EQUIVARIANT MAPS BETWEEN CERTAIN G-SPACES WITH $G = O(n-1,1)$.

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Abstract. In this note, there are determined all biscalars of a system of $s \leqslant n$ linearly independent contravariant vectors in n-dimensional pseudo-Euclidean geometry of index one. The problem is resolved by finding a general solution of the functional equation $F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{s}{u}) = (\operatorname{sign}(\det A))F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u})$ for an arbitrary pseudo-orthogonal matrix A of index one and the given vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}$.

Keywords: G-space, equivariant map, vector, scalar, biscalar

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1. Introduction

For $n \geqslant 2$ consider the matrix $E_1 = [e_{ij}] \in GL(n, \mathbb{R})$, where

$$e_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ +1 & \text{for } i = j \neq n \\ -1 & \text{for } i = j = n \end{cases}$$

Definition 1. A pseudo-orthogonal group of index 1 is a subgroup of the group $GL(n, \mathbb{R})$ satisfying

$$G = O(n-1,1) = \{A\colon\ A \in GL(n, \mathbb{R}) \wedge A^T \cdot E_1 \cdot A = E_1\}.$$

Denoting $\varepsilon(A) = \operatorname{sign}(\det A) = \det A$ we have $\varepsilon(A \cdot B) = \varepsilon(A) \cdot \varepsilon(B)$.

The class of G-spaces (M_α, G, f_α), where f_α is an action of G on the space M_α, constitutes a category if we take as morphisms equivariant maps $F_{\alpha\beta}\colon\ M_\alpha \longrightarrow M_\beta$, i.e. the maps which satisfy the condition

$$\bigwedge_{\alpha,\beta} \bigwedge_{x\in M_\alpha} \bigwedge_{A\in G} F_{\alpha\beta}(f_\alpha(x,A)) = f_\beta(F_{\alpha\beta}(x),A). \tag{1}$$

In particular, among the objects of this category are: the G-space of contravariant vectors

$$(\mathbb{R}^n, G, f_1), \text{ where } \bigwedge_{u\in\mathbb{R}^n} \bigwedge_{A\in G} f_1(u,A) = A\cdot u, \tag{2}$$

the G-space of scalars

$$(\mathbb{R}, G, f_2), \quad \text{where } \bigwedge_{x\in\mathbb{R}} \bigwedge_{A\in G} f_2(x,A) = x, \tag{3}$$

and the G-space of biscalars

$$(\mathbb{R}, G, f_3), \text{ where } \bigwedge_{x\in\mathbb{R}} \bigwedge_{A\in G} f_3(x,A) = \varepsilon(A)\cdot x. \tag{4}$$

For $s = 1, 2, \ldots, n$, let a system of linearly independent vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}$ be given. Every equivariant map F of this system into $M_2 = \mathbb{R}$ satisfies the equality (1), which applying the transformation rules (2) and (3) may be rewritten in the form

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{s}{u}) = F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}). \tag{5}$$

For a pair u, v of contravariant vectors the map $p(u,v) = u^T \cdot E_1 \cdot v$ satisfies (5), namely $p(Au, Av) = (Au)^T \cdot E_1 \cdot (Av) = u^T(A^T E_1 A)v = u^T E_1 v = p(u,v)$.

In [6] it was proved that the general solution of the equation (5) is of the form

$$F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}) = \Theta(p(\underset{i}{u}, \underset{j}{u})) \quad \text{for} \quad i \leqslant j = 1, 2, \ldots, s \leqslant n \tag{6}$$

where Θ is an arbitrary function of $\frac{s(s+1)}{2}$ variables.

In this paper we are going to determine all equivariant maps F of this system of vectors into $M_3 = \mathbb{R}$. The problem is equivalent to finding the general solution of the functional equation (1), which applying the transformations rules (2) and (4) may be rewritten in the form

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{s}{u}) = \varepsilon(A) F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}). \tag{7}$$

2. Type of a subspace

Let be given a sequence $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}, \dots, \underset{n}{u}$ of linearly independent vectors. Let $L_s = L(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u})$ denote the linear subspace generated by the vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$ and $p \mid L_s$ the restriction of the form p to the subspace L_s.

Definition 2. The subspace L_s is called:

1. Euclidean subspace if the form $p \mid L_s$ is positive definite,
2. pseudo-Euclidean subspace if the form $p \mid L_s$ is regular and indefinite,
3. singular subspace if the form $p \mid L_s$ is singular.

If we denote

$$p_{ij} = p(\underset{i}{u}, \underset{j}{u}) \quad \text{for} \quad i, j = 1, 2, \dots, n,$$

and

$$P(s) = P(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}) = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{vmatrix} = \det [p_{ij}]_1^s \qquad \text{for } s = 1, 2, \dots, n,$$

then the above three cases are equivalent to $P(s) > 0, P(s) < 0$ and $P(s) = 0$, respectively.

Let us consider an isotropic cone $K_0 = \{u\colon\, u \in \mathbb{R}^n \wedge p(u, u) = 0 \wedge u \neq 0\}$. It is an invariant and transitive subset. Every isotropic vector $v \in K_0$ determines an isotropic direction, which is, according to $v^n \neq 0$ and $v = v^n[\frac{v^1}{v^n}, \frac{v^2}{v^n}, \dots, \frac{v^{n-1}}{v^n}, 1]^T = u^n[q^1, q^2, \dots, q^{n-1}, 1]^T$ with $\sum\limits_{i=1}^{n-1} (q^i)^2 = 1$, equivalent to a point q belonging to the sphere S^{n-2}.

Let us recall that for $A \in G$

$$W' = \det(A\underset{1}{u}, \dots, A\underset{n}{u}) = \varepsilon(A) \det(\underset{1}{u}, \dots, \underset{n}{u}) = \varepsilon(A) \cdot W. \tag{8}$$

Therefore, for $s = n$ the mapping det satisfies the functional equation (7).

Let be given a system $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u}$ of $n-1$ linearly independent vectors for which $P(n-1) = 0$. The singular subspace $L(\underset{1}{u}, \dots, \underset{n-1}{u})$ determines exactly one isotropic direction $q \in S^{n-2}$ whose representative is of the form $v = v^n \cdot [q^1, \dots, q^{n-1}, 1]^T \in K_0$. From $p(\underset{i}{u}, v) = 0$ for $i = 1, 2, \dots, n-1$ it follows that each vector $\underset{i}{u}$ is of the form

$$\underset{i}{u} = \Big[\underset{i}{u}^1, \dots, \underset{i}{u}^{n-1}, \sum_{k=1}^{n-1} \underset{i}{u}^k q^k\Big]^T \quad \text{where} \quad \det [\underset{i}{u}^j]_1^{n-1} \neq 0. \tag{9}$$

Let us consider the two 1-forms $\det(\underset{1}{u},\dots,\underset{s-1}{u},v,\underset{s+1}{u},\dots,\underset{n-1}{u},x)$ and $p(v,x)$. Both these forms vanish on the subspace $L(\underset{1}{u},\dots,\underset{n-1}{u})$, and consequently there exists uniquely determined number $B_s(\underset{1}{u},\dots,\underset{s}{u},\dots,\underset{n-1}{u})$ such that

$$\det(\underset{1}{u},\dots,\underset{s-1}{u},v,\underset{s+1}{u},\dots,\underset{n-1}{u},x) = -B_s(\underset{1}{u},\dots,\underset{s}{u},\dots,\underset{n-1}{u})\cdot p(v,x). \tag{10}$$

Taking in mind the properties of the mappings p and det from (10) it follows immediately that for arbitrary $A\in G$ it holds that

$$B'_s = B_s(A\underset{1}{u},A\underset{2}{u},\dots,A\underset{n-1}{u}) = \varepsilon(A)\cdot B_s(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u}) = \varepsilon(A)\cdot B_s. \tag{11}$$

From (9) and (10) we get in terms of coordinates the formula

$$B_s(\underset{1}{u},\dots,\underset{n-1}{u}) = \begin{vmatrix} \underset{1}{u}^1 & \dots & \underset{1}{u}^{n-1} \\ \dots & \dots & \dots \\ \underset{s-1}{u}^{\,1} & \dots & \underset{s-1}{u}^{\,n-1} \\ q^1 & \dots & q^{n-1} \\ \underset{s+1}{u}^{\,1} & \dots & \underset{s+1}{u}^{\,n-1} \\ \dots & \dots & \dots \\ \underset{n-1}{u}^{\,1} & \dots & \underset{n-1}{u}^{\,n-1} \end{vmatrix} \quad \text{for} \quad s=1,2,\dots,n-1. \tag{12}$$

We have $B_s^2(\underset{1}{u},\dots,\underset{s}{u},\dots,\underset{n-1}{u}) = P(\underset{1}{u},\dots,\underset{s-1}{u},\underset{s+1}{u},\dots,\underset{n-1}{u})$, so at least one of the quantities B_s is different from zero (see [6], Theorem 15).

3. General solution of the functional equation (7)

Theorem 3. *The general solution of the functional equation* (7) *in the case* $s=n$ *is of the form*

$$F(\underset{1}{u},\underset{2}{u},\dots,\underset{n}{u}) = \Theta(p(\underset{i}{u},\underset{j}{u}))\cdot\det(\underset{1}{u},\underset{2}{u},\dots,\underset{n}{u})$$

where $i\leqslant j=1,2,\dots,n$ *and* Θ *is an arbitrary function of* $\frac{n(n+1)}{2}$ *variables.*

P r o o f. If $F(\underset{1}{u},\underset{2}{u},\dots,\underset{n}{u})$ is the general solution of the functional equation (7), then also $F(\underset{1}{u},\dots,\underset{n}{u})\cdot[\det(\underset{1}{u},\dots,\underset{n}{u})]^{-1}$ is the general solution of the equation (5). By virtue of (6) the statement of the theorem is true. □

Theorem 4. *The general solution of the functional equation* (7) *in the case* $s=n-1$ *and* $P(n-1)=0$ *is of the form*

$$F(\underset{1}{u},\dots,\underset{n-1}{u}) = \Theta(p(\underset{i}{u},\underset{j}{u}))\cdot B(\underset{1}{u},\dots,\underset{n-1}{u})$$

where $i \leqslant j = 1, 2, \dots, n-1$, Θ *is an arbitrary function of* $\frac{n(n-1)}{2}$ *variables and* B *is any nonzero equivariant among* $B_1, B_2, \dots, B_{n-1}$.

P r o o f. The proof runs analogously as the proof of Theorem 3. $\square$

Theorem 5. *The general solution of the functional equation* (7) *is trivial,*

$$F(\underset{1}{u}, \underset{1}{u}, \dots, \underset{s}{u}) \equiv 0,$$

if $s < n-1$ *or* $s = n-1$ *and* $P(n-1) \neq 0$.

P r o o f. If $P(n-1) \neq 0$, there exists a vector v such that $p(v, v) \neq 0$, and v is orthogonal (with respect to p) to the subspace W generated by $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u}$. The whole space coincides with the direct sum $[v] \oplus W$. If $s < n-1$ then there exists a vector v such that $p(v, v) \neq 0$ and v is orthogonal to the vectors $\underset{1}{u}, \underset{2}{u} \dots, \underset{s}{u}$. Let W denote this time the orthogonal complement of the vector v. Obviously, $\underset{1}{u}, \underset{2}{u} \dots, \underset{s}{u} \in W$, and the whole space coincides with the direct sum $[v] \oplus W$. Now, we take $A \in O(n-1, 1)$ defined by $A \cdot v = -v$ and $A\,|_W = \mathrm{id}$. We have $\varepsilon(A) = -1$. Then we get either

$$F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u}) = F(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{n-1}{u}) = -F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u})$$

or

$$F(\underset{1}{u}, \underset{1}{u}, \dots, \underset{s}{u}) = F(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{s}{u}) = -F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}).$$

In both cases we obtain $F \equiv 0$. $\square$

The statements proven in this section we can formulate in the following

Theorem 6. *The general solution of the functional equation* (7) *is of the form*

$$F(\underset{1}{u}, \dots, \underset{s}{u}) = \begin{cases} 0 \quad \text{if } s < n-1 \text{ or } s = n-1 \text{ and } P(n-1) \neq 0 \\ \sum\limits_{k=1}^{n-1} \Theta_k(p(\underset{i}{u}, \underset{j}{u})) \cdot B_k(\underset{1}{u}, \dots, \underset{n-1}{u}) \ \text{ if } s = n-1 \text{ and } P(n-1) = 0 \\ \Theta(p(\underset{i}{u}, \underset{j}{u})) \cdot \det(\underset{1}{u}, \dots, \underset{n}{u}) \ \text{ if } s = n \end{cases}$$

where $i \leqslant j = 1, 2, \dots, s$ *and* $\Theta, \Theta_1, \dots, \Theta_{n-1}$ *are arbitrary functions of* $\frac{s(s+1)}{2}$ *variables.*

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