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Eduard Feireisl

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On global well/ill-posedness of the Euler-Poisson system

Eduard Feireisl *

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Abstract

We discuss the problem of well-posedness of the Euler-Poisson system arising, for example, in the mathematical theory of semi-conductors, models of plasma and gaseous stars in astrophysics. We introduce the concept of *dissipative* weak solution satisfying, in addition to the standard system of integral identities replacing the original system of partial differential equations, the balance of total energy, together with the associated *relative entropy inequality*. We show that strong solutions are unique in the class of dissipative solutions (weak-strong uniqueness). Finally, we use the method of convex integration to show that the Euler-Poisson system may admit even infinitely many weak dissipative solutions emanating from the same initial data.

Keywords: Euler-Poisson system, weak solution, dissipative solution

Contents

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1 Introduction

We consider the *Euler-Poisson system* of partial differential equations in the form

$$\partial_t n + \mathrm{div}_x \mathbf{J} = 0, \tag{1.1}$$

$$\partial_t \mathbf{J} + \mathrm{div}_x \left(\frac{\mathbf{J} \times \mathbf{J}}{n} \right) + \nabla_x (nT) = \pm n \mathbf{V}, \tag{1.2}$$

$$\frac{3}{2}\partial_t (nT) + \frac{3}{2}\mathrm{div}_x (T\mathbf{J}) - \Delta T + nT \mathrm{div}_x \left(\frac{\mathbf{J}}{n} \right) = 0, \tag{1.3}$$

$$\mathbf{V} = \nabla_x \Phi, \ -\Delta \Phi = n - 1. \tag{1.4}$$

In specific applications, n is the density, $\mathbf{J}$ the flux, and T the (absolute) temperature of charged particles, driven by the potential volume force proportional to $n\nabla_x \Phi$, see Guo [10], Guo and Pausader [11], Juengel [13], among others. From the mathematical viewpoint, the equations (1.1), (1.2) represent a hyperbolic Euler system, with the density n and the velocity $\mathbf{J}/n$, coupled with a parabolic "heat equation" (1.3), and the elliptic Poisson equation (1.4).

To avoid the technical problems caused by the presence of a kinematic boundary, we restrict ourselves to the spatially periodic boundary conditions, specifically all quantities are functions of the time $t \in (0, \tau)$ and the position x, belonging to the flat torus

$$\mathcal{T} = \Big([-1, 1]|_{\{-1,1\}} \Big)^3 .$$

Accordingly, the problem is formally closed by prescribing the initial conditions

$$n(0,\cdot) = n_0,\ \mathbf{J}(0,\cdot) = \mathbf{J}_0,\ T(0,\cdot) = T_0. \tag{1.5}$$

For smooth and physically relevant initial data, meaning

$$n_0(x) \geq \underline{n} > 0,\ T_0(x) \geq \underline{T} > 0 \text{ for all } x \in \mathcal{T}, \tag{1.6}$$

the problem (1.1 - 1.5) admits a unique regular solution on a maximal existence interval $(0, \tau_{\max})$, see Alazard [1], Serre [15], [16]. On the other hand, the Euler system (1.1), (1.2) being hyperbolic, discontinuities in the form of shock waves are likely to develop in a finite time regardless the smoothness of the initial data, see Guo and Tahvildar-Zadeh [12]. However, as observed by Guo [10], the linearized Euler-Poisson system (1.1), (1.2), (1.4) (with $T = \text{const}$) coincides with the Klein-Gordon equation, where the dispersive effects due "plasma" oscillations prevents the formation of shocks in small irrotational solutions.

In view of these arguments, it is interesting to examine the problem of global existence in the class of weak solutions, satisfying, in addition, certain *admissibility criteria* that would guarantee well-posedness, that means, existence, uniqueness, and possibly stability for any physically relevant initial data.

Motivated by [8], we introduce a relative entropy (energy) functional associated to the system (1.1 - 1.5), together with a class of *dissipative* weak solutions. These are, roughly speaking, the weak solutions satisfying, in addition, the total energy balance. Then we show the weak-strong uniqueness property, namely, any dissipative solution coincides with the strong solution emanating from the same initial data as long as the latter exists. The strong solutions are unique within the class of weak solutions (cf. Berthelin and Vasseur [2], Dafermos [5], Germain [9] for related results).

The last part of the paper is devoted to the problem of well-posedness in the class of weak and/or dissipative solutions. Using an extension of the *variable coefficients* analogue of the results of DeLellis and Székelyhidi [6] developed in [4], we show that the Euler-Poisson system (1.1 - 1.5) admits infinitely many global-in-time weak solutions for any smooth initial data. Although one can still hope that some apparently non-physical solutions can be eliminated by imposing the total energy balance as an admissibility criterion (dissipative weak solutions), we identify a vast class of physically admissible initial data for which the problem possesses infinitely many dissipative weak solutions.

The paper is organized as follows. Section 2 contains some preliminary material including proper definitions of the weak and dissipative solutions. In Section 3, we show the weak-strong uniqueness property for the dissipative solutions. In Section 4, the existence of global-in-time weak solutions is established for any physically admissible smooth initial data. Some examples of ill-posedness within the class of dissipative weak solutions are discussed in Section 5. The various concepts of solutions and their basic properties are summarized in Section 6.

2 Preliminaries, weak and dissipative solutions

We start rewriting (1.3) as an entropy balance. Specifically, dividing (1.3) on T and using (1.1), we arrive at

$$\partial_t \left(nS(n,T) \right) + \mathrm{div}_x \left(S(n,T)\mathbf{J} \right) - \Delta \log(T) = |\nabla_x \log(T)|^2, \tag{2.1}$$

with the specific entropy

$$S(n,T) = \log \left(\frac{T^{3/2}}{n} \right).$$

2.1 Weak solutions

We say that $[n, \mathbf{J}, T]$ is a *weak solution* to the problem (1.1 - 1.5) in $(0,\tau) \times \mathcal{T}$ if:

- $n > 0$, $T > 0$ a.a. in $[0,\tau) \times \mathcal{T}$, $\int_{\mathcal{T}} (n-1)\ \mathrm{d}x = 0$,

$$n \in L^\infty((0,\tau)\times\mathcal{T}) \cap C([0,\tau]; L^1(\mathcal{T})),\ \mathbf{J} = L^\infty((0,\tau)\times\mathcal{T}; R^3) \cap C_{\mathrm{weak}}([0,\tau]; L^2(\mathcal{T}; R^3)),$$

$$T \in L^\infty((0,\tau)\times\mathcal{T}) \cap C([0,\tau]; L^1(\mathcal{T})),\ \nabla_x \vartheta \in L^2((0,\tau)\times\mathcal{T}; R^3);$$

- the equation of continuity (1.1) is replaced by a family of integral identities

$$\int_{\mathcal{T}} \left[n(s,\cdot)\varphi(s,\cdot) - n_0\varphi(0,\cdot) \right]\ \mathrm{d}x = \int_0^s \int_{\mathcal{T}} \left[n\partial_t\varphi + \mathbf{J}\cdot\nabla_x\varphi \right]\ \mathrm{d}x\ \mathrm{d}t \tag{2.2}$$

  for any $s \in [0,\tau)$ and any test function $\varphi \in C^\infty([0,\tau]\times\mathcal{T})$;

- the momentum balance (1.2) is satisfied in the sense that

$$\int_{\mathcal{T}} \left[\mathbf{J}(s,\cdot)\cdot\varphi(s,\cdot) - \mathbf{J}_0\cdot\varphi(0,\cdot) \right]\ \mathrm{d}x \tag{2.3}$$

$$= \int_0^s \int_{\mathcal{T}} \left[\mathbf{J}\cdot\partial_t\varphi + \frac{\mathbf{J}\otimes\mathbf{J}}{n} : \nabla_x\varphi + nT\mathrm{div}_x\varphi \pm n\nabla_x\Phi\cdot\varphi \right]\ \mathrm{d}x\ \mathrm{d}t$$

  holds for any $s \in [0,\tau)$ and any test function $\varphi \in C^\infty([0,\tau]\times\mathcal{T}; R^3)$;

- the heat equation (1.3) is replaced by a weak form of the entropy balance (2.1), specifically, the integral identity

$$\int_{\mathcal{T}} \left[nS(n,T)(s,\cdot)\varphi(s,\cdot) - n_0S(n_0,T_0)\varphi(0,\cdot) \right]\ \mathrm{d}x = \int_0^s \int_{\mathcal{T}} |\nabla_x \log(T)|^2\ \mathrm{d}x\ \mathrm{d}t \tag{2.4}$$

$$+ \int_0^s \int_{\mathcal{T}} \left[nS(n,T)\partial_t\varphi + S(n,T)\mathbf{J}\cdot\nabla_x\varphi - \nabla_x\log(T)\cdot\nabla_x\varphi \right]\ \mathrm{d}x\ \mathrm{d}t$$

  holds for any $s \in [0,\tau)$ and any test function $\varphi \in C^\infty([0,\tau]\times\mathcal{T})$;

- the potential Φ is the (unique) solution of the elliptic equation

$$-\Delta\Phi(s,\cdot) = n(s,\cdot) - 1 \text{ in } \mathcal{T},\ \int_{\mathcal{T}} \Phi(s,\cdot)\ \mathrm{d}x = 0 \text{ for all } s \in [0,\tau). \tag{2.5}$$

Remark 2.1 *The condition*

$$\int_{\mathcal{T}} (n-1)\ \mathrm{d}x = 0$$

can be replaced by

$$\int_{\mathcal{T}} (n-\overline{n})\ \mathrm{d}x = 0 \textit{ for a certain } \overline{n} > 0.$$

Accordingly, we have to take the potential Φ *such that*

$$-\Delta\Phi(s,\cdot) = n(s,\cdot) - \overline{n} \textit{ in } \mathcal{T}.$$

Remark 2.2 *Apparently, replacing the heat equation (1.3) by the entropy balance (2.1) may not be an equivalent operation within the framework of weak solutions. On the other hand, however, the density* n *as well as the temperature* T *considered in the present paper will be regular enough for (2.1) to imply (1.3) and vice versa. The entropy formulation (2.4) is more convenient for introducing the concept of dissipative solution discussed below.*

2.2 Relative entropy (energy), dissipative solutions

Similarly to [8], we introduce the ballistic free energy

$$H_\Theta(n,T) = n\left(\frac{3}{2}T - \Theta S(n,T)\right),$$

together with the relative entropy functional

$$\mathcal{E}\left(n,T,\mathbf{J}\middle|N,\Theta,\mathbf{V}\right) \tag{2.6}$$

$$= \int_{\mathcal{T}} \left[\frac{1}{2}n\left|\frac{\mathbf{J}}{n} - \frac{\mathbf{V}}{N}\right|^2 + \left(H_\Theta(n,T) - \frac{\partial H_\Theta(N,\Theta)}{\partial N}(n-N) - H_\Theta(N,\Theta)\right)\right]\ \mathrm{d}x.$$

Remark 2.3 *The relative entropy (2.6) coincides, modulo the multiplicative factor* Θ*, with the relative entropy introduced in the context of hyperbolic conservation laws by Dafermos [5]. Thus, correctly speaking, the physical dimension of* $\mathcal{E}$ *is* energy *rather than* entropy*.*

2.3 Relative entropy inequality

As we can check by direct manipulation, regular solutions of the system (1.1 - 1.5) satisfy the relative entropy inequality

$$\left[\mathcal{E}\left(n,T,\mathbf{J}\middle|N,\Theta,\mathbf{V}\right)\right]_{t=0}^{t=s}+\int_0^s\int_{\mathcal{T}}\Theta\frac{|\nabla_x T|^2}{T^2}\ \mathrm{d}x\ \mathrm{d}t \tag{2.7}$$

$$\leq\int_0^s\int_{\mathcal{T}}\left[\frac{1}{N}(n\mathbf{V}-N\mathbf{J})\cdot\partial_t\left(\frac{\mathbf{V}}{N}\right)+\frac{1}{nN}(n\mathbf{V}-N\mathbf{J})\otimes\mathbf{J}:\nabla_x\left(\frac{\mathbf{V}}{N}\right)-nT\mathrm{div}_x\left(\frac{\mathbf{V}}{N}\right)\right]\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^s\int_{\mathcal{T}}\left[n\Big(S(n,T)-S(N,\Theta)\Big)\partial_t\Theta+\Big(S(n,T)-S(N,\Theta)\Big)\mathbf{J}\cdot\nabla_x\Theta\right]\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^s\int_{\mathcal{T}}\left[\left(1-\frac{n}{N}\right)\partial_t(N\Theta)-\frac{\mathbf{J}}{N}\cdot\nabla_x(N\Theta)\right]\ \mathrm{d}x+\int_0^s\int_{\mathcal{T}}\frac{\nabla_x T}{T}\cdot\nabla_x\Theta\ \mathrm{d}x\ \mathrm{d}t$$

$$\pm\int_0^s\int_{\mathcal{T}}\frac{1}{N}\nabla_x\Delta^{-1}[n-1]\,(n\mathbf{V}-N\mathbf{J})\ \mathrm{d}x\ \mathrm{d}t$$

for a.a. $s\in[0,\tau)$ and any trio of smooth "test" functions

$$N,\ \Theta,\ \mathbf{V},\ N>0,\ \Theta>0, \tag{2.8}$$

cf. [8].

2.4 Dissipative solutions

We say that a trio $[n,\mathbf{J},T]$ is a *dissipative solution* to the problem (1.1 - 1.5) in $(0,\tau)\times\mathcal{T}$ if:

- $n>0$, $T>0$ a.a. in $[0,\tau)\times\mathcal{T}$, $\int_{\mathcal{T}}(n-1)\ \mathrm{d}x=0$,

$$n\in L^\infty((0,\tau)\times\mathcal{T})\cap C([0,\tau];L^1(\mathcal{T})),\ \mathbf{J}=L^\infty((0,\tau)\times\mathcal{T};R^3)\cap C_{\mathrm{weak}}([0,\tau];L^2(\mathcal{T};R^3)),$$

$$T\in L^\infty((0,\tau)\times\mathcal{T})\cap C([0,\tau];L^1(\mathcal{T})),\ \nabla_x\vartheta\in L^2((0,\tau)\times\mathcal{T};R^3);$$

- the relative entropy inequality (2.7) holds for any choice of smooth test functions $N,\Theta,\mathbf{V}$ satisfying (2.8).

3 Weak strong uniqueness

The concept of *dissipative solution* in the context of the incompressible Euler system was introduced by DiPerna and Lions (see [14]). It is interesting to note that the dissipative solutions apparently do not satisfy any system of differential equations but just the relative entropy inequality (2.4). However, the following *weak-strong uniqueness property* holds:

Theorem 3.1 *Let* $[n, T, \mathbf{J}]$ *be a dissipative solution of the problem (1.1 - 1.5) in* $(0, \tau) \times \mathcal{T}$*, with the initial data* $[n_0, T_0, \mathbf{J}_0]$ *satisfying (1.6). Suppose that the problem (1.1 - 1.5) admits also a regular solution* $[\tilde{n}, \tilde{T}, \tilde{\mathbf{J}}]$*,*

$$\partial_t \tilde{n},\ \partial_t \tilde{T},\ \partial_t \tilde{\mathbf{J}},\ \partial_x^m \tilde{n},\ \partial_x^m \tilde{T},\ \partial_x^m \tilde{\mathbf{J}} \in C([0, \tau) \times \mathcal{T}),\ m = 0, 1, 2,$$

emanating form the same initial data $[n_0, T_0, \mathbf{J}_0]$*.*

Then

$$n \equiv \tilde{n},\ T \equiv \tilde{T},\ \mathbf{J} \equiv \tilde{\mathbf{J}} \ in\ [0, \tau) \times \mathcal{T}.$$

The proof of Theorem 3.1 is based on taking $N = \tilde{n}$, $\Theta = \tilde{T}$, $\mathbf{V} = \tilde{\mathbf{J}}$ as test functions in the relative entropy inequality (2.4) and "absorbing" the terms on the right-hand side of the resulting expression by means of the Gronwall argument. Since the dissipative solutions introduced in Section 2.4 are *bounded*, the proof of Theorem 3.1 is essentially the same as that of Theorem 6.1 [7, Section 6]. Note that the extra term

$$\pm \int_0^s \int_{\mathcal{T}} \frac{1}{N} \nabla_x \Delta^{-1} [n - 1] \left(n\mathbf{V} - N\mathbf{J} \right)\ \mathrm{d}x\ \mathrm{d}t$$

in (2.4) can be handled without any additional difficulty.

Finally, note that the existence of *local-in-time* regular solutions to the problem (1.1 - 1.5) ranging in the standard energy Sobolev scale $W^{m,2}$ was established by Alazard [1], Serre [15], while the existence of possibly global-in-time dissipative solutions remains an oustanding open problem.

4 Existence of weak solutions for physically relevant data

In the remaining part of the paper, we focus on the class of *weak* solutions to (1.1 - 1.5), and, in particular, on their relation to the dissipative solutions.

4.1 Global-in-time weak solutions

We start with a rather striking result concerning the existence of global-in-time weak solutions in the sense specified in Section 2.1.

Theorem 4.1 *Let $\tau > 0$ be given. Suppose that the initial data $[n_0, T_0, \mathbf{J}_0]$,*

$$n_0, T_0, \mathbf{J}_0 \in C^3(\mathcal{T}),$$

satisfy (1.6).

Then the problem (1.1 - 1.5) possesses infinitely many weak solutions in $[0,\tau) \times \mathcal{T}$. In addition, the weak solutions $[n, T, \mathbf{J}]$ belong to the class

$$n \in C^2([0,\tau) \times \mathcal{T}),\ \partial_t T \in L^p(0,T; L^p(\mathcal{T})),\ \nabla_x^2 T \in L^p(0,\tau; L^p(\mathcal{T}; R^{3\times 3})) \text{ for any } 1 \le p < \infty,$$

$$\mathbf{J} \in C_{\rm weak}([0,\tau]; L^2(\mathcal{T}; R^3)) \cap L^\infty((0,\tau) \times \mathcal{T}; R^3),\ {\rm div}_x \mathbf{J} \in C^2([0,\tau) \times \mathcal{T}).$$

Remark 4.1 *It is easy to check that any weak solution enjoying the regularity properties specified in Theorem 4.1 satisfies the equation of continuity (1.1), the entropy balance equation (2.1) as well as the internal energy equation (1.3) a.a. in $[0,\tau) \times \mathcal{T}$. All possible singularities are therefore concentrated on the solenoidal component of the flux $\mathbf{J}$. On the other hand, the solutions are neither regular nor dissipative solutions of the problem in agreement with the conclusion of Theorem 3.1*

The remaining part of this section is devoted to the proof of Theorem 4.1.

4.1.1 Oscillatory lemma

Similarly to [4], the weak solutions claimed in Theorem 4.1 are obtained by the method of convex integration, in particular, an extension to "variable coefficients" of the following result of De Lellis and Székelyhidi [6, Proposition 3], Chiodaroli [3, Section 6, formula (6.9)]:

Lemma 4.1 *Let* $[T_1, T_2]$, $T_1 < T_2$, *be a time interval and* $B \subset R^3$ *a domain. Let* $\tilde{n} \in (0, \infty)$, $\tilde{\mathbf{Z}} \in R^3$, $\tilde{\mathbb{U}} \in R^{3\times 3}_{\rm sym,0}$ *be* constant *fields such that*

$$0 < \underline{n} < \tilde{n} < \overline{n}, |\tilde{\mathbf{Z}}| < \overline{Z},\ |\tilde{\mathbb{U}}| < \overline{U}.$$

Suppose that

$$\mathbf{v} \in C_{\rm weak}([T_1, T_2]; L^2(B, R^3)) \cap C^1((T_1, T_2) \times \overline{B}; R^3)$$

satisfies a linear system of equations

$$\partial_t \mathbf{v} + {\rm div}_x \mathbb{U} = 0,\ {\rm div}_x \mathbf{v} = 0 \ in\ (T_1, T_2) \times B$$

with some $\mathbb{U} \in C^1((T_1, T_2) \times \overline{B}; R^{3\times 3}_{\rm sym,0})$ *such that*

$$\frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \tilde{\mathbf{Z}}) \otimes (\mathbf{v} + \tilde{\mathbf{Z}})}{\tilde{n}} - \frac{1}{3} \frac{|\mathbf{v} + \tilde{\mathbf{Z}}|^2}{\tilde{n}} \mathbb{I} - (\mathbb{U} + \tilde{\mathbb{U}}) \right] < e - \frac{1}{2} \frac{|\mathbf{v} + \tilde{\mathbf{Z}}|^2}{\tilde{n}} \ in\ (T_1, T_2) \times B$$

for a certain function $e \in C([T_1; T_2] \times \overline{B})$.

Then there exist sequences $\{\mathbf{w}_n\}_{n=1}^\infty \subset C^\infty_c((T_1, T_2) \times B; R^3)$, $\{\mathbb{Y}_n\}_{n=1}^\infty \subset C^\infty_c((T_1, T_2) \times B; R^{3\times 3}_{\rm sym,0})$ *such that* $\mathbf{v}_n = \mathbf{v} + \mathbf{w}_n$, $\mathbb{U}_n = \mathbb{U} + \mathbb{Y}_n$ *satisfy*

$$\partial_t \mathbf{v}_n + {\rm div}_x \mathbb{U}_n = 0,\ {\rm div}_x \mathbf{v}_n = 0 \ in\ (T_1, T_2) \times B$$

$$\frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v}_n + \tilde{\mathbf{Z}}) \otimes (\mathbf{v}_n + \tilde{\mathbf{Z}})}{\tilde{n}} - \frac{1}{3} \frac{|\mathbf{v}_n + \tilde{\mathbf{Z}}|^2}{\tilde{n}} \mathbb{I} - (\mathbb{U}_n + \mathbb{U}) \right] < e - \frac{1}{2} \frac{|\mathbf{v}_n + \tilde{\mathbf{Z}}|^2}{\tilde{n}} \ in\ (T_1, T_2) \times B,$$

$$\mathbf{v}_n \to \mathbf{v} \in C_{\rm weak}([T_1, T_2]; L^2(B; R^3)),$$

and

$$\liminf_{n\to\infty} \int_{T_1}^{T_2} \int_B |\mathbf{v}_n - \mathbf{v}|^2 \ {\rm d}x \ {\rm d}t \geq \Lambda \left(\underline{r}, \overline{n}, \overline{Z}, \overline{U}, \|e\|_{L^\infty((T_1,T_2)\times B)} \right) \int_{T_1}^{T_2} \int_B \left(e - \frac{1}{2} \frac{|\mathbf{v} + \tilde{\mathbf{Z}}|^2}{\tilde{n}} \right)^2 \ {\rm d}x \ {\rm d}t. \tag{4.1}$$

Remark 4.2 *The symbol* $\lambda_{\max}[\mathbb{A}]$ *denotes the maximal eigenvalue of a symmetric traceless matrix* $\mathbb{A} \in R^{3\times 3}_{\rm sym,0}$.

Remark 4.3 *Since* $\tilde{\mathbf{v}}$, $\tilde{\mathbb{U}}$ *are constant, we have*

$$\partial_t (\mathbf{v} + \tilde{\mathbf{v}}) + {\rm div}_x (\mathbb{U} + \tilde{\mathbb{U}}) = 0.$$

Now, exactly as in [4, Section 3.4.1, Lemma 3.2], we can use the scale invariance of (4.1) to extend validity of Lemma 4.1 to *non-constant* fields $\tilde{n}$, $\tilde{\mathbf{Z}}$, $\tilde{\mathbb{U}}$:

Lemma 4.2 *Let $[T_1, T_2]$, $T_1 < T_2$, be a time interval and $B \subset R^3$ a domain. Let*

$$\tilde{n} \in C^1([T_1, T_2] \times \overline{B}), \ \ \tilde{\mathbf{Z}} \in C^1([T_1, T_2] \times \overline{B}; R^3), \ \ \tilde{\mathbb{U}} \in C^1([T_1, T_2] \times \overline{B}; R^{3\times 3}_{\mathrm{sym},0})$$

be given such that

$$0 < \underline{n} < \tilde{n} < \overline{n}, |\tilde{\mathbf{Z}}| < \overline{Z}, \ \ |\tilde{\mathbb{U}}| < \overline{U} \ \ in \ (T_1, T_2) \times B.$$

Suppose that

$$\mathbf{v} \in C_{\mathrm{weak}}([T_1, T_2]; L^2(B, R^3)) \cap C^1((T_1, T_2) \times \overline{B}; R^3)$$

satisfies a linear system of equations

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \ \mathrm{div}_x \mathbf{v} = 0 \ \ in \ (T_1, T_2) \times B$$

with some $\mathbb{U} \in C^1((T_1, T_2) \times \overline{B}; R^{3\times 3}_{\mathrm{sym},0})$ such that

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\tilde{\mathbf{Z}})\otimes(\mathbf{v}+\tilde{\mathbf{Z}})}{\tilde{n}} - \frac{1}{3}\frac{|\mathbf{v}+\tilde{\mathbf{Z}}|^2}{\tilde{n}}\mathbb{I} - (\mathbb{U}+\tilde{\mathbb{U}})\right] < e - \frac{1}{2}\frac{|\mathbf{v}+\tilde{\mathbf{Z}}|^2}{\tilde{n}} - \delta \ \ in \ (T_1, T_2) \times B$$

for a certain function $e \in C([T_1; T_2] \times \overline{B})$ and some $\delta > 0$.

Then there exist sequences $\{\mathbf{w}_n\}_{n=1}^{\infty} \subset C^{\infty}_c((T_1, T_2) \times B; R^3)$, $\{\mathbb{Y}_n\}_{n=1}^{\infty} \subset C^{\infty}_c((T_1, T_2) \times B; R^{3\times 3}_{\mathrm{sym},0})$ such that $\mathbf{v}_n = \mathbf{v} + \mathbf{w}_n$, $\mathbb{U}_n = \mathbb{U} + \mathbb{Y}_n$ satisfy

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0, \ \ \mathrm{div}_x \mathbf{v}_n = 0 \ \ in \ (T_1, T_2) \times B$$

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_n+\tilde{\mathbf{Z}})\otimes(\mathbf{v}_n+\tilde{\mathbf{Z}})}{\tilde{n}} - \frac{1}{3}\frac{|\mathbf{v}_n+\tilde{\mathbf{Z}}|^2}{\tilde{n}}\mathbb{I} - (\mathbb{U}_n+\tilde{\mathbb{U}})\right] < e - \frac{1}{2}\frac{|\mathbf{v}_n+\tilde{\mathbf{Z}}|^2}{\tilde{n}} \ \ in \ (T_1, T_2) \times B,$$

$$\mathbf{v}_n \to \mathbf{v} \in C_{\mathrm{weak}}([T_1, T_2]; L^2(B; R^3)),$$

and

$$\liminf_{n\to\infty} \int_{T_1}^{T_2}\int_B |\mathbf{v}_n - \mathbf{v}|^2 \ \mathrm{d}x \ \mathrm{d}t \geq \Lambda\left(\underline{r}, \overline{n}, \overline{Z}, \overline{U}, \|e\|_{L^\infty((T_1,T_2)\times B)}\right)\int_{T_1}^{T_2}\int_B \left(e - \frac{1}{2}\frac{|\mathbf{v}+\tilde{\mathbf{Z}}|^2}{\tilde{n}}\right)^2 \ \mathrm{d}x \ \mathrm{d}t. \tag{4.2}$$

Remark 4.4 *The present result may be viewed as a generalization of [4, Section 3.4.1, Lemma 3.2] to the case, where* both $\mathbf{v}$ *and* $\mathbb{U}$ *are perturbed by smooth fields $\tilde{Z}$, $\tilde{\mathbb{U}}$, respectively.*

4.1.2 Reformulation of the problem

Step 1:

We start by writing the field $\mathbf{J}$ is terms of its Helmholtz decomposition,

$$\mathbf{J} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0.$$

We observe that

$$\partial_t n = -\Delta \Psi,$$

in other words, the time evolution of n is governed by the gradient component of $\mathbf{J}$. Consequently, we can fix a smooth trajectory $t \mapsto n(t, \cdot)$ in such a way that

$$n(0, \cdot) = n_0, \ \int_{\mathcal{T}} (n(t, \cdot) - 1) \ \mathrm{d}x = 0, \ 0 < \underline{n} < n(t, \cdot) < \overline{n} \text{ for all } t \in [0, \tau],$$

and compute

$$\Psi(t, \cdot) = -\Delta^{-1}[\partial_t n], \ \int_{\mathcal{T}} \Psi(t, \cdot) \ \mathrm{d}x = 0.$$

Finally, we choose $\partial_t n(0, \cdot)$ in such a way that

$$\nabla_x \Psi(0, t) = \nabla_x \Psi_0, \ \mathbf{J}_0 = \mathbf{v}_0 + \nabla_x \Psi_0, \ \mathrm{div}_x \mathbf{v}_0 = 0.$$

Step 2:

Given n, we compute the potential Φ,

$$-\Delta \Phi(t, \cdot) = n(t, \cdot) - 1, \ \int_{\mathcal{T}} \Phi(t, \cdot) \ \mathrm{d}x = 0,$$

where, incidentally,

$$\partial_t \Phi = \Psi.$$

Step 3:

Having fixed n, Ψ, and Φ, we may express the temperature $T = T[\mathbf{v}]$ using the internal energy balance (1.3). Exactly as in [4, Section 3.2], we deduce that for given n, Ψ, and

$$\mathbf{v} \in L^\infty((0, \tau) \times \mathcal{T}; R^3), \ \mathbf{J} = \mathbf{v} + \nabla_x \Psi,$$

the equation (1.3) admits a unique solution $T = T[\mathbf{v}]$ such that

$$T(t, x) > 0 \text{ for all } t \in [0, \tau], \ x \in \mathcal{T},$$

$$\partial_t T \in L^p(0,\tau; L^p(\mathcal{T})),\ \nabla_x^2 T \in L^p(0,\tau; L^p(\mathcal{T}; R^{3\times 3}) \text{ for any } 1 \le p < \infty, \tag{4.3}$$

where the bounds depend only on the data and on $\|\mathbf{v}\|_{L^\infty((0,\tau)\times\mathcal{T};R^3)}$. Moreover, it follows from the entropy balance (2.1) that

$$0 < \underline{T} \le T(t,x) < \overline{T} \text{ for all } t \in [0,\tau],\ x \in \mathcal{T}, \tag{4.4}$$

where the constants $\underline{T}$, $\overline{T}$ are *independent* of $\mathbf{v}$.

Step 4:

Finally, we rewrite the momentum equation as follows:

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{n} \pm \nabla_x \Phi \otimes \nabla_x \Phi \right) + \nabla_x \left(nT + \partial_t \Psi \mp \left(\Phi + \frac{1}{2} |\nabla_x \Phi|^2 \right) \right) = 0, \tag{4.5}$$

$$\mathrm{div}_x \mathbf{v} = 0,\ \mathbf{v}(0,\cdot) = \mathbf{v}_0, \tag{4.6}$$

where we have used the relation

$$n \nabla_x \Phi = \nabla_x \Phi + \frac{1}{2} \nabla_x |\nabla_x \Phi|^2 - \mathrm{div}_x \left(\nabla_x \Phi \otimes \nabla_x \Phi \right).$$

4.1.3 Application of the method of convex integration

We are ready to complete the proof of Theorem 4.1 following step by step the arguments of [4, Section 3.3]. To begin, we introduce the energy

$$\overline{e}[\mathbf{v}] = \chi - \frac{3}{2} nT[\mathbf{v}] - \frac{3}{2} \left(\partial_t \Psi \mp \left(\Phi + \frac{1}{6} |\nabla_x \Phi|^2 \right) \right),\ \chi \in C^1[0,\tau], \tag{4.7}$$

together with the associated space of subsolutions:

$$X_0 = \Bigg\{ \mathbf{v} \ \Big| \ \mathbf{v} \in L^\infty((0,\tau)\times\mathcal{T}; R^3) \cap C^1((0,\tau)\times\mathcal{T}; R^3) \cap C_{\mathrm{weak}}([0,\tau]; L^2(\Omega; \mathcal{T})), \tag{4.8}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0,\ \mathrm{div}_x \mathbf{v} = 0 \text{ in } (0,\tau)\times\mathcal{T} \text{ for some } \mathbb{U} \in C^1((0,\tau)\times\mathcal{T}; R^{3\times 3}_{\mathrm{sym},0}),$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}(\tau,\cdot) = \mathbf{v}_0,$$

$$\inf_{t\in(\varepsilon,T), x\in\mathcal{T}} \left\{ \overline{e}[\mathbf{v}] - \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{n} - \left(\mathbb{U} \mp \nabla_x \Phi \otimes \nabla_x \Phi - \frac{1}{3} |\nabla_x \Phi|^2 \mathbb{I} \right) \right] \right\} > 0$$

$$\text{for any } 0 < \varepsilon < T \Bigg\}.$$

Next, as a consequence of (4.4), we can fix $\chi \in C^1[0,\tau]$ in (4.7) so that

$$\frac{3}{2}\lambda_{\max}\left[\frac{(\mathbf{v}_0+\nabla_x\Psi)\otimes(\mathbf{v}_0+\nabla_x\Psi)}{n}\mp\left(\nabla_x\Phi\otimes\nabla_x\Phi-\frac{1}{3}|\nabla_x\Phi|^2\mathbb{I}\right)\right]<\bar{e}[\mathbf{v}_0] \text{ in } [0,\tau]\times\mathcal{T},$$

in particular, the constant function $\mathbf{v}\equiv\mathbf{v}_0$, together with $\mathbb{U}\equiv 0$, belong to X_0. Note that, as shown by De Lellis and Székelyhidi [6],

$$\frac{1}{2}|\mathbf{w}|^2\leq\frac{3}{2}\lambda_{\max}\left[\mathbf{w}\otimes\mathbf{w}-\mathbb{U}\right],\ \mathbf{w}\in R^3,\ \mathbb{U}\in R^{3\times 3}_{\mathrm{sym},0},$$

where the identity holds only if

$$\mathbb{U}=\mathbf{w}\otimes\mathbf{w}-\frac{1}{3}|\mathbf{w}|^2\mathbb{I}.$$

Now, exactly as in [4, Section 3.3], we define a *topological* space X as a completion of X_0 with respect to the metrizable topology of $C_{\mathrm{weak}}([0,\tau];L^2(\mathcal{T};R^3))$, noting that the space X_0 is non-empty as $\mathbf{v}=\mathbf{v}_0\in X_0$.

We introduce a family of functionals

$$I_\varepsilon[\mathbf{v}]=\int_\varepsilon^\tau\int_{\mathcal{T}}\left(\frac{1}{2}\frac{|\mathbf{v}+\nabla_x\Psi|^2}{n}-\bar{e}[\mathbf{v}]\right)\,\mathrm{d}x\,\mathrm{d}t \text{ for } \mathbf{v}\in X,\ 0<\varepsilon<\tau. \tag{4.9}$$

As a consequence of (4.3), (4.4), the functionals I_ε are lower-semicontinuous in X.

Finally, we claim the following assertion that can be deduced from Lemma 4.2 by means of the arguments used in [4, Section 3.4]:

Lemma 4.3 *Let $\mathbf{v}\in X_0$ such that*

$$I_\varepsilon[\mathbf{v}]<-\alpha<0,\ 0<\varepsilon<\tau/2.$$

There there is $\beta=\beta(\alpha)>0$ and a sequence $\{\mathbf{v}_n\}_{n=1}^\infty\subset X_0$ such that

$$\mathbf{v}_n\to\mathbf{v} \textit{ in } C_{\mathrm{weak}}([0,\tau];L^2(\mathcal{T};R^3)),\ \liminf_{n\to\infty} I_\varepsilon[\mathbf{v}_n]\geq I_\varepsilon[\mathbf{v}]+\beta.$$

Following the arguments of [6] we obtain:

(i) cardinality of the space X_0 is infinite;
(ii) the points of continuity of each I_ε form a residual set in X;
(iii) the set

$$\mathcal{C}=\bigcap_{m>1}\left\{\mathbf{v}\in X \mid I_{1/m}[\mathbf{v}] \text{ is continuous }\right\},$$

being an intersection of a countable family of residual sets, is residual, in particular of infinite cardinality;
(iv)

$$I_{1/m}[\mathbf{v}] = 0 \text{ for all } m > 1 \tag{4.10}$$

for each $\mathbf{v} \in \mathcal{C}$.

The relation (4.10) implies that

$$\frac{1}{2}\frac{|\mathbf{v} + \nabla_x \Psi|^2}{n} = \bar{e}[\mathbf{v}] = \chi - \frac{3}{2} n T[\mathbf{v}] - \frac{3}{2}\left(\partial_t \Psi \mp \left(\Phi + \frac{1}{6}|\nabla_x \Phi|^2\right)\right)$$

for any $\mathbf{v} \in \mathcal{C}$,

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0 \text{ in the sense of distributions in } (0,\tau) \times \mathcal{T},$$

where

$$\mathbb{U} = \frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{n} - \frac{1}{3}\frac{|\mathbf{v} + \nabla_x \Psi|^2}{n}\mathbb{I} \pm \left(\nabla_x \Phi \otimes \nabla_x \Phi - \frac{1}{3}|\nabla_x \Phi|^2 \mathbb{I}\right).$$

In other words, $\mathbf{v}$ satisfies (4.5), (4.6) in the sense of distributions. We have proved Theorem 4.1.

Remark 4.5 *The reader will have noticed that our construction of the weak solutions enables to* prescribe *the value of the density also for $t = \tau$.*

5 Well-posedness in the class of dissipative weak solutions

As we have seen in the previous part, although the problem (1.1-1.5) admits global-in-time weak solutions, it is not well-posed in this class. On the other hand, the dissipative solutions enjoy the property of weak-strong uniqueness, meaning they coincide with the unique (local) strong solution as long as the latter exists. We introduce an intermediate class of *dissipative weak* solutions, specifically, the weak solutions satisfying the relative entropy inequality (2.7).

As shown in [7], a weak solution is a dissipative solution as soon as it satisfies the total energy balance:

$$\int_{\mathcal{T}} \left[\frac{1}{2}\frac{|\mathbf{J}|^2}{n} + \frac{3}{2} n T \mp \frac{1}{2} n \Phi\right](s,\cdot)\ \mathrm{d}x = \int_{\mathcal{T}} \left[\frac{1}{2}\frac{|\mathbf{J}_0|^2}{n_0} + \frac{3}{2} n_0 T_0 \mp \frac{1}{2} n_0 \Phi_0\right]\ \mathrm{d}x \text{ for a.a. } s \in (0,\tau), \tag{5.1}$$

where we have use the identities

$$\int_{\mathcal{T}} \nabla_x \Phi \cdot \mathbf{J}\ \mathrm{d}x = -\int_{\mathcal{T}} \Phi \mathrm{div}_x \mathbf{J}\ \mathrm{d}x = \int_{\mathcal{T}} \Phi \partial_t n\ \mathrm{d}x = -\int_{\mathcal{T}} \Phi \partial_t \Delta \Phi\ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathcal{T}} |\nabla_x \Phi|^2\ \mathrm{d}x,$$

and

$$\int_{\mathcal{T}} |\nabla_x \Phi|^2\ \mathrm{d}x = \int_{\mathcal{T}} n\Phi\ \mathrm{d}x.$$

Note that (5.1) combined with the weak formulation (2.2 - 2.5), is, in fact, stronger than the relative entropy inequality (2.7), where the latter still holds if (5.1) is replaced by an *inequality*:

$$\int_{\mathcal{T}} \left[\frac{1}{2} \frac{|\mathbf{J}|^2}{n} + \frac{3}{2} nT \mp \frac{1}{2} n\Phi \right] (s, \cdot) \ \mathrm{d}x = \int_{\mathcal{T}} \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{n_0} + \frac{3}{2} n_0 T_0 \mp \frac{1}{2} n_0 \Phi_0 \right] \ \mathrm{d}x \text{ for a.a. } s \in (0, \tau).$$

A weak solution of the problem (1.1 - 1.5) satisfying the energy equality (5.1) will be termed *finite energy weak solution.* As a direct consequence of [7, Theorem 6.1] (cf. Theorem 3.1 above) we obtain:

Theorem 5.1 *Let* $[n, T, \mathbf{J}]$ *be a finite energy weak solution of the problem (1.1 - 1.5) in* $(0, \tau) \times \mathcal{T}$, *with the initial data* $[n_0, T_0, \mathbf{J}_0]$ *satisfying (1.6). Suppose that the problem (1.1 - 1.5) admits also a regular solution* $[\tilde{n}, \tilde{T}, \tilde{\mathbf{J}}]$,

$$\partial_t \tilde{n}, \ \partial_t \tilde{T}, \ \partial_t \tilde{\mathbf{J}}, \ \partial_x^m \tilde{n}, \ \partial_x^m \tilde{T}, \ \partial_x^m \tilde{\mathbf{J}} \in C([0, \tau) \times \mathcal{T}), \ m = 0, 1, 2,$$

emanating form the same initial data $[n_0, T_0, \mathbf{J}_0]$.

Then

$$n \equiv \tilde{n}, \ T \equiv \tilde{T}, \ \mathbf{J} \equiv \tilde{\mathbf{J}} \ \text{in} \ [0, \tau) \times \mathcal{T}.$$

Thus, the stipulation of the total energy conservation (5.1) seems to eliminate the "non-physical" weak solutions obtained in Theorem 4.1. On the other hand, however, there might still be "irregular" initial data for which the problem (1.1 - 1.5) admits infinitely many finite energy weak solution. The precise statement reads:

Theorem 5.2 *Let $\tau > 0$ and the initial data $[n_0, T_0]$,*

$$n_0, T_0 \in C^3(\mathcal{T}),$$

satisfying (1.6) be given.

Then there exists a flux

$$\mathbf{J}_0 \in L^\infty(\mathcal{T}; R^3)$$

such that the problem (1.1 - 1.5) possesses infinitely many finite energy *weak solutions in* $[0, \tau) \times \mathcal{T}$. *In addition, the weak solutions $[n, T, \mathbf{J}]$ belongs to the class specified in Theorem 4.1.*

The proof of Theorem 5.2 is essentially the same as that of [4, Theorem 4.2] and we leave it to the interested reader. Clearly, in accordance Theorem 3.1, the initial datum $\mathbf{J}_0$, the existence of which is claimed in Theorem 5.2, cannot be regular. On the other hand, the solutions obtained in Theorem 2.2 must have a "large" solenoidal part - to be compared with the result of Guo [10] on global existence of smooth solutions with *irrotational* initial data.

6 Conclusion

Unlike the standard Euler system, the Euler-Poisson system possesses global-in-time weak solutions for small irrotational initial data, see Guo [10]. On the other hand, we have shown that large data with a non-zero solenoidal component give rise to infinitely many weak solutions satisfying the global energy balance, see Theorem 5.2. In general, we have introduced several classes of solutions to the Euler-Poisson system (1.1 - 1.5), the properties of which can summarized as follows:

- **Strong (classical) solutions.** They are classical (differentiable) solutions of the system (1.1 - 1.5), emanating from regular initial data, that exist on a (possibly) short time interval for general (smooth) data $[n_0, \mathbf{J}_0, T_0]$. Global-in-time existence is to be expected for small irrotational data, see Guo [10].

- **Weak (distributional) solutions.** They satisfy (1.1 - 1.5) in the sense of distribution. Global-in-time weak solutions do exist for any initial data but they may be "unphysical" in the sense that they produce energy at the initial time.

- **Dissipative solutions.** The satisfy the relative entropy inequality and, consequently, they coincide with the (local) strong solution emanating from the same initial data as long as the latter exists. Global-in-time existence of dissipative solutions for general initial data is an open problem.

- **Finite energy weak solutions.** These are the weak solutions satisfying, in addition, the total energy balance. They are dissipative solutions so they coincide with a strong solutions as long as the latter exists. Global existence of finite energy weak solutions for general initial data is an open problem. On the other hand, there is a vast set of initial data (with an irregular flux $\mathbf{J}_0$), for which the problem (1.1 - 1.5) admits *infinitely many* global-in-time finite energy weak solutions.

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