SINGULAR DIRICHLET PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS WITH ϕ-LAPLACIAN

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Abstract. We provide sufficient conditions for solvability of a singular Dirichlet boundary value problem with ϕ-Laplacian

$$(\phi(u'))' = f(t, u, u'),$$
$$u(0) = A,\ u(T) = B,$$

where ϕ is an increasing homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$, f satisfies the Carathéodory conditions on each set $[a, b] \times \mathbb{R}^2$ with $[a, b] \subset (0, T)$ and f is not integrable on $[0, T]$ for some fixed values of its phase variables. We prove the existence of a solution which has continuous first derivative on $[0, T]$.

Keywords: singular Dirichlet problem, ϕ-Laplacian, existence of smooth solution, lower and upper functions

MSC 2000: 34B16, 34B15

1. Formulation of the problem

In a certain problem in fluid dynamics and boundary layer theory ([1], [2]), the generalized Emden-Fowler differential equation $u'' + \psi(t)u^\lambda = 0$, $t \in (0, 1)$, arises. This equation is singular, because ψ need not be Lebesgue integrable on the whole interval $[0, 1]$.

On the other hand, due to various applications, for example to diffusions of flows in porous media ([3], [4]), several authors have proposed the study of second order ordinary differential equations with the p-Laplacian $(\phi_p(u'))'$, where $p \in (1, \infty)$ and $\phi_p(y) = |y|^{p-2}y$ for $y \in \mathbb{R}$. Usually the p-Laplacian is replaced by its abstract and more general version called a ϕ-Laplacian.

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Throughout the paper ϕ will be an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$, $\phi(0) = 0$ and $A, B \in \mathbb{R}$, $[0,T] \subset \mathbb{R}$. We study the problem of existence of smooth solutions of the Dirichlet problem with ϕ-Laplacian

$$(\phi(u'))' = f(t,u,u'), \tag{1.1}$$

$$u(0) = A, \ u(T) = B, \tag{1.2}$$

where f satisfies the Carathéodory conditions on each set $[a,b] \times \mathbb{R}^2$, where $[a,b] \subset (0,T)$, but f does not satisfy the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$.

Recall that a real valued function f satisfies the Carathéodory conditions on the set $[a,b] \times \mathbb{R}^2$ if

(i) $f(\cdot,x,y)\colon [a,b] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathbb{R}^2$,
(ii) $f(t,\cdot,\cdot)\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,
(iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K \in L[a,b]$ such that $|f(t,x,y)| \leqslant m_K(t)$ for a.e. $t \in [a,b]$ and all $(x,y) \in K$.

We write $f \in \mathrm{Car}([a,b] \times \mathbb{R}^2)$. By the assumption $f \notin \mathrm{Car}([0,T] \times \mathbb{R}^2)$ we mean that condition (iii) is not fulfilled for $[a,b] = [0,T]$, i.e. that f has time singularities at the endpoints 0 and T.

Definition 1.1. We say that f has *time singularities at the points* 0 *and* T, respectively, if there exist $x, y \in \mathbb{R}$ such that

$$\int_0^{\varepsilon} |f(t,x,y)| \,\mathrm{d}t = \infty \quad \text{and} \quad \int_{T-\varepsilon}^{T} |f(t,x,y)| \,\mathrm{d}t = \infty \tag{1.3}$$

for each sufficiently small $\varepsilon > 0$. The points 0 and T are called *singular points of* f.

We will seek solutions of problem (1.1), (1.2) in the space of functions having continuous first derivatives on $[0,T]$, in particular at the singular points 0 and T.

Definition 1.2. A function $u\colon [0,T] \to \mathbb{R}$ with $\phi(u') \in \mathrm{AC}[0,T]$ is called a *solution* of problem (1.1), (1.2) if u satisfies

$$(\phi(u'(t)))' = f(t,u(t),u'(t))$$

for a.e. $t \in [0,T]$ and fulfils (1.2).

Note that the condition $\phi(u') \in \mathrm{AC}[0,T]$ implies $u \in C^1[0,T]$.

Majority of papers dealing with time singularities use an alternative approach to the solvability of problem (1.1), (1.2) (see [5]–[15]). These papers understand solutions as functions whose first derivatives need not exist at singular points. Here we will call them w-solutions. More precisely:

Definition 1.3. A function $u \in C[0,T]$ is called a *w-solution* of problem (1.1), (1.2) if $\phi(u') \in \mathrm{AC}_{\mathrm{loc}}(0,T)$, u satisfies

$$(\phi(u'(t)))' = f(t, u(t), u'(t))$$

for a.e. $t \in [0,T]$ and fulfils (1.2).

Since the condition $\phi(u') \in \mathrm{AC}_{\mathrm{loc}}(0,T)$ implies that a w-solution u belongs only to $C^1(0,T)$, we do not know the behaviour of u' at the singular endpoints 0, T.

Although most of the known existence results concern w-solutions, we often need the existence of solutions in the sense of Definition 1.2. For example, when searching for positive, radially symmetric solutions to the partial differential equation

$$-\Delta_p u = f(u) + h(x) \tag{1.4}$$

on an open ball Ω in $\mathbb{R}^n$ (centered at the origin), $n \geqslant 1$, where the n-dimensional p-Laplacian has the form

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

equation (1.4) reduces under the assumption $u'(0) = 0$ to the singular ordinary differential equation

$$(|u'|^{p-2}u')' + \frac{n-1}{t}|u'|^{p-2}u' + f(u) + h(t) = 0 \tag{1.5}$$

with a time singularity at $t = 0$. We can see that only solutions of (1.5) belonging to $C^1[0,T]$ have sense for the associated equation (1.4).

Further, we can check that the function $u(t) = 1 - t^2$ is a solution of the singular problem

$$\frac{1}{2}((u')^3)' + \frac{1}{t}u' + \frac{12t^2\sqrt{u}}{\sqrt{1-t^2}} + 2 = 0, \qquad u(0) = 1, \quad u(1) = 0. \tag{1.6}$$

We see that the function $f(t,x,y) = y/t + 12t^2\sqrt{x}/\sqrt{1-t^2} + 2$ has time singularities at $t = 0$ and $t = 1$ and problem (1.6) has at least one solution $u \in C^1[0,1]$.

In addition, numerical computations ([16], [17]) lead to smooth solutions of singular Dirichlet problems, as well.

Motivated by these facts we provide new existence principles which lead to sufficient conditions guaranteeing the existence of a solution of the singular problem (1.1), (1.2) in the sense of Definition 1.2.

Singular problems are usually investigated by means of auxiliary regular problems. To establish the existence of a solution of the singular problem (1.1), (1.2) we introduce a sequence of approximating regular problems which are solvable. Then we pass to the limit in the sequence of approximate solutions to get a solution (a w-solution) of the original problem (1.1), (1.2). In the next theorem we provide an existence principle which contains the main rules for the construction of such approximating sequences.

For $n \in \mathbb{N}$ consider equations

$$(\phi(u'))' = f_n(t,u,u'), \tag{2.1}$$

where $f_n \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$. Solutions of problem (2.1), (1.2) are understood in the sense of Definition 1.2. Denote

$$J_n = \begin{cases} [1/n, T-1/n] & \text{if } n \geqslant 3/T, \\ \emptyset & \text{if } n < 3/T. \end{cases} \tag{2.2}$$

Theorem 2.1. *Assume that*

(2.3) $f \in \mathrm{Car}((0,T) \times \mathbb{R}^2)$ *has time singularities at* $t=0$ *and* $t=T$,

(2.4) *for all* $n \in \mathbb{N}$, $f_n(t,x,y) = f(t,x,y)$ *for a.e.* $t \in J_n$ *and all* $x,y \in \mathbb{R}$,

(2.5) *there exists a bounded set* $\Omega \subset C^1[0,T]$ *such that for each* $n \in \mathbb{N}$ *the regular problem* (2.1), (1.2) *has a solution* $u_n \in \Omega$.

Then

- *there exist* $u \in C[0,T] \cap C^1(0,T)$ *and a subsequence* $\{u_{n_l}\} \subset \{u_n\}$ *such that* $\lim\limits_{l\to\infty} \|u_{n_l} - u\|_{C[0,T]} = 0$, $\lim\limits_{l\to\infty} u'_{n_l}(t) = u'(t)$ *locally uniformly on* $(0,T)$,
- u *is a* w*-solution of* (1.1), (1.2).

Moreover, assume that there exist $\eta \in (0, \frac{1}{2}T)$, $\lambda_1, \lambda_2 \in \{-1,1\}$, $y_1, y_2 \in \mathbb{R}$ *and* $\psi \in L[0,T]$ *such that for each* $n \in \mathbb{N}$

$$\begin{cases} \lambda_1 \operatorname{sign}(u_n'(t) - y_1) f_n(t, u_n(t), u_n'(t)) \geqslant \psi(t) & \text{a.e. on } (0,\eta), \\ \lambda_2 \operatorname{sign}(u_n'(t) - y_2) f_n(t, u_n(t), u_n'(t)) \geqslant \psi(t) & \text{a.e. on } (T-\eta,T). \end{cases} \tag{2.6}$$

Then u *is a solution of* (1.1), (1.2).

P r o o f. By (2.5) there exists $c = c(\Omega) > 0$ such that

$$\|u_n\|_{C[0,T]} \leqslant c, \quad \|u_n'\|_{C[0,T]} \leqslant c \quad \forall n \in \mathbb{N},$$

so the sequence $\{u_n\}$ is bounded in $C^1[0,T]$ and

$$\begin{aligned}&\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall n \in \mathbb{N} \ \forall t, s \in [0,T]\colon\\ &|t-s| < \delta \Rightarrow |u_n(t) - u_n(s)| = \left| \int_s^t u_n'(\tau)\,\mathrm{d}\tau \right| \leqslant c|t-s|,\end{aligned}$$

which means that the sequence $\{u_n\}$ is equicontinuous on $[0,T]$. Due to the Arzelà-Ascoli theorem, we can choose a subsequence $\{u_{n_k}\} \subset \{u_n\}$ which is converging uniformly on $[0,T]$ to a function $u \in C[0,T]$. Clearly, u satisfies (1.2), because each u_n does.

1. Let us take an arbitrary compact interval $K \subset (0,T)$. Then, by (2.3), $f \in \mathrm{Car}(K \times \mathbb{R}^2)$ and there exists $n_0 \in \mathbb{N}$ such that

$$K \subset J_n \quad \text{for all } n \geqslant n_0. \tag{2.7}$$

Further, by (2.4),

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t \in K, \text{ for all } x, y \in \mathbb{R} \text{ and all } n \geqslant n_0. \tag{2.8}$$

Then there exists $h_K \in L[0,T]$ such that for each $t, s \in K$

$$\begin{aligned}|\phi(u_{n_k}'(t)) - \phi(u_{n_k}'(s))| &= \left| \int_s^t f_{n_k}(\tau, u_{n_k}(\tau), u_{n_k}'(\tau))\,\mathrm{d}\tau \right| \\ &= \left| \int_s^t f(\tau, u_{n_k}(\tau), u_{n_k}'(\tau))\,\mathrm{d}\tau \right| \leqslant \left| \int_s^t h_K(\tau)\,\mathrm{d}\tau \right| \quad \forall n_k \geqslant n_0.\end{aligned}$$

So $\{\phi(u_{n_k}')\}$ is equicontinuous on K. By virtue of the uniform continuity of ϕ on compact intervals, the sequence $\{u_{n_k}'\}$ is equicontinuous and, by (2.5), equibounded on K. By the Arzelà-Ascoli theorem, we can find a subsequence of $\{u_{n_k}'\}$ uniformly converging on K. Now we will show that there exists a subsequence $\{u_{n_l}'\} \subset \{u_{n_k}'\}$ which converges locally uniformly on $(0,T)$ to $u' \in C(0,T)$.

Let us consider a sequence of compact intervals $K_i = [\alpha_i, \beta_i] \subset (0,T)$, $i \in \mathbb{N}$, $K_i \subset K_{i+1}$, $\lim\limits_{i\to\infty} \alpha_i = 0$, $\lim\limits_{i\to\infty} \beta_i = T$. Then there exists a subsequence $\{u_{1,n}'\}_{n=1}^\infty \subset \{u_{n_k}'\}$ which converges uniformly on K_1. We can choose $\{u_{2,n}'\}_{n=1}^\infty \subset \{u_{1,n}'\}$ which converges uniformly on K_2 and for each $i \in \mathbb{N}$, $\{u_{i,n}'\} \subset \{u_{i-1,n}'\}$ which converges uniformly on K_i, i.e.

$$\forall i \in \mathbb{N} \ \forall \varepsilon > 0 \ \exists k_i \in \mathbb{N} \ \forall t \in K_i \ \forall n > k_i\colon \ |u_{i,n}'(t) - u'(t)| < \varepsilon.$$

Let us take $\{u_{n,n}'\}$. This sequence converges uniformly on K_i for all $i \in \mathbb{N}$, because for each i there are only $i-1$ terms of $\{u_{n,n}'\}$ which do not belong to $\{u_{i,n}'\}$. Clearly, for each compact interval $K \subset (0,T)$ there exists $i_0 \in \mathbb{N}$ such that $K \subset K_i$ for all $i > i_0$. Hence

$$\forall \varepsilon > 0\ \forall t \in K\ \forall n > k_{i_0}\colon\ |u'_{n,n}(t) - u'(t)| < \varepsilon.$$

This implies that

$$\forall K \subset (0,T)\ \forall \varepsilon > 0\ \exists m \in \mathbb{N}\ \forall t \in K\ \forall n > m\colon\ |u'_{n,n}(t) - u'(t)| < \varepsilon.$$

So the sequence $\{u'_{n,n}\} \subset \{u'_{n_k}\}$ converges locally uniformly on $(0,T)$ to $u' \in C(0,T)$.

2. Let us denote $\{u'_{n,n}\}$ as $\{u'_{n_l}\}$ and take an arbitrary compact interval $K \subset (0,T)$. There exists $n_0 \in \mathbb{N}$ such that (2.7) and (2.8) are valid. Therefore for all $n_l \geqslant n_0$ we have

$$\phi(u'_{n_l}(t)) = \phi(u'_{n_l}(\tfrac{1}{2}T)) + \int_{\frac{T}{2}}^{t} f(s, u_{n_l}(s), u'_{n_l}(s))\,\mathrm{d}s \quad \text{for } t \in K. \tag{2.9}$$

Since f is continuous in its phase variables for a.e. $t \in (0,T)$ and because

$$u_{n_l} \rightrightarrows u \text{ on } [0,T] \quad \text{and} \quad u'_{n_l} \overset{\text{loc}}{\rightrightarrows} u' \text{ on } (0,T), \tag{2.10}$$

we get

$$\lim_{l\to\infty} f(t, u_{n_l}(t), u'_{n_l}(t)) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T].$$

Recall that

$$\exists h_K \in L[0,T]\colon\ |f(t, u_{n_l}(t), u'_{n_l}(t))| \leqslant h_K(t) \quad \text{for a.e. } t \in K \text{ and all } n_l \geqslant n_0.$$

Letting $l \to \infty$ in (2.9), using the Lebesgue dominated convergence theorem on K and the fact that $K \subset (0,T)$ is arbitrary, we get

$$\phi(u'(t)) = \phi(u'(\tfrac{1}{2}T)) + \int_{\frac{T}{2}}^{t} f(s, u(s), u'(s))\,\mathrm{d}s \quad \text{for } t \in (0,T).$$

Hence, $\phi(u') \in \mathrm{AC}_{\mathrm{loc}}(0,T)$ and u satisfies (1.1) for a.e. $t \in [0,T]$. It means that u is a w-solution of (1.1), (1.2).

3. Let (2.6) be fulfilled. It remains to prove that $\phi(u') \in \mathrm{AC}[0,T]$. It is sufficient to prove that $\phi(u') \in \mathrm{AC}[0,\eta] \cap \mathrm{AC}[T-\eta,T]$. Now, we will show that for a.e. $t \in (0,\eta)$

$$\lim_{l\to\infty} \operatorname{sign}(u'_{n_l}(t) - y_1) f_{n_l}(t, u_{n_l}(t), u'_{n_l}(t)) = \operatorname{sign}(u'(t) - y_1) f(t, u(t), u'(t)). \tag{2.11}$$

Put

$$
\begin{aligned}
V_1 &= \{t \in (0,\eta)\colon\ f(t,\cdot,\cdot)\colon\ \mathbb{R}^2 \to \mathbb{R} \text{ is not continuous}\},\\
V_2 &= \{t \in (0,\eta)\colon\ t \text{ is an isolated zero of } u' - y_1\},\\
V_3 &= \{t \in (0,\eta)\colon\ (\phi(u'(t)))' \text{ does not exist or (1.1) is not fulfiled}\}.
\end{aligned}
$$

Then $\operatorname{meas}(\mathscr{M}) = 0$, where $\mathscr{M} = V_1 \cup V_2 \cup V_3$. Choose an arbitrary $t_0 \in (0,\eta) \setminus \mathscr{M}$. If t_0 is an accumulation point of a set of zeros of $u' - y_1$, then there exists a sequence $\{t_n\} \subset (0,\eta)$ such that $u'(t_n) = y_1$ and $\lim\limits_{n\to\infty} t_n = t_0$. Since u' is continuous on $(0,\eta)$, we get $u'(t_0) = y_1$. Further, by virtue of $t_0 \notin V_3$,

$$
\lim_{t_n \to t_0} \frac{\phi(u'(t_n)) - \phi(u'(t_0))}{t_n - t_0} = 0 = (\phi(u'(t_0)))'
$$

and we get

$$
0 = (\phi(u'(t_0)))' = f(t_0, u(t_0), u'(t_0)).
$$

Since $t_0 \notin V_1$, we have by (2.4) and (2.10)

$$
\lim_{l\to\infty} f_{n_l}(t_0, u_{n_l}(t_0), u'_{n_l}(t_0)) = \lim_{l\to\infty} f(t_0, u_{n_l}(t_0), u'_{n_l}(t_0)) = f(t_0, u(t_0), u'(t_0)) = 0
$$

and

$$
\begin{aligned}
&\lim_{l\to\infty} \operatorname{sign}(u'_{n_l}(t_0) - y_1) f_{n_l}(t_0, u_{n_l}(t_0), u'_{n_l}(t_0))\\
&\qquad\qquad = 0 = \operatorname{sign}(u'(t_0) - y_1) f(t_0, u(t_0), u'(t_0)).
\end{aligned}
$$

Let $u'(t_0) \neq y_1$. If $u'(t_0) > y_1$ then there exists $n_0 \in \mathbb{N}$ such that $u'_{n_l}(t_0) > y_1$ for all $n_l \geqslant n_0$. It means that $\operatorname{sign}(u'_{n_l}(t_0) - y_1) = 1$ and (2.11) is satisfied. Similarly if $u'(t_0) < y_1$. Therefore we have proved that (2.11) is fulfilled for a.e. $t_0 \in (0,\eta)$. Further, for all $n_l \in \mathbb{N}$ we have

$$
\begin{aligned}
&\int_0^\eta \operatorname{sign}(u'_{n_l}(t) - y_1) f_{n_l}(t, u_{n_l}(t), u'_{n_l}(t))\,\mathrm{d}t\\
&\qquad = \int_0^\eta \operatorname{sign}(\phi(u'_{n_l}(t)) - \phi(y_1))(\phi(u'_{n_l}(t)) - \phi(y_1))'\,\mathrm{d}t\\
&\qquad = \int_0^\eta |\phi(u'_{n_l}(t)) - \phi(y_1)|'\,\mathrm{d}t\\
&\qquad = (|\phi(u'_{n_l}(\eta)) - \phi(y_1)| - |\phi(u'_{n_l}(0)) - \phi(y_1)|).
\end{aligned}
$$

By (2.6) there exists $d = d(c, y_1, \psi) > 0$ such that

$$
0 \leqslant \int_0^\eta [\lambda_1 \operatorname{sign}(u'_{n_l}(s) - y_1) f_{n_l}(s, u_{n_l}(s), u'_{n_l}(s)) + |\psi(s)|]\,\mathrm{d}s \leqslant d, \quad n_l \in \mathbb{N}. \tag{2.12}
$$

For a.e. $t \in (0,\eta)$ let us put

$$\begin{aligned}\varphi_{n_l}(t) &= \lambda_1 \operatorname{sign}(u'_{n_l}(t) - y_1) f_{n_l}(t, u_{n_l}(t), u'_{n_l}(t)) + |\psi(t)|,\\ \varphi(t) &= \lambda_1 \operatorname{sign}(u'(t) - y_1) f(t, u(t), u'(t)) + |\psi(t)|.\end{aligned}$$

According to (2.6) we can see that $\varphi_{n_l} \in L[0,\eta]$ are nonnegative a.e. on $(0,\eta)$. Further, by (2.11), $\varphi_{n_l} \xrightarrow{\text{a.e.}} \varphi$ on $(0,\eta)$ and, by (2.12), $\int_0^\eta \varphi_{n_l}(s)\,\mathrm{d}s \leqslant d$. Using the Fatou lemma, we conclude that $\varphi \in L[0,\eta]$. Then $|f(\cdot,u,u')| \in L[0,\eta]$ and also $f(\cdot,u,u') \in L[0,\eta]$. It means that $\phi(u') \in \mathrm{AC}[0,\eta]$. In an analogous way, we can prove that $\phi(u') \in \mathrm{AC}[T-\eta,T]$. The theorem is proved. $\square$

3. Regular Dirichlet BVP's

In order to fulfil the basic condition (2.5) in Theorem 2.1 we need existence results for regular problems (2.1), (1.2) and a priori estimates for their solutions. To this aim we consider a regular equation

$$(\phi(u'))' = h(t,u,u'), \tag{3.1}$$

$h \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$, and use the lower and upper functions method to get solvability of problem (3.1), (1.2).

Definition 3.1. Functions σ_1, $\sigma_2\colon [0,T] \to \mathbb{R}$ are respectively *lower and upper functions of problem* (3.1), (1.2) if $\phi(\sigma_i') \in \mathrm{AC}[0,T]$ for $i \in \{1,2\}$ and

$$\begin{gathered}(\phi(\sigma_1'(t)))' \geqslant f(t,\sigma_1(t),\sigma_1'(t)),\ (\phi(\sigma_2'(t)))' \leqslant f(t,\sigma_2(t),\sigma_2'(t)) \ \text{ for a.e. } t \in [0,T],\\ \sigma_1(0) \leqslant A,\ \sigma_1(T) \leqslant B,\ \sigma_2(0) \geqslant A, \sigma_2(T) \geqslant B.\end{gathered}$$

Lemma 3.2. *Let σ_1 and σ_2 be respectively lower and upper functions of problem* (3.1), (1.2) *and let $\sigma_1 \leqslant \sigma_2$ on $[0,T]$. Further assume that there is $h_0 \in L[0,T]$ such that*

$$|h(t,x,y)| \leqslant h_0(t) \text{ for a.e. } t \in [0,T] \text{ and for all } (x,y) \in [\sigma_1(t),\sigma_2(t)] \times \mathbb{R}.$$

Then problem (3.1), (1.2) *has a solution $u \in C^1[0,T]$ with $\phi(u') \in \mathrm{AC}[0,T]$ such that*

$$\sigma_1 \leqslant u \leqslant \sigma_2 \text{ on } [0,T]. \tag{3.2}$$

Since the lower and upper functions method for regular problems with ϕ-Laplacian can be found in literature (see e.g. [18]–[20]), we present just the main ideas of the proof of Lemma 3.2.

S k e t c h o f t h e p r o o f. For a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$ define an auxiliary function

(3.3)
$$g(t,x,y) = \begin{cases} h(t,\sigma_2,y) + \omega_2\left(t, \dfrac{x-\sigma_2}{x-\sigma_2+1}\right) + \dfrac{x-\sigma_2}{x-\sigma_2+1} & \text{for } x > \sigma_2(t), \\ h(t,x,y) & \text{for } \sigma_1(t) \leqslant x \leqslant \sigma_2(t), \\ h(t,\sigma_1,y) - \omega_1\left(t, \dfrac{\sigma_1-x}{\sigma_1-x+1}\right) - \dfrac{\sigma_1-x}{\sigma_1-x+1} & \text{for } x < \sigma_1(t), \end{cases}$$

where

$$\omega_i(t,\varepsilon) = \sup\{|h(t,\sigma_i,\sigma_i') - h(t,\sigma_i,y)| \colon\ |y-\sigma_i'| \leqslant \varepsilon\},\ i = 1,2,\ \varepsilon \in [0,1]$$

and consider the equation

$$(\phi(u'))' = g(t,u,u'). \tag{3.4}$$

We see that $\omega_i \in \mathrm{Car}([0,T]\times[0,1])$ are nonnegative, nondecreasing in their second variable and $\omega_i(t,0) = 0$ for a.e. $t \in [0,T]$, $i = 1,2$. Further we see that $g \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$ and there exists $\tilde g \in L[0,T]$ such that

$$|g(t,x,y)| \leqslant \tilde g(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}.$$

We will prove the existence of a solution of the auxiliary problem (3.4), (1.2).

For fixed $v \in C^1[0,T]$ define $\gamma_v \colon \mathbb{R} \to \mathbb{R}$ by

$$\gamma_v(x) = \int_0^T \phi^{-1}\left(x + \int_0^r g_v(s)\,\mathrm{d}s\right)\mathrm{d}r$$

where $g_v(s) \equiv g(s,v(s),v'(s))$ for a.e. $s \in [0,T]$. The properties of ϕ and g imply that for each $v \in C^1[0,T]$ there exists a unique τ_v satisfying

$$\gamma_v(\tau_v) = \int_0^T \phi^{-1}\left(\tau_v + \int_0^r g_v(s)\,\mathrm{d}s\right)\mathrm{d}r = B - A, \tag{3.5}$$

and that there exists $m > 0$ such that $|\tau_v| \leqslant m$ for every $v \in C^1[0,T]$. Now define an operator $\mathscr{T} \colon C^1[0,T] \to C^1[0,T]$ by the formula

$$(\mathscr{T}u)(t) = A + \int_0^t \phi^{-1}\left(\tau_u + \int_0^r g_u(s)\,\mathrm{d}s\right)\mathrm{d}r.$$

Then we can check that if u is a fixed point of the operator $\mathscr{T}$ then u is a solution of (3.4), (1.2). Using the Lebesgue theorem and the Arzelà-Ascoli theorem we prove that $\mathscr{T}$ is continuous and compact. Further, for all $u \in C^1[0,T]$ the following estimate holds:

$$\|\mathscr{T}u\|_{C^1[0,T]} \leqslant |A| + (T+1)\max\{|\phi^{-1}(-m - \|\tilde{g}\|_{L[0,T]})|, |\phi^{-1}(m + \|\tilde{g}\|_{L[0,T]})|\} = Q.$$

Define $\Omega = \{u \in C^1[0,T]\colon \|u\|_{C^1[0,T]} \leqslant Q\}$. Then Ω is a nonempty closed bounded and convex set. The compact operator $\mathscr{T}$ sends the set Ω into Ω. By the Schauder fixed point theorem, operator $\mathscr{T}$ has a fixed point u. This fixed point is a solution of problem (3.4), (1.2).

It remains to prove that u satisfies (3.2). We put $v(t) = \sigma_1(t) - u(t)$ for all $t \in [0,T]$. By (1.2), we have $v(0) \leqslant 0$ and $v(T) \leqslant 0$, and then we can show that v does not have a positive maximum at any point of $(0,T)$. The second inequality in (3.2) can be proved similarly.

(3.2) and (3.3) yield that u is also a solution of problem (3.1), (1.2). The lemma is proved. □

Lemma 3.2 gives the existence result for (3.1), (1.2) provided the function h has a Lebesgue integrable majorant h_0. The method of a priori estimates enables us to extend this result to more general right-hand sides h.

Lemma 3.3 (An a priori estimate)**.** *Assume that $a, b \in [0,T], a \leqslant b$, $y_1, y_2 \in \mathbb{R}$, $c_0 \in (0,\infty)$. Let $g_0 \in L[0,T]$ be nonnegative and let $\omega \in C[0,\infty)$ be positive and*

$$\int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty. \tag{3.6}$$

Then there exists $\varrho_0 \in (c_0,\infty)$ such that for each function $u \in C^1[0,T]$ satisfying the conditions

$$\phi(u') \in \mathrm{AC}[0,T],$$

$$|u(t)| \leqslant c_0 \quad \text{for each } t \in [0,T], \tag{3.7}$$

$$|u'(\xi)| \leqslant c_0 \quad \text{for some } \xi \in [a,b], \tag{3.8}$$

$$(\phi(u'(t)))' \operatorname{sign}(u'(t) - y_1) \geqslant -\omega(|\phi(u'(t)) - \phi(y_1)|)(g_0(t) + |u'(t) - y_1|)$$
$$\text{for a.e. } t \in [0,b] \text{ and for } |\phi(u'(t))| > |\phi(y_1)| \tag{3.9}$$

and

$$(\phi(u'(t)))' \operatorname{sign}(u'(t) - y_2) \leqslant \omega(|\phi(u'(t)) - \phi(y_2)|)(g_0(t) + |u'(t) - y_2|)$$
$$\text{for a.e. } t \in [a,T] \text{ and for } |\phi(u'(t))| > |\phi(y_2)|, \tag{3.10}$$

the estimate

$$|u'(t)| \leqslant \varrho_0 \quad \textit{for each } t \in [0,T] \tag{3.11}$$

is valid.

Proof. We can see that $\operatorname{sign}(\phi(u'(t)) - \phi(y_i)) = \operatorname{sign}(u'(t) - y_i)$, $i = 1,2$. Put $v_i'(t) = \phi(u'(t)) - \phi(y_i)$, $i = 1,2$. Then

$$|v_i'(\xi)| = |\phi(u'(\xi)) - \phi(y_i)| \leqslant \max\{|\phi(-c_0)|, \phi(c_0)\} + |\phi(y_i)| = c_i, \quad i = 1,2.$$

Condition (3.6) implies that there exists $\varrho_i \in (c_i, \infty)$ such that

$$\int_{c_i}^{\varrho_i} \frac{\mathrm{d}s}{\omega(s)} > \|g_0\|_{L[0,T]} + 2c_0 + T|y_i|, \quad i = 1,2. \tag{3.12}$$

First, let us prove the estimate

$$|v_1'(t)| \leqslant \varrho_1 \quad \text{for } t \in [0,\xi]. \tag{3.13}$$

By (3.9) we get

$$-\frac{v_1''(t)\operatorname{sign} v_1'(t)}{\omega(|v_1'(t)|)} \leqslant g_0(t) + |u'(t) - y_1| \quad \text{for a.e. } t \in [0,\xi]. \tag{3.14}$$

Asume that (3.13) is not valid, i.e. that there exists an interval $[\alpha,\beta] \subset [0,\xi]$ such that

$$|v_1'(\beta)| \leqslant c_1, \quad |v_1'(\alpha)| > \varrho_1 \quad \text{and} \quad v_1'(t) \neq 0 \text{ on } [\alpha,\beta].$$

Integrating (3.14) over $[\alpha,\beta]$ we arrive at

$$\int_\alpha^\beta -\frac{v_1''(t)\operatorname{sign} v_1'(t)}{\omega(|v_1'(t)|)}\,\mathrm{d}t \leqslant \int_\alpha^\beta g_0(t)\,\mathrm{d}t + \int_\alpha^\beta |u'(t) - y_1|\,\mathrm{d}t.$$

Using a substitution $s = |v_1'(t)|$, we obtain

$$\int_{c_1}^{\varrho_1} \frac{\mathrm{d}s}{\omega(s)} \leqslant \int_{|v_1'(\beta)|}^{|v_1'(\alpha)|} \frac{\mathrm{d}s}{\omega(s)} \leqslant \|g_0\|_{L[0,T]} + \left|\int_\alpha^\beta u'(t)\,\mathrm{d}t\right| + \left|\int_\alpha^\beta y_1\,\mathrm{d}t\right| \leqslant \|g_0\|_{L[0,T]} + 2c_0 + T|y_1|,$$

which contradicts (3.12). Therefore (3.13) is valid. Similiarly we can prove the estimate

$$|v_2'(t)| \leqslant \varrho_2 \quad \text{for } t \in [\xi,T]. \tag{3.15}$$

Now (3.13) and (3.15) imply that

$$|\phi(u'(t)) - \phi(y_1)| \leqslant \varrho_1 \text{ on } [0,\xi], \quad |\phi(u'(t)) - \phi(y_2)| \leqslant \varrho_2 \text{ on } [\xi,T],$$

wherefrom

$$\begin{aligned}&|u'(t)| \leqslant \max\{|\phi^{-1}(-\max\{|\phi(y_1)|,|\phi(y_2)|\} - \max\{\varrho_1,\varrho_2\})|,\\ &|\phi^{-1}(\max\{|\phi(y_1)|,|\phi(y_2)|\} + \max\{\varrho_1,\varrho_2\})|\} = \varrho_0 \quad \text{on } [0,T].\end{aligned}$$

□

Using Lemma 3.2 and Lemma 3.3 we get the existence result for (3.1), (1.2) under one-sided growth restrictions of the Nagumo type (3.19), (3.20). Note that for $\phi(y) \equiv y$ similar results can be found in [8] and for $\phi = \phi_p$ in the papers [10], [19], [20], where both-sided growth restrictions are assumed.

Theorem 3.4. *Assume that the following conditions are fulfilled:*

(3.16) σ_1 *and* σ_2 *are respectively lower and upper functions of* (3.1), (1.2) *and* $\sigma_1 \leqslant \sigma_2$ *on* $[0,T]$,

(3.17) $a,b \in (0,T),\ a < b,\ y_1,y_2 \in \mathbb{R},\ c_0 \geqslant 2(1+b-a)(b-a)^{-1}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty)$,

(3.18) $g \in L[0,T]$ *is nonnegative,* $\omega \in C[0,\infty)$ *is positive and fulfils* (3.6),

(3.19) $h(t,x,y)\operatorname{sign} y \geqslant -\omega(|\phi(y) - \phi(y_1)|)(g(t) + |y|)$ *for a.e.* $t \in [0,b], \forall x \in [\sigma_1(t),\sigma_2(t)], \forall y \in \mathbb{R}$ *such that* $|\phi(y)| > |\phi(y_1)|$

and

(3.20) $h(t,x,y)\operatorname{sign} y \leqslant \omega(|\phi(y) - \phi(y_2)|)(g(t) + |y|)$ *for a.e.* $t \in [a,T],\ \forall x \in [\sigma_1(t),\sigma_2(t)],\ \forall y \in \mathbb{R}$ *such that* $|\phi(y)| > |\phi(y_2)|$.

Then problem (3.1), (1.2) *has a solution* u *satisfying*

(3.21) $$\sigma_1 \leqslant u \leqslant \sigma_2 \quad \text{on } [0,T]$$

and

(3.22) $$|u'(t)| \leqslant \varrho_0 \quad \text{for } t \in [0,T],$$

where $\varrho_0 \in (0,\infty)$ *is the constant from Lemma 3.3 with* $g_0 = g + |y_1| + |y_2|$.

P r o o f. Put $r_0 = \varrho_0 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty$ where ϱ_0 is given by Lemma 3.3 and define functions

$$\chi(y) = \begin{cases} 1 & \text{if } \ |y| \leqslant r_0, \\ 2 - y/r_0 & \text{if } \ r_0 < |y| < 2r_0, \\ 0 & \text{if } \ |y| \geqslant 2r_0 \end{cases}$$

and

$$\tilde{h}(t,x,y) = \chi(y)h(t,x,y) \tag{3.23}$$

for a.e. $t \in [0,T]$, x, $y \in \mathbb{R}$.

For $(x,y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$, the function $\tilde{h}$ is bounded on $[0,T]$ by a Lebesgue integrable function. In addition, σ_1, σ_2 are respectively lower and upper functions of the problem

$$(\phi(x'(t)))' = \tilde{h}(t, x(t), x'(t)), \quad (1.2). \tag{3.24}$$

According to Lemma 3.2 there exists a solution u of problem (3.24) satisfying (3.2).

Let us prove that u is also a solution of problem (3.1), (1.2). Conditions (3.2) and (3.17) imply that u fulfils (3.7) and (3.8). Moreover, by (3.19) and (3.23),

$$\begin{aligned} \tilde{h}(t,x,y)\operatorname{sign}(\phi(y) - \phi(y_1)) &= \chi(y)h(t,x,y)\operatorname{sign} y \\ &\geqslant -\omega(|\phi(y) - \phi(y_1)|)(g(t) + |y|) \\ &\geqslant -\omega(|\phi(y) - \phi(y_1)|)(g_0(t) + |y - y_1|) \end{aligned}$$

for a.e. $t \in [0,T]$, all $x \in [\sigma_1(t), \sigma_2(t)]$ and every $y \in \mathbb{R}$ such that $|\phi(y)| > |\phi(y_1)|$, where $g_0(t) = g(t) + |y_1| + |y_2|$. (Note that $|\phi(y)| > |\phi(y_1)|$ implies $\operatorname{sign}(y - y_1) = \operatorname{sign} y$.) It means that (3.9) is valid. Similarly, using (3.20), we can derive that (3.10) is valid. We have shown that all conditions of Lemma 3.3 are satisfied. So, the estimate (3.11) is true and u is a solution of problem (3.1), (1.2). This concludes the proof of Theorem 3.4. □

4. Main result

In this section we prove our main result about the solvability of the singular Dirichlet boundary value problem (1.1), (1.2).

Theorem 4.1. *Assume that conditions* (3.18),

(4.1) $a, b \in (0,T),\ a < b, \quad r_1, r_2, y_1, y_2 \in \mathbb{R}$,

$$\text{(4.2)} \qquad \begin{cases} r_1 + ty_1 \leqslant \min\{A,B\},\ r_2 + ty_2 \geqslant \max\{A,B\} & \textit{for } t \in [0,T], \\ f(t, r_1 + ty_1, y_1) \leqslant 0, \quad f(t, r_2 + ty_2, y_2) \geqslant 0 & \textit{for a.e. } t \in [0,T], \end{cases}$$

(4.3) $f(t,x,y)\operatorname{sign} y \geqslant -\omega(|\phi(y) - \phi(y_1)|)(g(t) + |y|)$
for a.e. $t \in [0,b], \forall x \in [r_1 + ty_1, r_2 + ty_2], \forall y \in \mathbb{R}$ *such that* $|\phi(y)| > |\phi(y_1)|$

and

(4.4) $f(t,x,y)\operatorname{sign} y \leqslant \omega(|\phi(y) - \phi(y_2)|)(g(t) + |y|)$
for a.e. $t \in [a,T], \forall x \in [r_1 + ty_1, r_2 + ty_2],\ \forall y \in \mathbb{R}$ *such that* $|\phi(y)| > |\phi(y_2)|$

are satisfied. Then there exists $\varrho_0 > 0$ *such that problem* (1.1), (1.2) *has a* w*-solution* $u \in C^1(0,T)$ *satisfying*

$$\text{(4.5)} \qquad r_1 + ty_1 \leqslant u(t) \leqslant r_2 + ty_2 \qquad \textit{for } t \in [0,T]$$

and

$$\text{(4.6)} \qquad |u'(t)| \leqslant \varrho_0 \quad \textit{for each } t \in (0,T).$$

Further, let there exist $\eta \in (0, T/2)$, $\lambda_1, \lambda_2 \in \{-1, 1\}$, $\psi_0 \in L[0,T]$ *such that*

$$\text{(4.7)} \qquad \begin{cases} \lambda_1 \operatorname{sign}(y - y_1) f(t,x,y) \geqslant \psi_0(t) \\ \qquad \textit{for a.e. } t \in [0,\eta], \forall x \in [r_1 + ty_1, r_2 + ty_2], \forall y \in [-\varrho_0, \varrho_0], \\ \lambda_2 \operatorname{sign}(y - y_2) f(t,x,y) \geqslant \psi_0(t) \\ \qquad \textit{for a.e. } t \in [T-\eta, T], \forall x \in [r_1 + ty_1, r_2 + ty_2], \forall y \in [-\varrho_0, \varrho_0]. \end{cases}$$

Then $u \in C^1[0,T]$ *is a solution of problem* (1.1), (1.2).

P r o o f. For each $n \in \mathbb{N}$ define J_n by (2.2),

$$\text{(4.8)} \qquad f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{for a.e. } t \in J_n,\ \forall x, y \in \mathbb{R}, \\ 0 & \text{for a.e. } t \in [0, 1/n) \cup (T - 1/n, T],\ \forall x, y \in \mathbb{R}. \end{cases}$$

Then $f_n \in \operatorname{Car}([0,T] \times \mathbb{R}^2)$ for each $n \in \mathbb{N}$. Choose $n \in \mathbb{N}$ and show that problem (2.1), (1.2) satisfies the assumptions of Theorem 3.4. Let us put $\sigma_1(t) = r_1 + ty_1$ and $\sigma_2(t) = r_2 + ty_2$ for $t \in [0,T]$. Then, according to (4.2), σ_1 and σ_2 are lower and upper function of problem (2.1), (1.2), i.e. (3.16) holds. From inequalities (4.3) and (4.4) we get

$$\begin{aligned}
&f_n(t,x,y)\operatorname{sign}y = f(t,x,y)\operatorname{sign}y \geqslant -\omega(|\phi(y)-\phi(y_1)|)(g(t)+|y|)\\
&\qquad\text{for a.e. } t\in[0,b]\cap J_n, \forall x\in[r_1+ty_1, r_2+ty_2], \forall y\in\mathbb{R}, |\phi(y)|>|\phi(y_1)|,\\
&f_n(t,x,y)\operatorname{sign}y = 0 \geqslant -\omega(|\phi(y)-\phi(y_1)|)(g(t)+|y|)\\
&\qquad\text{for a.e. } t\in[0,b]\setminus J_n, \forall x\in[r_1+ty_1, r_2+ty_2], \forall y\in\mathbb{R},\\
&f_n(t,x,y)\operatorname{sign}y = f(t,x,y)\operatorname{sign}y \leqslant \omega(|\phi(y)-\phi(y_2)|)(g(t)+|y|)\\
&\qquad\text{for a.e. } t\in[a,T]\cap J_n, \forall x\in[r_1+ty_1, r_2+ty_2], \forall y\in\mathbb{R}, |\phi(y)|>|\phi(y_2)|,\\
&f_n(t,x,y)\operatorname{sign}y = 0 \leqslant \omega(|\phi(y)-\phi(y_2)|)(g(t)+|y|)\\
&\qquad\text{for a.e. } t\in[a,T]\setminus J_n, \forall x\in[r_1+ty_1, r_2+ty_2], \forall y\in\mathbb{R}.
\end{aligned}$$

It means that conditions (3.19) and (3.20) are fulfilled. By Theorem 3.4, problem (2.1), (1.2) has a solution $u_n\in C^1[0,T]$ with $\phi(u_n')\in \mathrm{AC}[0,T]$. Moreover, u_n satisfies (4.5) and

$$|u_n'(t)|\leqslant \varrho_0 \qquad \text{for } t\in[0,T], \tag{4.9}$$

where $\varrho_0\in(0,\infty)$ is the constant from Lemma 3.3 with $g_0=g+|y_1|+|y_2|$. By virtue of Lemma 3.3, ϱ_0 does not depend on u_n. Therefore condition (2.5) is fulfilled, where

$$\Omega=\{u\in C^1([0,T])\colon\ \|u\|_\infty\leqslant\|\sigma_1\|_\infty+\|\sigma_2\|_\infty+\varrho_0\}.$$

Hence Theorem 2.1 yields the existence of a w-solution u of problem (1.1), (1.2). Moreover, u satisfies (4.5) and (4.6).

Now, moreover, assume (4.7). Let us define

$$\psi(t)=\min\{\psi_0(t),0\} \quad \text{for } t\in[0,T].$$

Then $\psi\in L[0,T]$. We can see that

$$\begin{aligned}
&\lambda_1 f_n(t,u_n(t),u_n'(t))\operatorname{sign}(u_n'(t)-y_1) = \lambda_1 f(t,u_n(t),u_n'(t))\operatorname{sign}(u_n'(t)-y_1)\\
&\qquad\qquad\geqslant\psi_0(t)\geqslant\psi(t) \text{ for a.e. } t\in[0,\eta]\cap J_n,\\
&\lambda_1 f_n(t,u_n(t),u_n'(t))\operatorname{sign}(u_n'(t)-y_1) = 0\geqslant\psi(t) \text{ for a.e. } t\in[0,\eta]\setminus J_n,\\
&\lambda_2 f_n(t,u_n(t),u_n'(t))\operatorname{sign}(u_n'(t)-y_2) = \lambda_2 f(t,u_n(t),u_n'(t))\operatorname{sign}(u_n'(t)-y_2)\\
&\qquad\qquad\geqslant\psi_0(t)\geqslant\psi(t) \text{ for a.e. } t\in[T-\eta,T]\cap J_n,\\
&\lambda_2 f_n(t,u_n(t),u_n'(t))\operatorname{sign}(u_n'(t)-y_2) = 0\geqslant\psi(t) \text{ for a.e. } t\in[T-\eta,T]\setminus J_n.
\end{aligned}$$

It means that condition (2.6) is satisfied. By Theorem 2.1, u is a solution of (1.1), (1.2). $\square$

R e m a r k 4.2. A similar result about the existence of a w-solution of problem (1.1), (1.2) with $\phi(y) = y$ can be found in [8], Theorem 3.1, but we do not know another result about the existence of a solution to (1.1), (1.2) in literature.

E x a m p l e 4.3 (Existence of solution). Let $p > 1$ and $\phi_p(y) = |y|^{p-2}y$ for $y \in \mathbb{R}$. Consider the equation

$$(\phi_p(u'))' = q(t)(u^k - r^k) + c\phi_p(u')u' + \Big(\frac{1}{t^\alpha} - \frac{1}{(T-t)^\beta}\Big)(\phi_p(u') - \phi_p(d)), \tag{4.10}$$

where $r, c, d \in \mathbb{R}$, $k \in \mathbb{N}$ is odd, α, $\beta \in (1,\infty)$, $q \in L[0,T]$ is nonnegative. Then, by Theorem 4.1, problem (4.10), (1.2) has a solution $u \in C^1[0,T]$ with $\phi_p(u') \in \mathrm{AC}[0,T]$. We will show that all the conditions of Theorem 4.1 are satisfied. Let $r_1, r_2 \in \mathbb{R}$. Then

$$f(t, r_i + td, d) = q(t)((r_i + td)^k - r^k) + c\phi_p(d)d \quad \text{for a.e. } t \in [0,T].$$

Since q is nonnegative on [0, T], we can find a large positive r_2 and a negative r_1 with large absolute value such that (4.2) holds. Denote

$$q_1(t) = q(t)\max\{|x^k - r^k|\colon\ r_1 + td \leqslant x \leqslant r_2 + td\} \quad \text{for a.e. } t \in [0,T],$$
$$q_2(t) = \begin{cases} (T-t)^{-\beta} & \text{for a.e. } t \in [0,a), \\ (T-t)^{-\beta} + t^{-\alpha} & \text{for a.e. } t \in [a,b], \\ t^{-\alpha} & \text{for a.e. } t \in (b,T]. \end{cases}$$

Then for a.e. $t \in [0,b]$, each $x \in [r_1 + td, r_2 + td]$ and each $y \in \mathbb{R}$, $|\phi_p(y)| > |\phi_p(d)|$ we have

$$\begin{aligned} f(t,x,y)\operatorname{sign} y &= f(t,x,y)\operatorname{sign}(\phi_p(y) - \phi_p(d)) \\ &> -q_1(t) - |c||\phi_p(y) - \phi_p(d)||y| - |c||\phi_p(d)||y| - \frac{1}{(T-t)^\beta}|\phi_p(y) - \phi_p(d)| \\ &> -(|\phi_p(y) - \phi_p(d)| + 1)((|c|+1)(|\phi_p(d)|+1))(|q_1(t)| + |q_2(t)| + |y|). \end{aligned}$$

Therefore, if we put

$$\omega(s) = (1+s)c_0, \quad c_0 = (|c|+1)(|\phi_p(d)|+1), \quad g(t) = |q_1(t)| + |q_2(t)|,$$

we get (4.3). Similarly we obtain (4.4) Further, for a.e. $t \in [0,T]$, each $x \in [r_1 + td, r_2 + td]$ and each $y \in [-\varrho_0, \varrho_0]$ we get

$$\begin{aligned} \operatorname{sign}(b-t)f(t,x,y)\operatorname{sign}(y-d) &= f(t,x,y)\operatorname{sign}(\phi_p(y) - \phi_p(d)) \\ &> -q_1(t) - |c|\phi_p(\varrho_0)\varrho_0 - q_2(t)(\phi_p(\varrho_0) + |\phi_p(d)|) = \psi_0(t), \end{aligned}$$

where $\psi_0 \in L[0,T]$, which means that (4.7) is satisfied.

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