THE CONNECTION BETWEEN QUADRATIC FORMS AND THE EXTENDED MODULAR GROUP

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Abstract. In this paper some properties of quadratic forms whose base points lie in the point set $F_{\overline{\Pi}}$, the fundamental domain of the modular group, and transforming these forms into the reduced forms with the same discriminant $\Delta < 0$ are given.

Keywords: binary quadratic forms, reduced forms, extended modular group

MSC 2000: 11E25

1. Introduction

A real binary quadratic form F (or just a form) is a polynomial in two variables of the shape

$$F(X,Y) = aX^2 + bXY + cY^2$$

with real coefficients a, b, c. We denote F briefly by $[a,b,c]$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta(F)$. F is an integral form iff $a,b,c \in \mathbb{Z}$, and F is positive definite iff $\Delta(F) < 0$ and $a,c > 0$.

Let Π be the modular group $PSL(2,\mathbb{Z})$, i.e. the set of the transformations

$$S(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{Z},\ ad-bc = 1.$$

Π is generated by the transformations $T(z) = -1/z$ and $V(z) = z+1$; let $U = T \cdot V$. Then $U(z) = -1/(z+1)$. Then Π has a representation

$$\Pi = \left\langle T, U\colon\ T^2 = U^3 = I \right\rangle.$$

T and U are elliptic transformations and their fixed points in the upper halfplane are i and $\varrho = \mathrm{e}^{2\pi \mathrm{i}/3}$.

We denote the symmetry with respect to the imaginary axis by R, that is $R(z) = -\bar{z}$. Then the group $\overline{\Pi} = \Pi \cup R\Pi$ is generated by the transformations R, T and U, and has a representation

$$\overline{\Pi} = \left\langle R, T, U \colon\ R^2 = T^2 = U^3 = I \right\rangle .$$

$\overline{\Pi}$ is called the extended modular group and Π is a subgroup of index 2 in $\overline{\Pi}$. Therefore Π is a normal subgroup of $\overline{\Pi}$.

There is a strong connection between transformations and matrices. Throughout this paper, we identify each matrix A with $-A$ so that both represent the same transformation, and we use the matrix representation of transformations.

2. The connections between forms and $\overline{\Pi}$

In this section we examine the connections between forms and $\overline{\Pi}$. Now we give the relation of forms to the extended modular group. We define the form gF by the formula

$$gF(X,Y) = (ar^2+brs+cs^2)X^2+(2art+bru+bts+2csu)XY+(at^2+btu+cu^2)Y^2$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Pi}$ and $F(X,Y) = aX^2 + bXY + cY^2$.

This definition of gF is a group action on the set of forms, that is, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} F = F$ and $g(hF) = (gh)F$ for every $g, h \in \overline{\Pi}$. Furthermore, $\Delta(F) = \Delta(gF)$ for $g \in \overline{\Pi}$, that is, the action of $\overline{\Pi}$ on the set of forms leaves the discriminant invariant. If F is positive definite or integral then so is gF for all $g \in \overline{\Pi}$.

Let $\mathbb{U} = \{z \in \mathbb{C} \colon\ \mathrm{Im}\,(z) > 0\}$ be the complex upper halfplane. For a positive definite form $F = [a,b,c]$ with discriminant $\Delta(F) < 0$, there exists a unique $z = z(F) \in \mathbb{U}$ such that

$$F = a(X + zY)(X + \bar{z}Y).$$

We call z the *base point* of F in $\mathbb{U}$. Indeed, for $z = x + \mathrm{i}y$ we have

$$F = a(X + zY)(X + \bar{z}Y) = aX^2 + 2axXY + a|z|^2Y^2.$$

Then we obtain $2ax = b$ and $a|z|^2 = c$, so $x = \frac{1}{2}b/a$ and $y = \frac{1}{2}\sqrt{-\Delta(F)}/a$. Since y is positive,

$$z = \frac{b + \mathrm{i}\sqrt{-\Delta(F)}}{2a} \in \mathbb{U}.$$

Conversely, for a given point $z \in \mathbb{U}$ there exists a positive definite quadratic form $F = [a, b, c]$ whose base point is z. For $z = x + \mathrm{i}y$ let $a = 1/|z|^2$, $b = 2x/|z|^2$ and $c = 1$. Then we have the positive definite quadratic form

$$F = [a, b, c] = \left[\frac{1}{|z|^2}, \frac{2x}{|z|^2}, 1\right] \tag{2.1}$$

with discriminant $\Delta(F) = -4y^2/|z|^4 < 0$ and its base point is z.

So the map $\Phi\colon F \to z(F)$ is a bijection between the set of positive definite forms with a fixed discriminant and the points of $\mathbb{U}$. There is a relation between positive definite forms with the same discriminant and their base points in $\mathbb{U}$.

Let F and G be two positive definite forms with the same discriminant. Then F and G are properly equivalent iff there exists a $g \in \overline{\Pi}$ such that $gF = G$. Moreover, F and G are properly equivalent iff $z(F)$ and $z(G)$ are in the same orbit in $\overline{\Pi}$. The proper equivalence classes of forms are just the orbits of the action of $\overline{\Pi}$ on the set of forms.

Let $F = [a, b, c]$ be a positive definite form with discriminant $\Delta(F)$. Then F is said to be *almost reduced* if $|b| \leqslant a \leqslant c$. We denote the set of almost reduced forms with discriminant Δ by $\underline{h}(\Delta)$. Flath showed in [2] that the number of almost reduced forms with given discriminant $\Delta < 0$ is finite, i.e. $\underline{h}(\Delta)$ is finite.

Any positive definite integral form can be transformed into an almost reduced one by an element of $\overline{\Pi}$. Let us prove that a positive definite integral form $F = aX^2 + bXY + cY^2$ that does not satisfy $|b| \leqslant a \leqslant c$ can be modified within its proper equivalence class as follows. If $c < a$, permute X and Y by replacing F by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F = cX^2 - bXY + aY^2$. If $|b| > a$, replace F by

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} F = aX^2 + (b + 2an)XY + (an^2 + bn + c)Y^2$$

where $n \in \mathbb{Z}$ is chosen such that $|b + 2an| \leqslant a$. By alternating these two modification procedures we are led to a sequence of forms $F_n = a_nX^2 + b_nXY + c_nY^2$ such that $a_n \geqslant a_{n+1}$ and $a_n > a_{n+2}$. Since all of the a_n's are positive integers, the sequence must stop. This can only happen when $|b| \leqslant a \leqslant c$ as desired, i.e. every positive definite integral form is equivalent to an almost reduced form.

For any given positive definite quadratic form $F = [a, b, c]$ we know that F can be writtten as $F = a(X + zY)(X + \bar{z}Y)$ for some complex number z. We may assume that $\mathrm{Im}\,(z) > 0$ since z and $\bar{z}$ play symmetric roles. The condition $|b| \leqslant a$ is equivalent to $|z + \bar{z}| \leqslant 1$, that is $|\mathrm{Re}\,(z)| \leqslant 1/2$. The condition $a \leqslant c$ translates to $z\bar{z} \geqslant 1$, that is $|z| \geqslant 1$. In other words, the form $F = [a, b, c]$ is almost reduced when z lies in the region pictured in Figure 1, which is the fundamental region of Π.

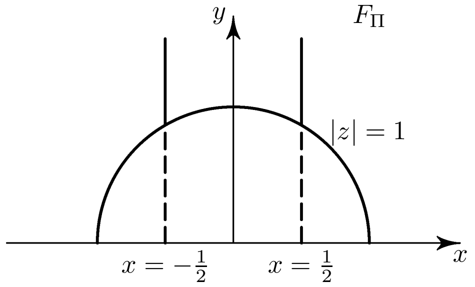

Figure 1

We denote this region by F_{Π}. The transformation R is a symmetry with respect to the imaginary axis, so the points on the right side of the imaginary axis are equivalent to the points on the left side. Hence the positive definite quadratic form F is said to be *reduced* if z lies in the region pictured in Figure 2. This region is the fundamental region of $\overline{\Pi}$, we denote it by $F_{\overline{\Pi}}$ and call it the *fundamental region* of the form.

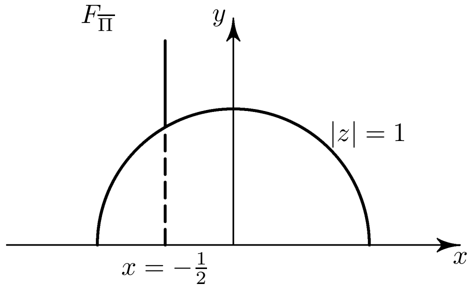

Figure 2

This definition has been made so that there exists a unique reduced form in each equivalence class of $\overline{\Pi}$.

Now we will give some properties of positive definite forms whose base points lie in some point set in $F_{\overline{\Pi}}$.

Theorem 2.1. *For $m \geqslant 2$ consider the line $x = -1/m$. Then there exists an integral positive definite quadratic form $F = [a, b, c]$ with discriminant $\Delta(F) = -D$, where $0 < D < m^2$, whose base point $z(F)$ lies on the line $x = -1/m$.*

Proof. We know that for a given point $z = x + iy$ in $\mathbb{U}$ there exists a positive definite quadratic form $F = [a, b, c] = [1/|z|^2, 2x/|z|^2, 1]$ with discriminant $\Delta(F) = -4y^2/|z|^4$. So we have $y = m + \sqrt{m^2 - D}/m\sqrt{D}$ from the equation $-4y^2/|z|^4 = -D$ for $x = -1/m$. Thus we have the positive definite quadratic form

$$F = [a, b, c] = \left[\frac{mD}{2\left(m + \sqrt{m^2 - D}\right)},\ \frac{-D}{m + \sqrt{m^2 - D}}, 1\right] \tag{2.2}$$

with discriminant $\Delta(F) = -D$ whose base point is $z = x + iy \in \mathbb{U}$.

Now we want the form F in (2.2) to be an integral form. To achieve this we have two cases:

Case 1. If m is odd, then D is even. In this case let $m = 2k + 1$ for $k \in \mathbb{Z}^+$. Then F is an integral form, i.e. $a, b \in \mathbb{Z}$ if and only if $D = m^2 - (2l - 1)^2$, $1 \leqslant l \leqslant k$. Let $a, b \in \mathbb{Z}$. We want to determine D numbers such that $a, b \in \mathbb{Z}$. Since $\sqrt{m^2 - D}$ is odd we have $\sqrt{m^2 - D} = 2l - 1$ for $l \in \mathbb{Z}$. Then $l \geqslant 1$, $m^2 - D = (2l - 1)^2$ and thus $D = m^2 - (2l - 1)^2$. Since D is positive, $m^2 - (2l - 1)^2 = (m - (2l - 1))(m + (2l - 1))$ must be positive. Since $l \geqslant 1$, we have $m + (2l - 1)$ is positive. Therefore $m - (2l - 1)$ must be positive. This shows that $k + 1 > l$. Thus we obtain $D = m^2 - (2l - 1)^2$ for $1 \leqslant l \leqslant k$. Conversely, let $D = m^2 - (2l - 1)^2$ for $1 \leqslant l \leqslant k$. Then we have

$$a = \frac{mD}{2\left(m + \sqrt{m^2 - D}\right)} = \frac{m\left(m^2 - (2l - 1)^2\right)}{2\left(m + (2l - 1)\right)} = \frac{m\left(m - (2l - 1)\right)}{2} \in \mathbb{Z}$$

since $m - (2l - 1)$ is even, and similarly

$$b = \frac{-D}{m + \sqrt{m^2 - D}} = \frac{-\left(m^2 - (2l - 1)^2\right)}{m + (2l - 1)} = -m + (2l - 1) \in \mathbb{Z}.$$

Case 2. Let m be even, say $m = 2k$ for $k \in \mathbb{Z}$. Then F is an integral form, i.e. $a, b \in \mathbb{Z}$ if and only if $D = m^2 - t^2$ for $1 \leqslant t \leqslant m - 1$. Let $a, b \in \mathbb{Z}$. We will determine D numbers such that $a, b \in \mathbb{Z}$. Let $\sqrt{m^2 - D} = t$ for $t \in \mathbb{Z}^+$ since $m^2 - D > 0$. Then $D = m^2 - t^2$. Since D is positive, $m^2 - t^2 = (m - t)(m + t)$ must be positive. Since $m + t$ is positive, $m - t$ must be positive. Thus we have $m > t$, i.e. $t \leqslant m - 1$. Therefore $D = m^2 - t^2$ for $1 \leqslant t \leqslant m - 1$. Conversely, let $D = m^2 - t^2$ for $1 \leqslant t \leqslant m - 1$. Then we have

$$a = \frac{mD}{2\left(m + \sqrt{m^2 - D}\right)} = \frac{m\left(m^2 - t^2\right)}{2\left(m + t\right)} = \frac{m\left(m - t\right)}{2} \in \mathbb{Z}$$

since m is even and

$$b = \frac{-D}{m + \sqrt{m^2 - D}} = \frac{-(m^2 - t^2)}{m + t} = -(m - t) \in \mathbb{Z}.$$

□

From Theorem 2.1 we obtain

Corollary 2.2. 1. *If m is odd, say $m = 2k+1$ for $k \in \mathbb{Z}^+$, then there exist k positive definite integral forms of the type*

$$F_j = [mj, -2j, 1], \ 1 \leqslant j \leqslant k$$

with discriminant $\Delta(F_j) = -D_j = -4j(m-j)$ whose base points $z(F_j)$ lie on the line $x = -1/m$.

2. *If m is even, say $m = 2k$ for $k \in \mathbb{Z}^+$, then there exist $m-1$ positive definite integral forms of the type*

$$F_j = [kj, -j, 1], \ 1 \leqslant j \leqslant m-1$$

with discriminant $\Delta(F_j) = -D_j = -j(2m-j)$ whose base points $z(F_j)$ lie on the line $x = -1/m$.

Example 2.1. Let $m = 7$. In this case there exist three positive definite integral forms which are

$$\begin{aligned} F_1 &= [7, -2, 1] \ \text{ with discriminant } \Delta(F_1) = -24, \\ F_2 &= [14, -4, 1] \ \text{ with discriminant } \Delta(F_2) = -40, \\ F_3 &= [21, -6, 1] \ \text{ with discriminant } \Delta(F_3) = -48. \end{aligned}$$

Now we want to find out which positive definite integral forms of the type given in Corollary 2.2. are reduced, i.e. whose base points lie in $F_{\overline{\Pi}}$.

Corollary 2.3. *There are two positive definite integral forms of the type* (2.2) *whose base points lie in $\Gamma_{\overline{\Pi}}$.*

Proof. For $m = 2$, we have the positive definite integral form $F_2 = [1, -1, 1]$ with discriminant $\Delta(F_2) = -3$. The base point of F_2 is $z(F_2) = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$; it lies on the line $x = -1/2$ which is in $\Gamma_{\overline{\Pi}}$. Similarly for $m = \infty$, we have the positive definite integral form $F_\infty = [1, 0, 1]$ with discriminant $\Delta(F_\infty) = -4$. The base point of F_∞ is $z(F_\infty) = \frac{1}{2}\mathrm{i}\sqrt{4} = \mathrm{i}$; it lies on the line $x = 0$ which is in $\Gamma_{\overline{\Pi}}$. $\square$

The positive definite integral forms we have obtained in Corollary 2.2 are not reduced except for F_2 and F_∞. We know that every positive definite integral form can be transformed into a reduced form with the same discriminant by an element $g \in \overline{\Pi}$.

Theorem 2.4. *Let $F = [a, b, c]$ be a positive definite integral form of the type (2.2). Then there exists a $g \in \overline{\Pi}$ such that $gF = F_R$, where F_R is the reduced form with the same discriminant.*

Proof. Consider two cases.

Case 1. If m is odd, say $m = 2k+1$ for $k \in \mathbb{Z}^+$, then we know from Corollary 2.2 that there exist k-integral forms of the type $F_j = [mj, -2j, 1], 1 \leqslant j \leqslant k$ with discriminant $\Delta(F_j) = -D_j$. One of the reduced forms with discriminant $\Delta(F_{Rj}) = -D_j$ is of the type

$$F_{Rj} = \left[1, 0, \frac{D_j}{4}\right].$$

We want to find a $g_j \in \overline{\Pi}$ such that $g_j F_j = F_{Rj}$. From the definition of gF, we have the system of equations

$$\begin{aligned} mjr^2 - 2jrs + s^2 &= 1, \\ 2mjrt - 2jru - 2jts + 2su &= 0, \\ mjt^2 - 2jtu + u^2 &= \frac{D_j}{4} \end{aligned}$$

for $F_j = [mj, -2j, 1]$ and $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Pi}$. This system has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = j$. So $g_j F_j = F_{Rj}$ for $g_j = \begin{pmatrix} 0 & 1 \\ 1 & j \end{pmatrix}$.

Case 2. If m is even, say $m = 2k$ for $k \in \mathbb{Z}^+$, then by Corollary 2.2 there exist $m - 1$ integral forms of the type $F_j = [kj, -j, 1]$, $1 \leqslant j \leqslant m - 1$ with discriminant $\Delta_j = -D_j$. In this case there are two possibilities:

(a) If j is odd, then one of the reduced forms with discriminant $\Delta(F_{Rj}) = -D_j$ is of the type

$$F_{Rj} = \left[1, 1, \frac{1 + D_j}{4}\right].$$

Let us consider the system of equations

$$\begin{aligned} kjr^2 - jrs + s^2 &= 1, \\ 2kjrt - jru - jts + 2su &= 1, \\ kjt^2 - jtu + u^2 &= \frac{1 + D_j}{4}. \end{aligned}$$

This system has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = \frac{1}{2}(j + 1)$. Therefore $g_j F_j = F_{Rj}$ for $g_j = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{2}(j+1) \end{pmatrix}$.

(b) If j is even, then one of the reduced forms with discriminant $\Delta(F_{Rj}) = -D_j$ is of the type

$$F_{Rj} = \left[1, 0, \frac{D_j}{4}\right].$$

Thus we have the system of equations

$$\begin{aligned} kjr^2 - jrs + s^2 &= 1, \\ 2kjrt - jru - jts + 2su &= 0, \\ kjt^2 - jtu + u^2 &= \frac{D_j}{4}. \end{aligned}$$

This system has a solution for $r = 0$, $s = -1$, $t = 1$ and $u = \frac{1}{2}j$. Therefore $g_j F_j = F_{Rj}$ for $g_j = \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{2}j \end{pmatrix}$. □

Example 2.2. Let $m = 7$. Then there exist three positive definite integral forms which are

$$\begin{aligned} F_1 &= [7, -2, 1] \text{ with discriminant } \Delta(F_1) = -24, \\ F_2 &= [14, -4, 1] \text{ with discriminant } \Delta(F_2) = -40, \\ F_3 &= [21, -6, 1] \text{ with discriminant } \Delta(F_3) = -48 \end{aligned}$$

and the reduced forms are

$$\begin{aligned} F_{R1} &= [1, 0, 6] \text{ with discriminant } \Delta(F_{R1}) = -24, \\ F_{R2} &= [1, 0, 10] \text{ with discriminant } \Delta(F_{R2}) = -40, \\ F_{R3} &= [1, 0, 12] \text{ with discriminant } \Delta(F_{R3}) = -48. \end{aligned}$$

Thus we have

$$\begin{aligned} g_1 F_1 = F_{R1} &\text{ for } g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ g_2 F_2 = F_{R2} &\text{ for } g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \\ g_3 F_3 = F_{R3} &\text{ for } g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}. \end{aligned}$$

Let $m = 8$. Then there exist seven positive definite integral forms. When j is odd, the integral forms are

$$\begin{aligned} F_1 &= [4, -1, 1] \text{ with discriminant } \Delta(F_1) = -15, \\ F_3 &= [12, -3, 1] \text{ with discriminant } \Delta(F_3) = -39, \\ F_5 &= [20, -5, 1] \text{ with discriminant } \Delta(F_5) = -55, \\ F_7 &= [28, -7, 1] \text{ with discriminant } \Delta(F_7) = -63 \end{aligned}$$

and the reduced forms are

$$\begin{aligned}
F_{R1} &= [1,1,4] \ \text{ with discriminant } \Delta(F_{R1}) = -15,\\
F_{R3} &= [1,1,10] \ \text{ with discriminant } \Delta(F_{R3}) = -39,\\
F_{R5} &= [1,1,14] \ \text{ with discriminant } \Delta(F_{R5}) = -55,\\
F_{R7} &= [1,1,16] \ \text{ with discriminant } \Delta(F_{R7}) = -63.
\end{aligned}$$

So

$$\begin{aligned}
g_1F_1 = F_{R1} \ \text{ for } g_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\\
g_3F_3 = F_{R3} \ \text{ for } g_3 &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},\\
g_5F_5 = F_{R5} \ \text{ for } g_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix},\\
g_7F_7 = F_{R7} \ \text{ for } g_7 &= \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}.
\end{aligned}$$

When j is even the integral forms are

$$\begin{aligned}
F_2 &= [8,-2,1] \ \text{ with discriminant } \Delta(F_2) = -28,\\
F_4 &= [16,-4,1] \ \text{ with discriminant } \Delta(F_4) = -48,\\
F_6 &= [24,-6,1] \ \text{ with discriminant } \Delta(F_6) = -60
\end{aligned}$$

and the reduced forms are

$$\begin{aligned}
F_{R2} &= [1,0,7] \ \text{ with discriminant } \ \Delta(F_{R2}) = -28,\\
F_{R4} &= [1,0,12] \ \text{ with discriminant } \Delta(F_{R4}) = -48,\\
F_{R6} &= [1,0,15] \ \text{ with discriminant } \Delta(F_{R6}) = -60.
\end{aligned}$$

So

$$\begin{aligned}
g_2F_2 = F_{R2} \ \text{ for } g_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},\\
g_4F_4 = F_{R4} \ \text{ for } g_4 &= \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix},\\
g_6F_6 = F_{R6} \ \text{ for } g_6 &= \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}.
\end{aligned}$$

Now we consider a positive definite quadratic form whose base points lie on a circle centered at $(0,0)$.

Theorem 2.5. *For $m \geqslant 1$ consider the circle $C\colon\ x^2+y^2=1/m^2$. Let $\widetilde{C}=C\cap\mathbb{U}$ and $0<D<4m^2$. Then there exist two types of positive definite quadratic forms, namely $F=[a,-b,c]$ and $G=[a,b,c]$ with the same discriminant $\Delta(F)=\Delta(G)=-D$ whose base points $z(F)$ and $z(G)$ lie on $\widetilde{C}$.*

Proof. We know that for any given number $z\in\mathbb{U}$ there exists a form $F=[a,b,c]=[1/|z|^2, 2x/|z|^2, 1]$ with discriminant $\Delta(F)=-4y^2/|z|^4<0$ whose base point is z. For $z=x+\mathrm{i}y$ the circle C is the circle $|z|=1/m$. Thus we have $y=\frac{1}{2}\sqrt{D}/m^2$ from the equation $-4y^2/|z|^4=-D$ and $x=\pm\frac{1}{2}\sqrt{4m^2-D}/m^2$ for $|z|=1/m$. Therefore we have the following two types of positive definite quadratic forms:

$$\begin{aligned} F&=[a,-b,c]=[m^2,-\sqrt{4m^2-D},1],\\ G&=[a,b,c]=[m^2,\sqrt{4m^2-D},1]\end{aligned}\tag{2.3}$$

with discriminant $\Delta(F)=\Delta(G)=-D$ whose points $z(F)$ and $z(G)$ lie on $\widetilde{C}$.

We want the quadratic forms F and G to be of integral form. To get this let $b=\sqrt{4m^2-D}$. Then $b\in\mathbb{Z}$, i.e. F and G are integral forms iff $D=4m^2-t^2$ for $1\leqslant t\leqslant 2m-1$. Let $b\in\mathbb{Z}$. Then $\sqrt{4m^2-D}\in\mathbb{Z}$. Since $4m^2-D$ is positive, $\sqrt{4m^2-D}$ must be positive. Let $\sqrt{4m^2-D}=t$ for $t\in\mathbb{Z}^+$. Then $4m^2-D=t^2$ and thus $D=4m^2-t^2$. Since D is positive, $D=(2m-t)(2m+t)$ must be positive. Since m and t are positive, $2m+t$ is positive. Therefore $m-t$ must be positive. Hence $t<2m$. Thus we get $D=4m^2-t^2$ for $1\leqslant t\leqslant 2m-1$. Conversely, let $D=4m^2-t^2$. Then $b=\sqrt{4m^2-D}=t\in\mathbb{Z}$. □

From the above theorem we have the following corollary.

Corollary 2.6. *For $m\geqslant 1$ there exist two types of positive definite integral forms of the type $F_j=[m^2,-j,1]$ and $G_j=[m^2,j,1]$ with the same discriminant $\Delta(F_j)=\Delta(G_j)=j^2-4m^2$, $1\leqslant j\leqslant 2m-1$ whose base points $z(F_j)$ and $z(G_j)$ lie on $\widetilde{C}$.*

Example 2.3. Let $m=3$. Then $C\colon\ x^2+y^2=1/g$ and $\widetilde{C}=C\cap\mathbb{U}$. Therefore there exist five positive definite integral forms of the types

$$\begin{aligned}
&F_j=[9,-1,1] \text{ and } G_1=[9,1,1] \text{ with discriminant } \Delta(F_1)=\Delta(G_1)=-35,\\
&F_2=[9,-2,1] \text{ and } G_2=[9,2,1] \text{ with discriminant } \Delta(F_2)=\Delta(G_2)=-32,\\
&F_3=[9,-3,1] \text{ and } G_3=[9,3,1] \text{ with discriminant } \Delta(F_3)=\Delta(G_3)=-27,\\
&F_4=[9,-4,1] \text{ and } G_4=[9,4,1] \text{ with discriminant } \Delta(F_4)=\Delta(G_4)=-20,\\
&F_5=[9,-5,1] \text{ and } G_5=[9,5,1] \text{ with discriminant } \Delta(F_5)=\Delta(G_5)=-11
\end{aligned}$$

whose base points lie on $\widetilde{C}$.

The transformation $R(z) = -\bar{z}$ is a reflection in the imaginary axis. So the base points $z(G_j)$ of G_j on the right side of the imaginary axis lie on $\widetilde{C}$ are equivalent to base points $z(F_j)$ of F_j on the left side of the imaginary axis lie on $\widetilde{C}$. We know that two positive definite quadratic forms of the same discriminant F and G are equivalent iff $z(F)$ and $z(G)$ are in the same orbit of $\overline{\Pi}$. Therefore $R(z(F_j)) = z(G_j)$ for $R(z) = -\bar{z} \in \overline{\Pi}$. Moreover, $g_j F_j = G_j$ for $g_j = \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} \in \overline{\Pi}$.

Hence we have proved the following theorem.

Theorem 2.7. *F_j and G_j are equivalent.*

For $m = 1$ consider the positive definite integral forms $F_1 = [1, -1, 1]$ and $G_1 = [1, 1, 1]$ with discriminant $\Delta(F_1) = \Delta(G_1) = -3$. The base point of F_1 is $z(F_1) = \frac{1}{2}(-1 + \mathrm{i}\sqrt{3})$ in $\Gamma_{\overline{\Pi}}$. Therefore F_1 is reduced. The base point of G_1 is $z(G_1) = \frac{1}{2}(1 + \mathrm{i}\sqrt{3}$ in Γ_{Π}. Therefore G_1 is almost reduced.

For $m > 1$, consider the positive definite integral forms $F_j = [m^2, -j, 1]$ and $G_j = [m^2, j, 1]$ with discriminant $\Delta(F_j) = \Delta(G_j) = -(4m^2 - j^2)$. The base point of F_j is $z(F_j) = (-j + \mathrm{i}\sqrt{4m^2 - j^2})/2m^2 \notin \Gamma_{\overline{\Pi}}$. Similarly the base point of G_j is $z(G_j) = (j + \mathrm{i}\sqrt{4m^2 - j^2})/2m^2 \notin \Gamma_{\overline{\Pi}}$. Therefore F_j and G_j are not reduced.

Hence we have proved the following theorem.

Theorem 2.8. *There exists a unique positive definite integral form of the type* (2.3) *whose base point lies in $\Gamma_{\overline{\Pi}}$ which is $F_1 = [1, -1, 1]$ with discriminant $\Delta(F_1) = -3$ for $m = 1$.*

Similarly we can transform quadratic forms F_j and G_j of the type (2.3) into reduced forms. To get this we only show that F_j can be transformed into a reduced form, since F_j and G_j are equivalent.

Theorem 2.9. *Let F_j be a quadratic form of the type* (2.3). *Then there exists a $g \in \overline{\Pi}$ such that $gF_j = F_{Rj}$.*

P r o o f. When j is odd, the reduced forms with discriminant $-D_j$ are of the type

$$F_{Rj} = \left(1, 1, \frac{1 + D_j}{4}\right).$$

The system of equations

$$\begin{aligned} m^2 r^2 - jrs + s^2 &= 1, \\ 2m^2 rt - jru - jts + 2su &= 1, \\ m^2 t^2 - jtu + u^2 &= \frac{1 + D_j}{4} \end{aligned}$$

has a solution for $r = 0$, $s = 1$, $t = 1$ and $u = \frac{1}{2}(j+1)$. So $g_j F_j = F_{Rj}$ for $g_j = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{2}(j+1) \end{pmatrix} \in \overline{\Pi}$.

When j is even the reduced forms with discriminant $-D_j$ are of the type

$$F_{Rj} = \left(1, 0, \frac{D_j}{4}\right).$$

The system of equations

$$\begin{aligned} m^2r^2 - jrs + s^2 &= 1, \\ 2m^2rt - jru - jts + 2su &= 0, \\ m^2t^2 - jtu + u^2 &= \frac{D_j}{4} \end{aligned}$$

has a solution for $r = 0$, $s = -1$, $t = 1$ and $u = \frac{1}{2}j$. So $g_j F_j = F_{Rj}$ for $g_j = \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{2}j \end{pmatrix} \in \overline{\Pi}$. □

E x a m p l e 2.4. Let $m = 3$, then we have the non-reduced forms

$$F_1 = [9, -1, 1],\ F_2 = [9, -2, 1],\ F_3 = [9, -3, 1],\ F_4 = [9, -4, 1],\ F_5 = [9, -5, 1]$$

and reduced forms

$$F_{R1} = [1, 1, 9],\ F_{R2} = [1, 0, 8],\ F_{R3} = [1, 1, 7],\ F_{R4} = [1, 0, 5],\ F_{R5} = [1, 1, 3].$$

When j is odd we have

$$\begin{aligned} g_1F_1 &= F_{R1} \quad \text{for } g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ g_3F_3 &= F_{R3} \quad \text{for } g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \\ g_5F_5 &= F_{R5} \quad \text{for } g_5 = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \end{aligned}$$

and when j is even we have

$$\begin{aligned} g_2F_2 &= F_{R2} \quad \text{for } g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \\ g_4F_4 &= F_{R4} \quad \text{for } g_4 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

[1] *Coxeter, H. S. M., Moser W. O. J.*: Generators and Relations for Discrete Groups. Springer, Berlin, 1957.

[2] *Flath, D. E.*: Introduction to Number Theory. Wiley, 1989.

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