

Because the case C_7 holds, we have $Z_2 = \infty$. Putting $\lambda = \alpha$ and $A_n = (1/a_n)^{1/\alpha}$, $B_n = b_n$, from (4) we obtain $S_\lambda(A, b) \geqslant Z_2 = \infty$. Since $\alpha < 1$, we have $\alpha < \alpha/(1-\alpha)$ and so, by applying Lemma 2 with $\mu = \alpha/(1-\alpha)$, we obtain

$$\infty = T_\mu(A, b) = \sum_{n=1}^{\infty} \Big(\frac{1}{a_n}\Big)^{1/\alpha} \Big(\sum_{k=n}^{\infty} b_k\Big)^{(1-\alpha)/\alpha}$$

which, in view of (13), gives the assertion.

S t e p 4: $\Omega_u = \infty \Longrightarrow u \in \mathbb{M}^-$. By contradiction, suppose $u \in \mathbb{M}^+$. Since $u^{[1]}$ is positive increasing for large n, there exists a positive constant h such that $u_n^{[1]} \geqslant h$ for $n \geqslant n_0 \geqslant 1$. Then

$$\sum_{n=n_0}^{\infty} \frac{|\Delta u_n|}{|u_n^{[1]}| u_n u_{n+1}} \leqslant \frac{1}{h} \sum_{n=n_0}^{\infty} \frac{|\Delta u_n|}{u_n u_{n+1}} = \sum_{n=n_0}^{\infty} \Delta\Big(\frac{1}{u_n}\Big) < \infty,$$

a contradiction.

From Steps 1–4, the assertion follows. $\square$

In Theorem 1, the condition (8) is assumed. If both the series W_2, Z_2 are convergent, the possible cases are C_2 or C_4 and the situation is different.

When the case C_2 holds, then

$$u \in \mathbb{M}^- \iff \Gamma_u = \infty \iff \Lambda_u = \infty, \tag{14}$$

as can be proved by using [3, Th. 4] and an argument similar to that given in the proof of Theorem 1. In a similar way, the statement

$$\Omega_u = \infty \Longrightarrow u \in \mathbb{M}^-$$

continues to hold, but the opposite implication can fail, as the following example shows.

E x a m p l e 1. Consider the equation

$$\Delta\big(|\Delta x_n|^{1/2} \operatorname{sgn} \Delta x_n\big) = (2 - \sqrt{2}) 2^{-3/2} \frac{2^{-n/2}}{(1 + 2^{-n-1})^{1/2}} |x_{n+1}|^{1/2} \operatorname{sgn} x_{n+1}. \tag{15}$$

It is easy to verify that the sequence u, where

$$u_n = 1 + 2^{-n},$$

is a solution of (15) in the class $\mathbb{M}^-$. Clearly, the case C_2 holds and

$$\Omega_u \sim \sum^{\infty} 2^{-n/2} < \infty.$$

Now consider the case C_4. In such a case we have

$$x \in \mathbb{M}^+ \Longrightarrow \Gamma_x < \infty,\ \Lambda_x < \infty,\ \Omega_x < \infty, \tag{16}$$

i.e. any of the conditions $\Gamma_u = \infty, \Lambda_u = \infty, \Omega_u = \infty$ yields $u \in \mathbb{M}^-$.

To prove this, let $x \in \mathbb{M}^+$. From [3, Th. 4] we obtain $\Gamma_x < \infty$. Since $x^{[1]}$ is positive increasing for large n, there exists $n_0 \geqslant 1$ such that $x_n^{[1]} x_{n+1}^{[1]} \geqslant x_{n_0}^{[1]} x_{n_0+1}^{[1]} > 0$ for $n \geqslant n_0$ and so

$$\sum_{n=n_0}^{\infty} \frac{b_n}{x_n^{[1]} x_{n+1}^{[1]}} \leqslant \frac{1}{x_{n_0}^{[1]} x_{n_0+1}^{[1]}} \sum_{n=n_0}^{\infty} b_n. \tag{17}$$

Since C_4 holds, from Remark 1 we have $Y_b < \infty$ and so (17) yields $\Lambda_x < \infty$. Finally, by the same reasoning as in the proof of Theorem 1, Step 4, we obtain $\Omega_x < \infty$.

In addition, in the case C_4 also the statement

$$\Gamma_u = \infty \Longrightarrow \Omega_u = \infty \tag{18}$$

continues to hold. Indeed, because $\Gamma_u = \infty$, u is a recessive solution of (1) ([3, Th. 4]). Then from Corollary 1 we obtain $\lim\limits_n u_n^{[1]} = l \neq 0$ and the assertion follows from (11) by using the comparison criterion for series.

Notice that, when the case C_4 occurs, the vice-versa of (16) and (18) are not true, as the following examples show.

Example 2. Consider the equation

$$\Delta\big(n(n+1)^3 |\Delta x_n|^2 \operatorname{sgn} \Delta x_n\big) = \frac{n+1}{n(n+2)^2} |x_{n+1}|^2 \operatorname{sgn} x_{n+1}. \tag{19}$$

It is easy to verify that the sequence u, where

$$u_n = \frac{n+1}{n}, \tag{20}$$

is a solution of (19) in the class $\mathbb{M}^-$ satisfying $\lim\limits_n u_n^{[1]} < 0$. Clearly the case C_4 holds and

$$\Lambda_u = \sum^{\infty} \frac{n+1}{(n+2)^3} < \infty, \qquad \Gamma_u = \sum^{\infty} \frac{n^{1/2}}{(n+1)^{3/2}(n+2)} < \infty,$$
$$\Omega_u = \sum^{\infty} \frac{n}{(n+1)^2(n+2)} < \infty.$$

Example 3. Consider the equation

$$\Delta\big(n^2(n+1)^4 |\Delta x_n|^4 \operatorname{sgn} \Delta x_n\big) = \frac{(2n+1)(n+1)^2}{n^2(n+2)^4} |x_{n+1}|^4 \operatorname{sgn} x_{n+1}. \tag{21}$$

Then the sequence u defined by (20) is a solution of (21) such that $u \in \mathbb{M}^-$ and $\lim_n u_n^{[1]} = 0$. Again the case C_4 holds and

$$\Lambda_u = \sum^{\infty} \frac{(2n+1)(n+1)^4}{(n+2)^2} = \infty, \quad \Gamma_u = \sum^{\infty} \frac{n^{1/2}}{(n+1)(n+2)} < \infty,$$
$$\Omega_u = \sum^{\infty} \frac{n^2}{(n+1)(n+2)} = \infty.$$

Hence the vice versa of (18) does not hold. Notice that in (21) we have $\alpha > 1$. If $\alpha \leqslant 1$, then, by using the comparison criterion for series, it is easy to show that $\Omega_u = \infty \Longrightarrow \Gamma_u = \infty$.

A closer examination of Examples 1–3 shows that the partial lack of equivalency between statements in (9) originates from the existence of asymptotically constant solutions of (1) in the class $\mathbb{M}^-$, i.e. solutions $u \in \mathbb{M}^-$ satisfying

$$\lim_n u_n = l \neq 0, \ \ \lim_n u_n^{[1]} = 0. \tag{22}$$

As follows from Theorem A and Lemma 3, these solutions exist when any of the cases C_2, C_4, C_7 occur. In the case C_7 Theorem 1 holds, while the remaining cases are described in the following theorem.

Theorem 2. *Let $u \in \mathbb{M}^-$ satisfy* (22).

If C_2 holds, then $\Gamma_u = \Lambda_u = \infty$ and, when $\alpha \geqslant 1$, $\Omega_u = \infty$. If $\alpha < 1$, we have $\Omega_u = \infty$ if and only if $Y_{ab} = \infty$.

If C_4 holds, then $\Gamma_u < \infty$, $\Lambda_u = \infty$ and, when $\alpha \leqslant 1$, $\Omega_u < \infty$. If $\alpha > 1$, we have $\Omega_u = \infty$ if and only $Y_{ab} = \infty$.

P r o o f. Consider the case C_2. As we have noticed above, (14) holds. Concerning the series Ω_u, when $\alpha \geqslant 1$, by using (11) and the comparison criterion for series Γ_u, Ω_u, we obtain $\Omega_u = \infty$. If $\alpha < 1$, from (13) we obtain

$$\frac{1}{a_n} \frac{1}{|\Delta u_n|^{\alpha-1} u_n u_{n+1}} \sim \left(\frac{1}{a_n}\right)^{1/\alpha} \left(\sum_{k=n}^{\infty} b_k\right)^{(1-\alpha)/\alpha},$$

which yields $\Omega_u = \infty$ if and only if $Y_{ab} = \infty$.

Now consider the case C_4. By Remark 1 we have $Y_a < \infty$ and so $\Gamma_u < \infty$. Taking into account that u is positive decreasing, from (1) we obtain $\Delta u_n^{[1]} \leqslant h b_n$, where $h = u_2^{\alpha}$, or

$$b_n \geqslant \frac{1}{h} \Delta u_n^{[1]}.$$

Then

$$\Lambda_u \geqslant \frac{1}{h}\sum^{\infty}\frac{\Delta u_n^{[1]}}{u_n^{[1]}u_{n+1}^{[1]}} = -\frac{1}{h}\sum^{\infty}\Delta\Big(\frac{1}{u_n^{[1]}}\Big) = \infty.$$

Concerning the series Ω_u, when $\alpha \leqslant 1$, by using (11) and the comparison criterion for series, we obtain $\Omega_u < \infty$.

If $\alpha > 1$, applying the argument used in the final part of the previous case, we have $\Omega_u = \infty$ if and only if $Y_{ab} = \infty$. □

Concerning the class $\mathbb{M}^+$, summarizing the above results, we obtain the following.

Corollary 2. *If* (8) *holds, i.e. if any of the cases* C_i, $i \in \{1,3,5,6,7,8\}$ *occurs, then*

$$x \in \mathbb{M}^+ \iff \Gamma_x < \infty \iff \Lambda_x < \infty \iff \Omega_x < \infty.$$

If the case C_2 *holds, then*

$$x \in \mathbb{M}^+ \iff \Gamma_x < \infty \iff \Lambda_x < \infty$$

and, if $\alpha \geqslant 1$,

$$x \in \mathbb{M}^+ \iff \Omega_x < \infty.$$

If the case C_4 *holds, then* (16) *holds.*

P r o o f. The assertion follows from Theorems A, 1 and 2. □

R e m a r k 2. Except for the case C_4, solutions in $\mathbb{M}^-$ or $\mathbb{M}^+$ are recessive or dominant, respectively (see, e.g., Corollary 1). Hence the above results can be used for improving asymptotic properties of recessive and dominant solutions of (1). In view of Corollary 1, the solutions considered in Theorem 2 are recessive solutions when C_2 occurs, and dominant solutions in the case C_4. In the case C_4, recessive solutions belong to $\mathbb{M}^-_{0,l}$ and in view of (11), (12) and (17) they satisfy

$$u \in \mathbb{M}^-_{0,l} \implies \Gamma_u = \infty,\ \Lambda_u < \infty,\ \Omega_u = \infty.$$

5. Riccati difference equation

In this section we describe asymptotic properties of solutions of (2). Here the notation $f \to 0+$ means that $\lim_n f_n = 0$, whereby $f_n > 0$ for all large n, and, similarly, $f \to 0-$ means that $\lim_n f_n = 0$ and $f_n < 0$ for all large n.

Theorem 3. *Let v be the minimal solution and w any other solution of* (2).
If C_2 holds, then $w \to 0+$, $v \to 0-$;
if C_3 holds, then $w \to \infty$, $v \to -\infty$;
if C_4 holds, then $w \to c_w$, $c_w \in \mathbb{R}$, $v \to -\infty$;
if C_5 holds, then $w \to 0+$;
if C_6 holds, then $v \to -\infty$;
if C_7 holds, then $v \to 0-$;
if C_8 holds, then $w \to \infty$.

Proof. First consider the minimal solution v. Since any solution u of (7) is a recessive solution of (1), from Corollary 1 we have $u \in \mathbb{M}^-$. From (7) we obtain

$$v_n = \frac{u_n^{[1]}}{|u_n|^\alpha \operatorname{sgn} u_n}, \tag{23}$$

and so the sequence $\{v_n\}$ is negative. If C_2 holds, then by Corollary 1, $v \to 0-$. Using the same argument, we obtain the assertion of Theorem 3 for v also in the remaining cases C_3, C_4, C_6 and C_7.

Now consider any other solution w of (2) and let x be the solution of

$$\Delta x_n = \bigl(|w_n|/a_n\bigr)^{1/\alpha} x_n \operatorname{sgn} w_n, \quad x_N = 1.$$

Clearly, (3) holds. Moreover, x can be defined for $n \geqslant 1$ and we have

$$0 < \frac{x_{n+1}}{x_n} = 1 + \frac{\Delta x_n}{x_n}$$

and therefore

$$1 + \frac{|\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n}{|x_n|^\alpha \operatorname{sgn} x_n} > 0,$$

which implies $a_n + w_n > 0$ for large n. In view of the quoted result of [10], we have $v_n < w_n$ for large n, that is

$$\frac{u_n^{[1]}}{|u_n|^\alpha \operatorname{sgn} u_n} < \frac{x_n^{[1]}}{|x_n|^\alpha \operatorname{sgn} x_n} \quad \text{for large } n.$$

Hence, by [3, Theorem 4], x is a dominant solution of (1). Applying Theorem A and Corollary 1 we have

$$\begin{aligned} &x \in \mathbb{M}^+_{\infty,l} \quad \text{in the cases } C_2,\ C_5;\\ &x \in \mathbb{M}^+_l \quad \text{in the cases } C_3,\ C_8;\\ &x \in \mathbb{M}^+_l \cup \mathbb{M}^-_l \quad \text{in the case } C_4, \end{aligned}$$

and the assertion follows from Lemma 3. □

In some cases Theorem 3 does not describe the asymptotic behavior of v or w. The following result is related to the recent ones stated in [1] for $\alpha = 1$ and gives an answer under additional assumptions.

Theorem 4. *Assume that $\{a_n\}$ is bounded and $Y_b < \infty$. Let v be the minimal solution and w any other solution of* (2). *If v has a limit, then $v \to 0-$. Similarly, if w has a limit, then $w \to 0+$.*

Proof. Reasoning as in the proof of Theorem 3 we obtain that $a_n + v_n > 0$ for large n, i.e. $\{v_n\}$ is bounded from below. From Corollary 1 and (23), the sequence $\{v_n\}$ is negative. Hence $\lim_n v_n = c_w$, $0 \geqslant c_w > -\infty$. Assume, by contradiction, $c_v < 0$. By summation of (2) we have

$$v_n - v_{n_0} - \sum_{i=n_0}^{n} b_i = \sum_{i=n_0}^{n} \big(S(a_i, v_i) - 1\big) v_i,$$

which implies that the series

$$\sum_{i=n_0}^{\infty} \big(S(a_i, v_i) - 1\big) v_i$$

converges. Then $\lim_i \big(S(a_i, v_i) - 1\big) v_i = 0$ and so $\lim_i S(a_i, v_i) = 1$. Since for large n

$$|S(a_n, v_n)|^{1/\alpha} = \frac{1}{\big|1 + (|v_n|/a_n)^{1/\alpha} \operatorname{sgn} v_n\big|},$$

we obtain

$$\lim_n \frac{v_n}{a_n} = 0,$$

a contradiction.

It remains to prove that any other solution w, which has a limit, must tend to zero. Since $\{a_n\}$ is bounded, the case C_4 does not occur. In virtue of Corollary 1 and (3), the sequence $\{w_n\}$ is positive for large n. If $\lim_n w_n = \infty$, we have $\lim_n S(a_n, w_n) = 0$ and from

$$w_n - w_N + \sum_{i=N}^{n} \big(1 - S(a_i, w_i)\big) w_i - \sum_{i=N}^{n} b_i = 0$$

we obtain a contradiction as $n \to \infty$. Hence $\lim_n w_n = c_w \geqslant 0$ and the case $c_w > 0$ can be eliminated by the same argument as above. $\square$

Theorem 4 provides a partial answer concerning the asymptotic behavior of v in the case C_1 or C_5 ($\alpha > 1$). When C_8 ($\alpha < 1$) holds, Theorem 4 cannot be used, because in this case it is easy to show that $Y_a < \infty$ and $Y_b = \infty$.

As regards other solutions w of (2), a partial answer concerning $\lim_n w_n$ follows from Theorem 4 if C_1 or C_7 ($\alpha < 1$) holds. When C_6 ($\alpha > 1$) holds, Theorem 4 cannot be used, because we have $Y_a < \infty$ and $Y_b = \infty$.

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Authors' addresses: *Mariella Cecchi*, Depart. of Electronics and Telecomunications, University of Florence, Via S. Marta 3, 50139 Firenze, Italy, e-mail: mariella.cecchi@unifi.it; *Zuzana Došlá*, Depart. of Mathematics, Masaryk University, Janáčkovo nám. 2a, 602 00 Brno, Czech Republic, e-mail: dosla@math.muni.cz; *Mauro Marini*, Depart. of Electronics and Telecomunications, University of Florence, Via S. Marta 3, 50139 Firenze, Italy, e-mail: mauro.marini@unifi.it; *Ivo Vrkoč*, Mathematical Institute of the Academy of Sciences, Žitná 25, 115 67 Praha, Czech Republic, e-mail: vrkoc@math.cas.cz.