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Abstract

We consider asymptotic regimes for a simplified model of compressible Navier-Stokes-Fourier system coupled to the radiation, when the radiative intensity is driven either to equilibrium or to non-equilibrium diffusion limit, depending the scaling performed, and we study the convergence of the system toward the aforementioned limits.

Key words: Radiation hydrodynamics, Navier-Stokes-Fourier system, weak solution, Oberbeck-Boussinesq.

1 Introduction

We consider a model of radiation hydrodynamics introduced by Teleaga, Seaïd, Gasser, Klar and Struckmeier in [21], incorporating the effects of radiation in a simplified setting. The motion of the fluid is governed by the standard field equations of classical continuum fluid mechanics describing the evolution of the mass density ϱ, the velocity field $\vec{u}$, and the absolute temperature ϑ as functions of the time $t \in \mathbb{R}_+$ and the Eulerian spatial coordinate $x \in \Omega$, where Ω is a bounded region of $\mathbb{R}^3$. The effect of radiation is incorporated in the radiative intensity $I = I(t, x, \vec{\omega}, \nu)$, depending on the direction vector $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$. The effect of radiation is then expressed in terms of integral means with respect to the variables $\vec{\omega}$ and ν of quantities depending on I.

The evolution of the compressible viscous heat conductive flow is coupled to radiation through the radiative transfer equation which reads

$$\frac{1}{c}\,\partial_t I + \vec{\omega} \cdot \nabla_x I = S \qquad \text{in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2, \tag{1.1}$$

where c is the speed of light. The radiative source $S := S_a + S_s$ is the sum of an emission-absorption term $S_{a,e} := \sigma_a\,(B(\nu,\vartheta) - I)$ and a scattering contribution

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$S_s := \sigma_s \left(\tilde{I} - I\right)$ where $\tilde{I} := \frac{1}{4\pi}\int_{\mathcal{S}^2} I \; d\vec{\omega}$, and the transport coefficients $\sigma_a(\vartheta)$ and $\sigma_a(\vartheta)$ are non negative functions. In the following we assume that σ_a and σ_s do not depend on $\vec{\omega}$ (isotropy) moreover we suppose that the so-called "grey hypothesis" is valid: σ_a and σ_s do not depend on the frequency ν. From the radiative intensity I and the source S we define the radiative energy $E_R = \frac{1}{c}\int_0^\infty \int_{\mathcal{S}^2} I \; d\vec{\omega} d\nu$, the radiative momentum $\vec{F}_R = \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} I \; d\vec{\omega} d\nu$, the radiative tensor $\mathbb{P}_R = \frac{1}{c}\int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \otimes \omega I \; d\vec{\omega} d\nu$, and from the source S we define the energy source $S_E = \int_0^\infty \int_{\mathcal{S}^2} S \; d\vec{\omega} d\nu$.

Finally the function $B(\nu, \vartheta)$ which appears in $S_{a,e}$, measuring the departure from equilibrium, is the Planck's function given by the formula $B(\nu, \vartheta) := \frac{2h\nu^3}{c^2}\left(\exp\left(\frac{h\nu}{k\vartheta}\right) - 1\right)^{-1}$ where h and k are the Planck and Boltzmann's constants. We also denote by the same symbol B the frequency average of $B(\nu, \vartheta)$ given by

$$B(\vartheta) := \int_0^\infty B(\nu, \vartheta) \; d\nu. \tag{1.2}$$

Observe that we get a conservation law by taking the first moment with respect to $\vec{\omega}$ in (1.1)

$$\partial_t E_R + \mathrm{div}_x \vec{F}_R = S_E \text{ in } (0, T) \times \Omega. \tag{1.3}$$

As for ordinary fluids, the mass conservation is expressed by the continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \text{ in } (0, T) \times \Omega. \tag{1.4}$$

Similarly, balance of linear momentum reads

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) = \; \mathrm{div}_x \mathbb{T} \quad \text{in } (0, T) \times \Omega, \tag{1.5}$$

where $\mathbb{T}$ is the Cauchy stress tensor characterized by the Stokes law $\mathbb{T} = -p\mathbb{I} + \mathbb{S}$, where p is the pressure and $\mathbb{S}$ is the viscous stress tensor. We consider that the state function $p = p(\varrho, \vartheta)$ is a given function of local thermodynamical variables ϱ and ϑ. So are the specific internal energy $e = e(\varrho, \vartheta)$ and the specific entropy for the fluid $s = s(\varrho, \vartheta)$. We also assume the Newton's rheological law for $\mathbb{S}$

$$\mathbb{S}(\vartheta, \nabla_x \vec{u}) = \mu(\vartheta)\left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3} \; \mathrm{div}_x \vec{u} \; \mathbb{I}\right) + \eta \; \mathrm{div}_x(\vec{u}\mathbb{I}),$$

where the shear and bulk viscosity coefficients μ and η may depend on ϑ. Taking the scalar product with $\vec{u}$ we get the kinetic energy balance

$$\partial_t \left(\frac{1}{2} \; \varrho |\vec{u}|^2\right) + \mathrm{div}_x \left(\frac{1}{2} \; \varrho |\vec{u}|^2 \vec{u} - \mathbb{T}\vec{u}\right) = -\mathbb{S} : \nabla_x \vec{u} + p \mathrm{div}_x \vec{u} \text{ in } (0, T) \times \Omega, \tag{1.6}$$

where $\mathbb{S} : \nabla_x \vec{u}$ describes the irreversible dissipation of mechanical energy. The balance of internal energy is

$$\partial_t \left(\varrho e\right) + \mathrm{div}_x \left(\varrho e \vec{u}\right) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u} - S_E \text{ in } (0, T) \times \Omega, \tag{1.7}$$

where the heat flux $\vec{q}$ is given by the Fourier's law $\vec{q} = -\kappa \nabla_x \vartheta$ and the heat conductivity κ may depend on ϑ. From (1.6) and (1.7) we get the total energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e + E_R \right) + \mathrm{div}_x \left(\left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e - \mathbb{T} \right) \vec{u} + \vec{q} + \vec{F}_R \right) = 0. \quad (1.8)$$

in $(0,T) \times \Omega$ and the system (1.4)(1.5)(1.8)(1.1) is finally supplemented with the boundary conditions:

$$\vec{u}|_{\partial\Omega} = 0, \ \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0, \quad (1.9)$$

and

$$I(t, x, \vec{\omega}, \nu) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0, \quad (1.10)$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$, and initial conditions

$$(\varrho(x,t), \ \vec{u}(x,t), \ \vartheta(x,t), I(x,t,\vec{\omega},\nu))|_{t=0} = \left(\varrho^0(x), \ \vec{u}^0(x), \ \vartheta^0(x), I^0(x,\vec{\omega},\nu) \right), \quad (1.11)$$

for any $x \in \Omega$ and $\nu \in (0, \infty)$. Integrating equation (1.8) on Ω and using the previous boundary conditions, we get

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e + E_R \right) \, dx + \int_0^\infty \int_{\Gamma_+} \vec{\omega} \cdot \vec{n} \ I \ d\vec{\omega} \ d\nu = 0, \quad (1.12)$$

where $\Gamma_+ = \{(x, \vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 \ : \ \vec{n}_x \cdot \vec{\omega} > 0\}$.

Just mention that system (1.4 - 1.10) can be viewed as a simplified model in radiation hydrodynamics [19], [18]. Such systems appear in astrophysics and their asymptotic regimes have been proposed by Lowrie, Morel and Hittinger [15] and revisited recently by Buet and Després [5]. For a more "complete system" including a radiative source in the right-hand side of (1.5), a global existence result has also recently been proved in [7] under some cut-off hypotheses on transport coefficients.

We are interested here in limit regimes of the previous system and our purpose is to show that the asymptotic theory developed in [9] can be adapted to problem (1.4 - 1.10). However, contrary to the low Mach number limit studied in [8], the diffusion limits correspond to a compressible system. This introduces a difficulty in estimates the differences between the solutions of the primitive and the target system, which can be overcome by using a relative entropy inequality introduced by Feireisl and Novotný [10] in their study of weak-strong uniqueness results for heat-conducting compressible fluids (see also [11]).

The paper is organized as follows. In Section 2, we list the principal hypotheses imposed on constitutive relations and state the existence result. In Section 3 we compute the formal asymptotic of the problem. Uniform bounds imposed on weak solutions by the data are derived in Section 4. In Section 4.1 a relative entropy inequality is derived and used in Section 4.2. The convergence Theorems are finally given in Section 5.

2 Hypotheses and the primitive system

Hypotheses imposed on constitutive relations and transport coefficients are motivated by the general existence theory for the Navier-Stokes-Fourier system developed in [9, Chapter 3] and reasonable physical assumptions [19]. We consider the pressure in the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{2.1}$$

where $P : [0, \infty) \to [0, \infty)$ is a given function with the following properties:

$$P \in C^1[0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.2}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$

The component $\frac{a}{3}\vartheta^4$ represents the effect of "equilibrium" radiation pressure (see [7] for motivations). The transport coefficients μ, η, and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1 + \vartheta) \leq \mu(\vartheta), \ \mu'(\vartheta) < c_2, \ 0 \leq \eta(\vartheta) \leq c(1 + \vartheta), \tag{2.5}$$

$$0 < c_1(1 + \vartheta^3) \leq \kappa(\vartheta) \leq c_2(1 + \vartheta^3) \tag{2.6}$$

for any $\vartheta \geq 0$.

Moreover we assume that σ_a and σ_s are continuous functions of ϑ such that

$$0 \leq \sigma_a(\vartheta), \sigma_s(\vartheta), |\partial_\vartheta \sigma_a(\vartheta)|, |\partial_\vartheta \sigma_a(\vartheta)| \leq c_1, \tag{2.7}$$

$$0 \leq \sigma_a(\vartheta)B(\nu, \vartheta), |\partial_\vartheta\{\sigma_a(\vartheta)B(\nu, \vartheta)\}| \leq c_2, \tag{2.8}$$

$$\sigma_a(\vartheta)B^m(\nu, \vartheta) \leq h(\nu), \ h \in L^1(0, \infty) \ \text{ for } m = 1, 2, \tag{2.9}$$

for any $\vartheta \geq 0$ and relations (2.7 - 2.9) represent "cut-off" hypotheses at large temperature.

We now just recall some definitions introduced in [7]. In the weak formulation of the Navier-Stokes-Fourier system the equation of continuity (1.4) is replaced by its renormalized version [6] represented by the integral identities

$$\int_\Omega \varrho(\tau, \cdot)\varphi(\tau, \cdot) \ dx - \int_\Omega \varrho_0\varphi(0, \cdot) \ dx = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\vec{u} \cdot \nabla_x\varphi\Big) \ dx \ dt \tag{2.10}$$

satisfied for any $\varphi \in C^1([0, T] \times \overline{\Omega})$ and any $\tau \in [0, T]$, and for $\varrho(0, \cdot) = \varrho_0$. Similarly, the momentum equation (1.5) is replaced by

$$\int_\Omega \varrho\vec{u}(\tau, \cdot)\phi(\tau, \cdot) \ dx - \int_\Omega \varrho_0\vec{u}_0\phi(0, \cdot) \ dx$$

$$= \int_0^\tau \int_\Omega \left(\varrho \vec{u} \cdot \partial_t \phi + \varrho \vec{u} \otimes \vec{u} : \nabla_x \phi + p \mathrm{div}_x \varphi - \mathbb{S} : \nabla_x \phi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.11}$$

for any $\phi \in C^1([0,T] \times \overline{\Omega}; \mathbb{R}^3)$ with $\phi|_{\partial\Omega} = 0$, any $\tau \in [0,T]$ and with $\vec{u}(0,\cdot) = \vec{u}_0$.

As the term $\mathbb{S}\vec{u}$ in the total energy balance (1.8) is not controlled on the (hypothetical) vacuum zones of vanishing density, we replace (1.8) by the internal energy equation

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \vec{u}) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u} - S_E. \tag{2.12}$$

Furthermore, dividing (2.12) on ϑ we get the entropy equation

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) =: \varsigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) - \frac{S_E}{\vartheta}, \tag{2.13}$$

where the first term $\varsigma_m := \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$ is the (positive) matter entropy production. In order to identify the second term we recall [1] the formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] d\vec{\omega} \, d\nu, \tag{2.14}$$

where $n = n(I) = \frac{c^2 I}{2h\nu^3}$ is the occupation number. Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] \vec{\omega} \, d\vec{\omega} \, d\nu, \tag{2.15}$$

and using the radiative transfer equation, we get the equation

$$\partial_t s^R + \mathrm{div}_x \vec{q}^R = -\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1} \; S \, d\vec{\omega} \, d\nu =: \varsigma^R. \tag{2.16}$$

Checking the identity $\log \frac{n(B)}{n(B)+1} = \frac{h\nu}{k\vartheta}$ with $B = B(\vartheta, \nu)$ the Planck's function, and using the definition of S, the right-hand side of (2.16) rewrites

$$\varsigma^R = \frac{S_E}{\vartheta} - \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (B - I) \, d\vec{\omega} \, d\nu$$
$$- \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (\tilde{I} - I) \, d\vec{\omega} d\nu,$$

where we used the hypothesis that the transport coefficients $\sigma_{a,s}$ do not depend on $\vec{\omega}$. So we obtain finally

$$\partial_t \left(\varrho s + s^R \right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \varsigma^m + \varsigma^R. \tag{2.17}$$

and equation (2.13) is replaced in the weak formulation by the inequality

$$\int_\Omega \left(\varrho_0 s_0 + s_0^R\right) \varphi(\tau,\cdot)\ dx - \int_\Omega \left(\varrho s + s^R\right)(\tau,\cdot)\varphi(\tau,\cdot)\ dx$$

$$\int_0^\tau \int_\Omega \left(\left(\varrho s + s^R\right)\partial_t \varphi + \varrho \vec{u}\cdot\nabla_x\varphi + \left(\frac{\vec{q}}{\vartheta} + \vec{q}^R\right)\cdot\nabla_x\varphi\right)\ \mathrm{d}x\ \mathrm{d}t \tag{2.18}$$

$$\leq -\int_0^\tau \int_0^\infty \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi\ \mathrm{d}t$$

$$-\frac{k}{h}\int_0^\tau \int_\Omega \left[\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)\ d\vec{\omega}\ d\nu\right.$$

$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I}-I)\ d\vec{\omega}\ d\nu\right]\varphi\ \mathrm{d}x\mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \geq 0$ and a.a. $\tau \in [0,T]$.

Since replacing equation (1.8) by inequality (2.18) would result in a formally under-determined problem, system (2.10), (2.11), (2.18) must be supplemented with the total energy balance

$$\int_\Omega \left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) + E^R\right)(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \left(\frac{1}{2\varrho_0}|(\varrho\vec{u})_0|^2 + (\varrho e)_0 + E_{R,0}\right)\ \mathrm{d}x$$

$$+\int_0^\tau \int_0^\infty \int_{\partial\Omega\times\mathcal{S}^2,\ \vec{\omega}\cdot\vec{n}\geq 0} \int_0^\infty \vec{\omega}\cdot\vec{n} I(t,x,\vec{\omega},\nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x\ \mathrm{d}t = 0, \tag{2.19}$$

where $E_{R,0} = \int_{\mathcal{S}^2}\int_0^\infty I_0(\cdot,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu$.

The existence result reads now [7]

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic function p satisfies hypotheses (2.1 - 2.4), and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.5 - 2.9).*

The problem (1.4 - 1.10) has a weak solution $(\varrho,\vec{u},\vartheta,I)$ such that

$$\varrho \geq 0,\ \vartheta > 0 \textit{ for a.a. } (t,x)\times\Omega,\ I \geq 0 \textit{ a.a. in } (0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty),$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)),$$
$$\vec{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),\ \vartheta \in L^2(0,T;W^{1,2}(\Omega)),$$
$$I \in L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),\ I(t,\cdot) \in L^\infty(0,T;L^1(\Omega\times\mathcal{S}^2\times(0,\infty)),$$

and if ϱ, $\vec{u}$, ϑ, I satisfy the integral identities (2.10), (2.11), (2.18), (2.19), together with the transport equation (1.1).

3 Formal scaling analysis

In order to identify the appropriate limit regimes we perform a general scaling, denoting by $L_{ref},\ T_{ref},\ U_{ref}, \rho_{ref},\ \vartheta_{ref},\ p_{ref},\ e_{ref},\ \mu_{ref},\ \kappa_{ref}$, the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity) and by $I_{ref},\ \nu_{ref},\ \sigma_{a,ref},\ \sigma_{s,ref}$, the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients). We also assume the compatibility conditions $p_{ref} \equiv \rho_{ref} e_{ref}$, $\nu_{ref} = \frac{k_B \vartheta_{ref}}{h}$, $I_{ref} = \frac{2h\nu_{ref}^3}{c^2}$ and we assume that the coefficients σ_a and σ_s are independent of the angular variable. We denote by $Sr := \frac{L_{ref}}{T_{ref} U_{ref}}$, $Ma = \frac{U_{ref}}{\sqrt{\rho_{ref} p_{ref}}}$, $Re = \frac{U_{ref}\rho_{ref} L_{ref}}{\mu_{ref}}$, $Pe = \frac{U_{ref} p_{ref} L_{ref}}{\vartheta_{ref}\kappa_{ref}}$, the Strouhal, Mach, Reynolds, Péclet (dimensionless) numbers corresponding to hydrodynamics, and by $\mathcal{C} = \frac{c}{U_{ref}}$, $\mathcal{L} = L_{ref}\sigma_{a,ref}$, $\mathcal{L}_s = \frac{\sigma_{s,ref}}{\sigma_{a,ref}}$, $\mathcal{P} = \frac{2k_B^4 \vartheta_{ref}^4}{h^3 c^3\ \rho_{ref} e_{ref}}$, various dimensionless numbers corresponding to radiation. Using these scalings and using carets to symbolize renormalized variables we get $S = \frac{I_{ref}}{L_{ref}}\ \hat{S}$, where $\hat{S} = \mathcal{L}\hat{\sigma}_a \left(b(\hat{\nu},\hat{\vartheta}) - \hat{I}\right) + \mathcal{L}\mathcal{L}_s\hat{\sigma}_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2}\hat{I}(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega} - \hat{I}\right)$, with $b(\nu,\vartheta) = \nu^3\left(e^{\frac{\nu}{\vartheta}} - 1\right)^{-1}$. Omitting the carets in the following, we get first the scaled equation for I, in the region $(0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2$

$$\frac{Sr}{\mathcal{C}}\ \partial_t I + \vec{\omega}\cdot\nabla_x I = S = \mathcal{L}\sigma_a\,(B - I) + \mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{3.1}$$

where $B := B(\nu,\vartheta) = \dfrac{\nu^3}{e^{\frac{\nu}{\vartheta}} - 1}$. Denoting also $E_R = \int_0^\infty\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu$ the renormalized energy and $S_E = \int_0^\infty\int_{\mathcal{S}^2} S\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu$, by

$$s^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right] d\vec{\omega}\ d\nu, \tag{3.2}$$

the radiative entropy, with $n = n(I) = \frac{I}{\nu^3}$ the occupation number, by

$$\vec{q}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega}\ d\nu, \tag{3.3}$$

the entropy flux, and taking the first two moments of (3.1) with respect to $\vec{\omega}$, we get also

$$\frac{Sr}{\mathcal{C}}\ \partial_t E_R + \mathrm{div}_x\vec{F}_R = S_E, \tag{3.4}$$

with nondimensional quantities $E_R = \int_0^\infty\int_{\mathcal{S}^2} I\ d\vec{\omega}d\nu$, $\vec{F}_R = \int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}I\ d\vec{\omega}d\nu$ and $S_E = \int_0^\infty\int_{\mathcal{S}^2} S\ d\vec{\omega}d\nu$.

The continuity equation is now

$$Sr\ \partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{3.5}$$

and the momentum equation

$$Sr\ \partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{Ma^2}\ \nabla_x p(\varrho,\vartheta) - \frac{1}{Re}\ \mathrm{div}_x\mathbb{T} = 0. \tag{3.6}$$

¿From (3.2), (3.3) and (3.1) we get the radiative entropy balance

$$\frac{Sr}{\mathcal{C}}\ \partial_t s^R + \mathrm{div}_x \vec{q}^R = -\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\ \log \frac{n}{n+1}\ s\ d\vec{\omega} d\nu. \tag{3.7}$$

The balance of internal energy rewrites

$$Sr\ \partial_t \left(\varrho e + \mathcal{P} E_R\right) + \mathrm{div}_x \left(\varrho e \vec{u} + \mathcal{P}\mathcal{C}\vec{F}_R\right) + \frac{1}{Pe}\ \mathrm{div}_x \vec{q} = \frac{Ma^2}{Re}\ \mathbb{S} : \nabla_x \vec{u} - p\mathrm{div}_x \vec{u},$$

and using (3.7), the balance of total entropy reads

$$Sr\ \partial_t \left(\varrho s + \mathcal{P} s^R\right) + \mathrm{div}_x \left(\varrho s \vec{u} + \mathcal{P}\mathcal{C}\vec{q}^R\right) + \frac{1}{Pe}\ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta}\right) = \varsigma, \tag{3.8}$$

with

$$\varsigma = \frac{1}{\vartheta}\left(\frac{Ma^2}{Re}\mathbb{S} : \nabla_x \vec{u} - \frac{1}{Pe}\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$
$$+\mathcal{P}\mathcal{C}\mathcal{L}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega}d\nu$$
$$+\mathcal{P}\mathcal{C}\mathcal{L}\mathcal{L}_s\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\vec{\omega}d\nu.$$

The scaled equation for total energy gives finally the total energy balance

$$Sr\frac{d}{dt}\int_\Omega \left(\frac{Ma^2}{2}\ \varrho|\vec{u}|^2 + \varrho e + \mathcal{P}E_R\right)\ dx + \mathcal{P}\mathcal{C}\int_0^\infty \int_{\Gamma_+} \vec{\omega}\cdot\vec{n} I\ d\Gamma_+ d\nu = 0, \tag{3.9}$$

with $E_R = \int_0^\infty \int_{\mathcal{S}^2} I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$.

Supposing that a small amount of radiation is present ($\mathcal{P} = O(\varepsilon)$) in our strongly under-relativistic flow ($\mathcal{C} = O(\varepsilon^{-1})$), where ε is a small positive number, we obtain the *equilibrium diffusion regime* described in [5] by $Ma = Sr = Pe = Re = 1,\ \mathcal{P} = \varepsilon,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L}_s = \varepsilon^2$ and $\mathcal{L} = \varepsilon^{-1}$, given by

$$\varepsilon\ \partial_t I + \vec{\omega}\cdot\nabla_x I = \frac{1}{\varepsilon}\sigma_a\left(B - I\right) + \varepsilon\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{3.10}$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{3.11}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) - \ \mathrm{div}_x\mathbb{T} = 0. \tag{3.12}$$

$$\partial_t\left(\varrho e + \varepsilon E_R\right) + \mathrm{div}_x\left(\varrho e\vec{u} + \vec{F}_R\right) + \mathrm{div}_x\vec{q} = \mathbb{S} : \nabla_x\vec{u} - p\mathrm{div}_x\vec{u}, \tag{3.13}$$

$$\partial_t\left(\varrho s + \varepsilon s_R\right) + \mathrm{div}_x\left(\varrho\vec{u}s + \vec{q}_R\right) + \ \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$
$$+\frac{1}{\varepsilon}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega}d\nu$$

$$+\varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \, d\vec{\omega} d\nu, \tag{3.14}$$

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \, \varrho |\vec{u}|^2 + \varrho e + E_R \right) \, dx + \frac{1}{\varepsilon} \int_0^\infty \int_{\Gamma_+} \vec{\omega} \cdot \vec{n} \; I \; d\Gamma_+ d\nu = 0. \tag{3.15}$$

In the same stroke, the " non-equilibrium diffusion regime" is defined by $Ma = Sr = Pe = Re = 1$, $\mathcal{P} = \varepsilon$, $\mathcal{C} = \varepsilon^{-1}$, $\mathcal{L} = \varepsilon^2$ and $\mathcal{L}_s = \varepsilon^{-1}$. One checks that equations (3.11) (3.12) (3.13) and (3.15) still hold is this scaling. However it will be convenient to rewrite (3.13), using (1.3) as

$$\partial_t \left(\varrho e \right) + \mathrm{div}_x \left(\varrho e \vec{u} \right) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u} - S_E. \tag{3.16}$$

The new system reads finally

$$\varepsilon \; \partial_t I + \vec{\omega} \cdot \nabla_x I = \varepsilon \sigma_a \left(B - I \right) + \frac{1}{\varepsilon} \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \; \mathrm{d}\vec{\omega} - I \right), \tag{3.17}$$

$$\partial_t \varrho + \mathrm{div}_x (\varrho \vec{u}) = 0, \tag{3.18}$$

$$\partial_t (\varrho \vec{u}) + \mathrm{div}_x (\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho, \vartheta) - \; \mathrm{div}_x \mathbb{T} = 0. \tag{3.19}$$

$$\partial_t \left(\varrho e \right) + \mathrm{div}_x \left(\varrho e \vec{u} \right) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u} - S_E, \tag{3.20}$$

$$\partial_t \left(\varrho s + \varepsilon s_R \right) + \mathrm{div}_x \left(\varrho \vec{u} s + \vec{q}_R \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$+\varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \, d\vec{\omega} d\nu$$

$$+\frac{1}{\varepsilon} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \, d\vec{\omega} d\nu. \tag{3.21}$$

3.1 The equilibrium-diffusion regime

In order to compute the limit system, we consider the formal expansions

$$\left\{ \begin{array}{l} I = I_0 + \varepsilon I_1 + \varepsilon^2 I_2 + O(\varepsilon^3), \\ \varrho = \rho_0 + \varepsilon \varrho_1 + \varepsilon^2 \varrho_2 + O(\varepsilon^3), \\ \vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + \varepsilon^2 \vec{u}_2 + O(\varepsilon^3), \\ \vartheta = \vartheta_0 + \varepsilon \vartheta_1 + \varepsilon^2 \vartheta_2 + O(\varepsilon^3). \end{array} \right. \tag{3.22}$$

Keeping the low order terms in (3.10) we get

$$I_0 = B(\nu, \vartheta_0), \tag{3.23}$$

and

$$I_1 = \partial_\vartheta B(\nu, \vartheta_0) \vartheta_1 - \frac{1}{\sigma_a(\vartheta_0)} \; \vec{\omega} \cdot \nabla_x I_0. \tag{3.24}$$

As the related radiative quantities are

$$\left\{\begin{array}{l} E_R = e_0 + \varepsilon e_1 + O(\varepsilon^2), \\ \vec{F}_R = \vec{f}_0 + \varepsilon \vec{f}_1 + O(\varepsilon^2), \end{array}\right. \tag{3.25}$$

using (3.23) and (3.24) we find $e_0(\vartheta_0) = B(\vartheta_0)$ where B is defined in (1.2) and

$$e_1 = \vartheta_1 \int_0^\infty \frac{1}{\sigma_a(\vartheta_0)} \left[\sigma_a(\vartheta_0)\partial_\vartheta B(\nu,\vartheta_0) + B(\nu,\vartheta_0)\partial_\vartheta \sigma_a(\vartheta_0)\right]\, d\nu, \tag{3.26}$$

together with $\vec{f}_0 = 0$ and

$$\vec{f}_1 = -\frac{1}{3\sigma_a(\vartheta_0)}\ \partial_\vartheta B(\vartheta_0)\nabla_x \vartheta_0. \tag{3.27}$$

The limit momentum equation is then

$$\partial_t(\varrho_0\vec{u}_0) + \mathrm{div}_x(\varrho_0\vec{u}_0 \otimes \vec{u}_0) + \nabla_x p(\varrho_0,\vartheta_0) -\ \mathrm{div}_x\left(\mu_0\left(\nabla_x\vec{u}_0 + \nabla_x^t\vec{u}_0\right)\right) = 0,$$

where $\mu_0 = \mu(\vartheta_0)$.

Let us observe that at lowest order, the limit energy equation is

$$\partial_t\left(\varrho\mathcal{E}(\varrho_0,\vartheta_0)\right) + \mathrm{div}_x\left(\varrho_0 e(\varrho_0,\vartheta_0)\vec{u}_0\right) + \mathrm{div}_x\left(\mathcal{K}(\varrho_0,\vartheta_0)\nabla_x\vartheta_0\right)$$

$$= \mathbb{S}(\varrho_0,\vartheta_0) : \nabla_x\vec{u}_0 - p(\varrho_0,\vartheta_0)\mathrm{div}_x\vec{u}_0,$$

where $\mathcal{E}(\varrho_0,\vartheta_0) = e(\varrho_0,\vartheta_0) + \frac{B(\vartheta_0)}{\varrho_0}$, and $\mathcal{K}(\vartheta_0) = \kappa(\vartheta_0) - \frac{1}{3\sigma_a(\vartheta_0)}\ \partial_\vartheta B(\vartheta_0)$.

Hence omitting the 0 index, we finally obtain the decoupled limit system in $(0,T)\times\Omega$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{3.28}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) = \mathrm{div}_x\mathbb{T}(\varrho,\vartheta), \tag{3.29}$$

$$\partial_t\left(\varrho\mathcal{E}(\varrho,\vartheta)\right) + \mathrm{div}_x\left(\varrho e(\varrho,\vartheta)\vec{u}\right) + \mathrm{div}_x\left(\mathcal{K}(\varrho,\vartheta)\nabla_x\vartheta\right)$$

$$= \mathbb{S}(\varrho,\vartheta) : \nabla_x\vec{u} - p(\varrho,\vartheta)\mathrm{div}_x\vec{u}, \tag{3.30}$$

$$I = B(\nu,\vartheta). \tag{3.31}$$

We also get boundary conditions

$$\vec{u}|_{\partial\Omega} = 0,\ \nabla\vartheta\cdot\vec{n}|_{\partial\Omega} = 0, \tag{3.32}$$

for (3.28)-(3.30), and initial conditions

$$\left(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t)\right)|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x)\right), \tag{3.33}$$

for any $x\in\Omega$.

This system corresponds to a renormalized compressible Navier-Stokes system, while radiative equilibrium is achieved between matter and radiation with radiative intensity $I = B(\nu,\vartheta)$, corresponding to the black-body radiation at temperature ϑ with radiative energy $E_R(\vartheta) = B(\vartheta)$.

¿From classical results of Matsumura and Nishida [16] [17](see also Jiang [14]) let us quote a global existence result for this system, for small data.

Let $(\overline{\varrho}, 0, \overline{\vartheta})$ be a given constant state with $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$. We note

$$e_0 := \|\varrho_0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vec{u}_0\|_{H^1(\Omega)} + \|\vartheta_0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|\mathbb{T}_0\|_{L^2(\Omega)} + \|\mathbb{V}_0\|_{L^4(\Omega)}, \quad (3.34)$$

where $\mathbb{V}_0$ is the initial vorticity (recall that $\mathbb{V}_{ij} = \partial_j u_i - \partial_i u_j$), and

$$E_0 := e_0 + \|\nabla_x \varrho_0\|_{L^2(\Omega)} + \|\nabla_x \varrho_0\|_{L^\alpha(\Omega)} + \|\nabla_x \mathbb{T}_0\|_{L^2(\Omega)}, \quad (3.35)$$

for an arbitrary fixed α such that $3 < \alpha < 6$. The following result holds

Theorem 3.1 *Let $(\varrho_0 - \overline{\varrho}, \vec{u}_0, \vartheta_0 - \overline{\vartheta}) \in (H^3(\Omega))^5$ and $\inf \vartheta_0 > 0$. There exists positive constants $\eta \leq 1$ and $\Gamma > 0$ depending on the data such that if $E_0 \leq \Gamma\eta$ $(\varrho, \vec{u}, \vartheta, I)$ is the unique classical solution to the Navier-Stokes-Fourier system (3.28)-(3.31) with boundary conditions (3.32) and initial conditions (3.33) in $(0,T) \times \Omega$ for any $T > 0$ such that*

$$(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}) \in C([0,T], H^3(\Omega)),$$

$$\sup_{t\geq 0} \|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} \leq \overline{\varrho}/2, \quad \inf_{x\in\overline{\Omega}, t\geq 0} \overline{\vartheta} > 0,$$

$$\partial_t \varrho \in C([0,T], H^2(\Omega)), \quad \partial_t \vec{u}, \partial_t \vartheta \in C([0,T], H^1(\Omega))$$

$$\partial_t \varrho, \partial_t \vec{u}, \partial_t \vartheta \in L^2([0,T], H^2(\Omega)).$$

Moreover, if $e_0 \leq \eta$

$$\sup_{0\leq t\leq T} \|\varrho - \overline{\varrho}\|^2_{L^2(\Omega)} + \|\vec{u}\|^2_{L^2(\Omega)} + \|\vartheta - \overline{\vartheta}\|^2_{L^2(\Omega)} + \|\nabla_x \vartheta\|^2_{L^2(\Omega)} \leq \Gamma e_0^2,$$

and

$$\sup_{0\leq t\leq T} \left(\|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vartheta - \overline{\vartheta}\|_{L^\infty(\Omega)} \right) \leq \Gamma e_0.$$

It is worth to note that when one considers the formal "nonconducting at rest" situation where $\kappa = 0$ and $\vec{u} = 0$ and in the no-scattering case ($\sigma_s \equiv 0$), one obtains the simplified system introduced by Bardos, Golse and Perthame [2] for which they proved global existence and diffusion limit (called "Rosseland approximation") under assumptions much more general than ours.

3.2 The non-equilibrium diffusion regime

Expanding as above using (3.22) in (3.17) and evaluating the lowest orders terms we get

$$\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0 \, \mathrm{d}\vec{\omega} = I_0, \quad (3.36)$$

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_s(\vartheta_0, \nu) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_1 \, \mathrm{d}\vec{\omega} - I_1 \right), \quad (3.37)$$

and

$$\partial_t I_0 + \vec{\omega} \cdot \nabla_x I_1 = \sigma_a(\vartheta_0)(B(\vartheta_0, \nu) - I_0) + \sigma_s(\vartheta_0) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_2 \, \mathrm{d}\vec{\omega} - I_2 \right)$$

$$+\partial_\vartheta \sigma_s(\vartheta_0) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_1 \, \mathrm{d}\vec{\omega} - I_1 \right) \vartheta_1. \tag{3.38}$$

Integrating on $\mathcal{S}^2$ and plugging the first two relations into the last one, we find

$$\partial_t I_0 + \vec{\omega} \cdot \nabla_x \tilde{I}_1 - \vec{\omega} \otimes \vec{\omega} \ \mathrm{div}_x \left(\frac{1}{\sigma_s(\vartheta_0)} \nabla_x I_0 \right)$$

$$= \sigma_a(\vartheta_0)(B(\vartheta_0, \nu) - I_0) + \sigma_s(\vartheta_0) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_2 \, \mathrm{d}\vec{\omega} - I_2 \right)$$

$$+\partial_\vartheta \sigma_s(\vartheta_0, \nu) \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_1 \, \mathrm{d}\vec{\omega} - I_1 \right) \vartheta_1.$$

Integrating in ν and using (3.36)(3.37), we get a diffusion equation for $N := \int_0^\infty I_0 \ d\nu$

$$\partial_t N - \frac{1}{3} \ \mathrm{div}_x \left(\frac{1}{\sigma_s(\vartheta_0)} \nabla_x N \right) = \sigma_a(\vartheta_0)(B(\vartheta_0) - N). \tag{3.39}$$

The limit continuity and momentum equations are the same as in the equilibrium case and the limit energy equation rewrites is, after (3.16)

$$\partial_t \left(\varrho e(\varrho_0, \vartheta_0) \right) + \mathrm{div}_x \left(\varrho_0 e(\varrho_0, \vartheta_0) \vec{u}_0 \right) + \mathrm{div}_x \left(\kappa(\vartheta_0) \nabla_x \vartheta \right)$$

$$= \mathbb{S}(\varrho_0, \vartheta_0) : \nabla_x \vec{u}_0 - p(\varrho_0, \vartheta_0) \mathrm{div}_x \vec{u}_0 - (S_E)_0,$$

where $(S_E)_0 = \int_0^\infty \int_{\mathcal{S}^2} s_0 \ \mathrm{d}\vec{\omega} \ d\nu = \sigma_a(\vartheta_0)[B(\vartheta_0) - N]$.

Noting as previously $\overline{\mu} = \mu(\vartheta_0)$ and $\overline{\kappa} = \kappa(\vartheta_0)$, we finally obtain a compressible Navier-Stokes-Fourier-type system in $(0, T) \times \Omega$ coupled to a diffusion equation for N.

Omitting the 0 index, we get finally the system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{3.40}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta), \tag{3.41}$$

$$\partial_t \left(\varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta) \vec{u} \right) + \mathrm{div}_x \left(\kappa(\vartheta) \nabla_x \vartheta \right)$$

$$= \mathbb{S}(\varrho, \vartheta) : \nabla_x \vec{u} - p(\varrho, \vartheta) \mathrm{div}_x \vec{u} - \sigma_a(\vartheta)[B(\vartheta) - N], \tag{3.42}$$

$$\partial_t N - \frac{1}{3} \ \mathrm{div}_x \left(\frac{1}{\sigma_s(\vartheta)} \nabla_x N \right) = \sigma_a(\vartheta) \left(B(\vartheta) - N \right). \tag{3.43}$$

with the boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \ \nabla \vartheta \cdot \vec{n}|_{\partial\Omega} = 0, \tag{3.44}$$

$$N|_{\partial\Omega} = 0. \tag{3.45}$$

and initial conditions

$$(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t), N(x,t))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x), N^0(x)\right), \tag{3.46}$$

for any $x \in \Omega$, with $N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu$.

In analogy with previous works on asymptotic analysis of radiative transfer equation (see [2][3][13]), we call (3.40)-(3.43) the Navier-Stokes-Rosseland system. As in the equilibrium case, we have a global existence result for solutions of the problem (3.40)-(3.43) for small data.

Let $(\overline{\varrho}, 0, \overline{\vartheta}, \overline{N})$ be a given constant state with $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{N} = B(\overline{\vartheta})$. We note

$$e_0 := \|\varrho^0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vec{u}^0\|_{H^1(\Omega)} + \|\vartheta^0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|N^0 - \overline{N}\|_{H^1(\Omega)}$$
$$+\|\mathbb{T}^0\|_{L^2(\Omega)} + \|\mathbb{V}^0\|_{L^4(\Omega)}, \tag{3.47}$$

and

$$E_0 := e_0 + \|\nabla_x \varrho^0\|_{L^2(\Omega)} + \|\nabla_x \varrho^0\|_{L^\alpha(\Omega)} + \|\nabla_x \mathbb{T}^0\|_{L^2(\Omega)}, \tag{3.48}$$

for an arbitrary fixed α such that $3 < \alpha < 6$. The following result holds

Theorem 3.2 *Let $(\varrho^0 - \overline{\varrho}, \vec{u}^0, \vartheta^0 - \overline{\vartheta}, N^0 - \overline{N}) \in (H^3(\Omega))^6$ and $\inf \vartheta^0, \inf N^0 > 0$. There exists positive constants $\eta \leq 1$ and $\Gamma > 0$ depending on the data such that if $E_0 \leq \Gamma\eta$, $(\varrho, \vec{u}, \vartheta, N)$ is the unique classical solution to the Navier-Stokes-Fourier system (3.40)-(3.43) with boundary conditions (3.44) and (3.45) and initial conditions (3.46) in $(0,T) \times \Omega$ for any $T > 0$ such that*

$$(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, N - \overline{N}) \in C([0,T], H^3(\Omega)),$$

$$\sup_{t\geq 0} \|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} \leq \overline{\varrho}/2, \quad \inf_{x\in\overline{\Omega}, t\geq 0} \overline{\vartheta} > 0,$$

$$\partial_t \varrho \in C([0,T], H^2(\Omega)), \quad \partial_t \vec{u}, \partial_t \vartheta, \partial_t N \in C([0,T], H^1(\Omega))$$
$$\partial_t \varrho, \partial_t \vec{u}, \partial_t \vartheta, \partial_t N \in L^2([0,T], H^2(\Omega)).$$

Moreover, if $e_0 \leq \eta$

$$\sup_{0\leq t\leq T} \|\varrho - \overline{\varrho}\|^2_{L^2(\Omega)} + \|\vec{u}\|^2_{L^2(\Omega)} + \|\vartheta - \overline{\vartheta}\|^2_{L^2(\Omega)} + \|N - \overline{N}\|^2_{L^2(\Omega)}$$
$$+\|\nabla_x \vartheta\|^2_{L^2(\Omega)} + \|\nabla_x N\|^2_{L^2(\Omega)} \leq \Gamma e_0^2.$$

and

$$\sup_{0\leq t\leq T} \left(\|\varrho - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\vartheta - \overline{\vartheta}\|_{L^\infty(\Omega)} + \|N - \overline{N}\|_{L^\infty(\Omega)}\right) \leq \Gamma e_0.$$

4 Uniform estimates for the primitive system

Let us recast the existence result of Theorem 2.1 in the rescaled context. For that purpose we choose initial data such that

$$\left\{\begin{array}{l} \varrho(0,x) \equiv \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \\ \vec{u}(0,x) \equiv \varepsilon \vec{u}^{(1)}_{0,\varepsilon}, \\ \vartheta(0,x) \equiv \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \\ I(0,x,\vec{\omega},\nu) \equiv I_{0,\varepsilon} = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon}, \end{array}\right. \tag{4.1}$$

with $\overline{I} = B(\nu, \overline{\vartheta})$, and

$$N(0,\cdot) \equiv N_{0,\varepsilon} = \overline{N} + \varepsilon N^{(1)}_{0,\varepsilon}, \tag{4.2}$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{N} = B(\overline{\vartheta})$ are the same as those appearing in Theorem 3.1 and Theorem 3.2.

We get immediately from Theorem 2.1 the following result (see [7] for details)

Proposition 4.1 *Suppose that the conditions of Theorem 2.1 are satisfied.*

Let initial data $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ *be given by (4.1), where* $(\varrho^{(1)}_{0,\varepsilon}, \vec{u}^{(1)}_{0,\varepsilon}, \vartheta^{(1)}_{0,\varepsilon}, I^{(1)}_{0,\varepsilon})$ *are bounded measurable functions.*

Then for any $\varepsilon > 0$ *small enough (in order to maintain positivity of* $\varrho^{(1)}_{0,\varepsilon}$ *and* $\vartheta^{(1)}_{0,\varepsilon}$*), there exits a weak solution* $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ *to the radiative Navier-Stokes system (1.4)(1.5)(1.7) (1.1) for* $(t,x,\vec{\omega},\nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$*, supplemented with the boundary conditions (1.9 - 1.10) and the initial conditions (4.1). More precisely we have*

$$\int_\Omega \varrho_\varepsilon(\tau,\cdot)\phi(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\phi(0,\cdot)\ dx = \int_0^\tau \int_\Omega \varrho_\varepsilon\left(\partial_t \phi + \vec{u}_\varepsilon \cdot \nabla_x \phi\right)\ dx\ dt \tag{4.3}$$

for any $\phi \in C^1([0,T) \times \overline{\Omega})$*, and any* $\tau \in [0,T]$*,*

$$\int_\Omega \varrho_\varepsilon \vec{u}_\varepsilon(\tau,\cdot)\ \phi(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\vec{u}_{0,\varepsilon}\ \phi(0,\cdot)\ dx$$

$$= \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t \phi + \varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla_x \phi + p_\varepsilon\ \mathrm{div}_x \phi - \mathbb{S}_\varepsilon : \nabla_x \phi\right)\ dx\ dt = 0, \tag{4.4}$$

for any $\phi \in C^1([0,T) \times \overline{\Omega}; \mathbb{R}^3)$ *, and any* $\tau \in [0,T]$*, such that* $\phi \cdot n|_{\partial\Omega} = 0$*, with* $p_\varepsilon = p(\varrho_\varepsilon, \vartheta_\varepsilon)$ *and* $\mathbb{S}_\varepsilon = \mathbb{S}(\vec{u}_\varepsilon, \vartheta_\varepsilon)$*,*

$$\int_0^\tau \int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon\right)\ dx\ dt + \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$

$$= \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon}\right)\ dx =: \mathcal{E}_0, \tag{4.5}$$

for a.a. $t \in [0,T]$ *with* $\Gamma_+ = \{(x,\vec\omega) \in \partial\Omega \times \mathcal{S}^2 \ : \ \vec\omega \cdot \vec n_x \geq 0\}$ *and with* $e_\varepsilon = e(\varrho_\varepsilon, \vartheta_\varepsilon)$ *and* $E^R_\varepsilon(t,x) = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec\omega,\nu)\ d\vec\omega\ d\nu$

$$\int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right) \partial_t \varphi + \left(\varrho_\varepsilon s_\varepsilon \vec u_\varepsilon + \vec q^R_\varepsilon\right) \cdot \nabla_x \varphi\right)\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \frac{\vec q_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \varphi\ dx\ dt + \left\langle \varsigma^m_\varepsilon + \varsigma^R_\varepsilon ; \phi \right\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})}$$

$$\leq - \int_\Omega \left(\varrho s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon}\right) \varphi(0,\cdot)\ dx + \int_\Omega \left(\varrho s_\varepsilon + \varepsilon s^R_\varepsilon\right))(\tau,\cdot)\varphi(\tau,\cdot)\ dx, \tag{4.6}$$

where

$$\varsigma^m_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \vec u_\varepsilon - \frac{\vec q_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon}\right), \tag{4.7}$$

and

$$\varsigma^R_\varepsilon \geq \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right] {\sigma_a}^{(j)}_\varepsilon (B_\varepsilon - I_\varepsilon)\ d\vec\omega d\nu$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde I_\varepsilon)}{n(\tilde I_\varepsilon)+1}\right] {\sigma_s}^{(j)}_\varepsilon (\tilde I_\varepsilon - I_\varepsilon)\ d\vec\omega d\nu, \tag{4.8}$$

for $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$ *and any* $\tau \in [0,T]$, *with* $\varsigma^m_\varepsilon \in \mathcal{M}^+([0,T) \times \overline{\Omega})$ *and* $\varsigma^R_\varepsilon \in \mathcal{M}^+([0,T) \times \overline{\Omega})$, *where* $\mathcal{M}(X)$ *is the set of signed Borel measures on* X *and* $\mathcal{M}^+(X)$ *is the cone of non-negative elements of* $\mathcal{M}(X)$.

We consider two possible values for the transport coefficients in the two cases $j = 1$ *(equilibrium case) or* $j = 2$ *(non-equilibrium case)*

$${\sigma_a}^{(j)}_\varepsilon = \left| \begin{array}{l} \dfrac{1}{\varepsilon}\ \sigma_a(\vartheta_\varepsilon) \quad \text{if } j = 1, \\ \varepsilon\sigma_a(\vartheta_\varepsilon) \quad \text{if } j = 2, \end{array}\right. \tag{4.9}$$

and

$${\sigma_s}^{(j)}_\varepsilon = \left| \begin{array}{l} \varepsilon\sigma_s(\vartheta_\varepsilon) \quad \text{if } j = 1, \\ \dfrac{1}{\varepsilon}\ \sigma_s(\vartheta_\varepsilon) \quad \text{if } j = 2. \end{array}\right. \tag{4.10}$$

Denoting $B_\varepsilon = B(\nu,\vartheta_\varepsilon)$, $\vec q_\varepsilon = \kappa(\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon$, $s_\varepsilon = s(\varrho_\varepsilon,\ \vartheta_\varepsilon)$, $s^R_\varepsilon = s^R(I_\varepsilon)$, $\vec q^R_\varepsilon = \vec q^R(I_\varepsilon)$, *and* $\tilde I_\varepsilon := \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(t,x,\nu,\vec\omega)\ \mathrm{d}\vec\omega$, *we have finally*

$$\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left(\varepsilon\partial_t\psi + \vec\omega \cdot \nabla_x\psi\right) I_\varepsilon\ d\vec\omega\ d\nu\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[{\sigma_a}^{(j)}_\varepsilon \left(B_\varepsilon - I_\varepsilon\right) + {\sigma_s}^{(j)}_\varepsilon \left(\tilde I_\varepsilon - I_\varepsilon\right)\right] \psi\ d\vec\omega\ d\nu\ dx\ dt,$$

$$= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_{0,\varepsilon}\psi(0,x,\vec\omega,\nu)\ d\vec\omega\ d\nu\ dx - \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_\varepsilon\psi(\tau,x,\vec\omega,\nu)\ d\vec\omega\ d\nu\ dx$$

$$+ \int_0^\tau \int_{\Gamma_+} \int_0^\infty \vec\omega \cdot \vec n_x I_\varepsilon \psi\ d\Gamma\ d\nu\ dt, \tag{4.11}$$

for any $\psi \in C^1([0,T) \times \overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$ *and any* $\tau \in [0,T]$.

Just mention that any weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ enjoys all of the regularity and integrability properties given in Theorem 2.1, and that the presence of ε is irrelevant in the existence theory.

4.1 Relative entropy inequality

We establish a relative entropy inequality satisfied by any weak solution $(\varrho, \vec{u}, \vartheta, I)$ of the radiative Navier-Stokes system, in the spirit of [10] (see also [20] in a more general context). Let us consider a set $\{r, \Theta, \vec{U}\}$ of smooth functions such that r and Θ are bounded below away from zero and $\vec{U}\Big|_{\partial\Omega} = 0$. We call *ballistic free energy* the thermodynamical potential given by

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta),$$

and *radiative ballistic free energy* the potential

$$H^R_\Theta(I) = E^R(I) - \Theta s^R(I).$$

The *relative entropy* is then defined by

$$\mathcal{E}(\varrho, \vartheta | r, \Theta) := H_\Theta(\varrho, \vartheta) - \partial_\varrho H_\Theta(r, \Theta)(\varrho - \Theta) - H_\Theta(r, \Theta).$$

Testing equation (4.3) with $\phi = \frac{1}{2}\, |\vec{U}|^2$, we get

$$\int_\Omega \frac{1}{2}\ \varrho_\varepsilon |\vec{U}|^2(\tau, \cdot)\ dx - \int_\Omega \frac{1}{2}\ \varrho_{0,\varepsilon} |\vec{U}(0, \cdot)|^2\ dx$$
$$= \int_0^\tau \int_\Omega \varrho_\varepsilon \left(\vec{U} \cdot \partial_t \vec{U} + \vec{u}_\varepsilon \cdot \nabla_x \vec{U} \cdot \vec{U}. \right)\ dx\ dt \tag{4.12}$$

Testing now equation (4.4) with $\phi = \vec{U}$, we get

$$\int_\Omega \varrho_\varepsilon \vec{u}_\varepsilon(\tau, \cdot) \cdot \vec{U}(\tau, \cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon} \vec{u}_{0,\varepsilon} \cdot \vec{U}(0, \cdot)\ dx.$$
$$= \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t \vec{U} + \varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla_x \vec{U} + p_\varepsilon\ \mathrm{div}_x \vec{U} - \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right)\ dx\ dt = 0, \tag{4.13}$$

Combining (4.12), (4.13) and (4.5), we get

$$\int_0^\tau \int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon \right)(\tau, \cdot)\, dx\, dt + \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t, x, \vec{\omega}, \nu)\, d\Gamma\, d\nu\, dt$$
$$= \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0, \cdot)|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} \right)\ dx$$
$$+ \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon \partial_t \vec{U} + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \vec{U} \right) \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) - p_\varepsilon\ \mathrm{div}_x \vec{U} + \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right)\ dx\ dt. \tag{4.14}$$

Testing finally equation (4.6) with $\varphi = \Theta$, we get

$$\int_\Omega \left(\varrho_0 s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon}\right)\Theta(0,\cdot)\ dx - \int_\Omega \left(\varrho s_\varepsilon + \varepsilon s^R_\varepsilon\right)\Theta(\tau,\cdot)\ dx$$

$$+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\right)\mathrm{d}x\ dt$$

$$+\int_0^\tau\int_\Omega \Theta\left\{\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a_\varepsilon}^{(j)}(B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\right.$$

$$\left. + \int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right]\sigma_{s_\varepsilon}^{(j)}(\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\right\}\ dx\ dt$$

$$\leq -\int_0^\tau\int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right)\partial_t\Theta + \left(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right)\cdot\nabla_x\Theta\right)\ dx\ dt$$

$$-\int_0^\tau\int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\Theta\ dx\ dt. \tag{4.15}$$

¿From (4.14) and (4.15) we get

$$\int_\Omega\left(\frac{1}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon - \vec{U}|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon - (\varrho s_\varepsilon + \varepsilon s^R_\varepsilon)\Theta\right)(\tau,\cdot)\ dx + \int_0^T\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$

$$+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\right)\mathrm{d}x\ dt$$

$$+\int_0^\tau\int_\Omega \Theta\left\{\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a_\varepsilon}^{(j)}(B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\right.$$

$$\left. + \int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right]\sigma_{s_\varepsilon}^{(j)}(\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\right\}\ dx\ dt$$

$$\leq \int_\Omega\left(\frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \varrho_{0,\varepsilon}e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} - (\varrho_0 s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon})\Theta(0,\cdot)\right)\ dx$$

$$+\int_0^\tau\int_\Omega\left(\left(\varrho_\varepsilon\partial_t\vec{U} + \varrho_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\vec{U}\right)\cdot\left(\vec{U} - \vec{u}_\varepsilon\right) - p_\varepsilon\ \mathrm{div}_x\vec{U} + \mathbb{S}_\varepsilon : \nabla_x\vec{U}\right)\ dx\ dt$$

$$-\int_0^\tau\int_\Omega\left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right)\partial_t\Theta + \left(\varrho_\varepsilon s_\varepsilon\vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right)\cdot\nabla_x\Theta\right)\ dx\ dt - \int_0^\tau\int_\Omega\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\Theta\ dx\ dt. \tag{4.16}$$

Testing equation (4.3) with $\phi = \partial_{\varrho_\varepsilon}H_\Theta(r,\Theta)$, we get

$$\int_\Omega \varrho_\varepsilon\partial_{\varrho_\varepsilon}H_\Theta(r,\Theta)(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\partial_{\varrho_\varepsilon}H_{\Theta(0,\cdot)}\left(r(0,\cdot),\Theta(0,\cdot)\right)\ dx$$

$$= \int_0^\tau\int_\Omega\left(\varrho_\varepsilon\partial_t\Big(\partial_{\varrho_\varepsilon}H_\Theta(r,\Theta)\Big) + \varrho_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\Big(\partial_{\varrho_\varepsilon}H_\Theta(r,\Theta)\Big)\right)\ dx\ dt. \tag{4.17}$$

¿From (4.16) and (4.17) we get

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - H_\Theta(r,\Theta) - \partial_{\varrho_\varepsilon} H_\Theta(r,\Theta)(\varrho_\varepsilon - r) + \varepsilon H^R_\Theta(I_\varepsilon) \right)(\tau,\cdot)\ dx$$

$$+ \int_0^T \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right) \mathrm{d}x\ dt$$

$$+ \int_0^\tau \int_\Omega \Theta \left\{ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right.$$

$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu \right\}\ dx\ dt$$

$$\leq \int_\Omega \frac{1}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2\ dx$$

$$+ \int_\Omega \Big(H_{\Theta(0,\cdot)}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot)) - \partial_{\varrho_{0,\varepsilon}} H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot))(\varrho_{0,\varepsilon} - r(0,\cdot))$$

$$+ \varepsilon H^R_{\Theta(0,\cdot)}(I_{0,\varepsilon}) \Big)\ dx$$

$$+ \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon \partial_t \vec{U} + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \vec{U} \right) \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) - p_\varepsilon\ \mathrm{div}_x \vec{U} + \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon s_\varepsilon \partial_t \Theta + \varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \Theta + \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta \right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varepsilon s^R_\varepsilon \partial_t \Theta + \vec{q}^R_\varepsilon \cdot \nabla_x \Theta \right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \partial_t \Big(\partial_{\varrho_\varepsilon} H_\Theta(r,\Theta) \Big) + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \Big(\partial_{\varrho_\varepsilon} H_\Theta(r,\Theta) \Big) \right)\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \partial_t \Big(r \partial_{\varrho_\varepsilon} H_\Theta(r,\Theta) - H_\Theta(r,\Theta) \Big)\ dx\ dt. \tag{4.18}$$

Observing finally that for $D = \partial_t$ or $D = \nabla_x$ one has

$$D\partial_{\varrho_\varepsilon} H_\Theta(r,\Theta) = -s(r,\Theta) D\Theta - r\partial_{\varrho_\varepsilon} s(r,\Theta) D\Theta + \partial^2_{\varrho_\varepsilon,\varrho_\varepsilon} H_\Theta(r,\Theta) D\varrho_\varepsilon + \partial^2_{\varrho_\varepsilon,\vartheta_\varepsilon} H_\Theta(r,\Theta) D\vartheta_\varepsilon,$$

and using the thermodynamical relations

$$\partial^2_{\varrho_\varepsilon,\varrho_\varepsilon} H_\Theta(r,\Theta) = \frac{1}{r}\ \partial_{\varrho_\varepsilon} p(r,\Theta), \quad r\partial_{\varrho_\varepsilon} s(r,\Theta) = -\frac{1}{r}\ \partial_{\vartheta_\varepsilon} p(r,\Theta),$$

and

$$\partial^2_{\varrho_\varepsilon,\vartheta_\varepsilon} H_\Theta(r,\Theta) = \partial_{\varrho_\varepsilon} \Big(\varrho_\varepsilon (\vartheta_\varepsilon - \Theta) \partial_{\vartheta_\varepsilon} s \Big)(r,\Theta) = (\vartheta_\varepsilon - \Theta) \partial_{\varrho_\varepsilon} \Big(\varrho_\varepsilon \partial_{\vartheta_\varepsilon} s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big)(r,\Theta) = 0,$$

equation (4.18) rewrites after some algebraic rearrangements (see [10] for details)

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) \right)(\tau, \cdot)\ dx + \int_0^\tau \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right)\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt,$$

$$- \int_\Omega \frac{1}{2} \left(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon}) \right)\ dx$$

$$\leq \int_0^\tau \int_\Omega \varrho_\varepsilon (\vec{u}_\varepsilon - \vec{U}) \cdot \nabla_x \vec{U} \cdot \left(\vec{U} - \vec{u}_\varepsilon \right)\ dx\ dt + \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \left(\vec{U} - \vec{u}_\varepsilon \right) \cdot \nabla_x \Theta\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(\partial_t \vec{U} + \vec{U} \cdot \nabla_x \vec{U} \right) \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) \right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(p_\varepsilon\ \mathrm{div}_x \vec{U} - \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right)\ dx\ dt - \int_0^\tau \int_\Omega \left(\varepsilon s_\varepsilon^R \partial_t \Theta + \vec{q}_\varepsilon^{\,R} \cdot \nabla_x \Theta \right)\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \partial_t \Theta \right)\ dx\ dt - \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \vec{U} \cdot \nabla_x \Theta\ dx\ dt$$

$$- \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta\ dx\ dt + \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_\varepsilon}{r} \right) \partial_t p(r,\Theta) - \frac{\varrho_\varepsilon}{r}\ \vec{u}_\varepsilon \nabla_x p(r,\Theta) \right)\ dx\ dt$$

$$=: K_0 + \sum_{j=1}^{9} \int_0^\tau K_j(t)\ dt. \tag{4.19}$$

4.2 Uniform estimates

Our intention is to apply the previous relative entropy inequality (4.19) with $(r = \rho, \vec{U} = \vec{u}, \Theta = \vartheta)$, where $(\varrho, \vec{u}, \vartheta)$ is a classical solution of the target system (in the equilibrium case or in the non equilibrium case), in order to bound the quantities $v_\varepsilon - v$, for $v = \varrho, \vec{u}, \vartheta, I$. Note that in the equilibrium case: $E^R = \int_0^\infty B(\nu, \vartheta)\ d\nu$ while accordingly in the non-equilibrium case, we note $E^R = I$, where I is the solution of the diffusion equation (3.43).

Just mention that the existence of classical solutions of the previous target systems is either local (for T small enough) or corresponds to a small departure from an equilibrium state. This last possibility corresponding to the kind of regime we are interested in (diffusion limits), we suppose in the following that the data of the problem satisfy the smallness requirements of Theorems 3.1 and 3.2.

We adapt from [9] the necessary definitions to the formalism of essential and residual sets.

Given three numbers $\overline{\varrho} \in \mathbb{R}_+$, $\overline{\vartheta} \in \mathbb{R}_+$ and $\overline{E} \in \mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values

$$\mathcal{O}^H_{ess} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \ : \ \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho}, \ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\}, \tag{4.20}$$

and $\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess} := \left\{ E^R \in \mathbb{R} \ : \ \frac{\overline{E}}{2} < E^R < 2\overline{E} \right\}, \tag{4.21}$$

with $\mathcal{O}_{ess} := \mathcal{O}^H_{ess} \cup \mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2 \backslash \mathcal{O}^H_{ess}, \quad \mathcal{O}^R_{res} := \mathbb{R}_+ \backslash \mathcal{O}^R_{ess}, \quad \mathcal{O}_{res} := (\mathbb{R}_+)^3 \backslash \mathcal{O}_{ess}. \tag{4.22}$$

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0})$ be a family of solutions of the scaled radiative Navier-Stokes system given in Theorem 4.1. We call $\mathcal{M}^\varepsilon_{ess} \subset (0,T) \times \Omega$ the set

$$\mathcal{M}^\varepsilon_{ess} = \left\{ (t,x) \in (0,T) \times \Omega \ : \ (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x), E^R_\varepsilon(t,x)) \in \mathcal{O}_{ess} \right\},$$

and $\mathcal{M}^\varepsilon_{res} = (0,T) \times \Omega \backslash \mathcal{M}^\varepsilon_{ess}$ the corresponding residual set.

To any measurable function h we decompose it into essential and residual parts $h = [h]_{ess} + [h]_{res}$ where $[h]_{ess} = h \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}$ and $[h]_{res} = h \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{res}}$. In the same way as [9], we have the following properties for matter and radiation

Lemma 4.1 *Let $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$ be two given constants and let $\mathcal{O}_{ess}$ and $\mathcal{O}_{res}$ be the corresponding sets of essential and residual values introduced in (4.22). Suppose that $(\varrho, \vartheta, E^R) \in \mathcal{O}_{ess}$.*

There exist positive constants $C_j = C_j(\overline{\varrho}, \overline{\vartheta})$ for $j = 1, \cdots, 8$ such that

1.

$$C_1 \left(|\varrho_\varepsilon - \varrho|^2 + |\vartheta_\varepsilon - \vartheta|^2 \right) \le \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \le C_2 \left(|\varrho_\varepsilon - \varrho|^2 + |\vartheta_\varepsilon - \vartheta|^2 \right), \tag{4.23}$$

for all $(\varrho_\varepsilon, \vartheta_\varepsilon) \in \mathcal{O}_{ess}$,

2.

$$\mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ge C_3, \tag{4.24}$$

for all $(\varrho_\varepsilon, \vartheta_\varepsilon) \in \mathcal{O}_{res}$,

3.

$$\mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ge C_4 \left(\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + \varrho |s(\varrho_\varepsilon, \vartheta_\varepsilon)| \right), \tag{4.25}$$

for all $(\varrho_\varepsilon, \vartheta_\varepsilon) \in \mathcal{O}_{res}$, *4.*

$$C_5 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu,\vartheta)|^2 d\vec{\omega}\, d\nu \le H^R(I_\varepsilon) \le C_6 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu,\vartheta)|^2 d\vec{\omega}\, d\nu, \quad (4.26)$$

for all $I_\varepsilon \in \mathcal{O}_{ess}$,

5.

$$H^R(I_\varepsilon) \ge C_7, \quad (4.27)$$

for all $I_\varepsilon \in \mathcal{O}_{res}$

6.

$$H^R(I_\varepsilon) \ge C_8 \left(E^R(I_\varepsilon) + |s^R(I_\varepsilon)| \right), \quad (4.28)$$

for all $I_\varepsilon \in \mathcal{O}_{res}$

Sketch of the proof:

1. The points $1, 2$ and 3 are proved in [9].
2. We have

$$H^R(I_\varepsilon) = E^R(I_\varepsilon) - \Theta\ s^R(I_\varepsilon) = \int_0^\infty \int_{\mathcal{S}^2} \psi_\varepsilon\ d\vec{\omega}\ d\nu,$$

   where $\psi_\varepsilon(t,x,\vec{\omega},\nu;\Theta,I_\varepsilon) = I_\varepsilon + \nu^2 \Theta \left(n(I_\varepsilon) \log n(I_\varepsilon) - (n(I_\varepsilon)+1) \log(n(I_\varepsilon)+1) \right)$, with $n(I_\varepsilon) = \frac{I_\varepsilon}{\nu^3}$. Computing $\partial_{I_\varepsilon} \psi_\varepsilon = 1 + \frac{\vartheta}{\nu}\ \log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1}$ and $\partial^2_{I_\varepsilon I_\varepsilon} \psi_\varepsilon = \frac{\vartheta}{\nu^4}\ \frac{I_\varepsilon}{n_\varepsilon(n_\varepsilon+1)}$, we observe that $\partial_{I_\varepsilon} \psi_\varepsilon|_{I_\varepsilon = B(\nu,\vartheta)} = 0$ and that $\partial^2_{I_\varepsilon I_\varepsilon} \psi_\varepsilon\big|_{I_\varepsilon = B(\nu,\vartheta)} > 0$. Moreover one computes $\psi_\varepsilon|_{I_\varepsilon} = -\nu^2 \vartheta \log(n(B(\nu,\vartheta)) + 1) < 0$. Then applying Taylor formula near $I_\varepsilon = B(\nu,\vartheta)$ and integrating, we get (4.26).
3. The convexity of ψ implies (4.27) and (4.28).

The previous result shows that all of the terms in the left-hand side of (4.19) are positive. Then, following the lines of [10], we have to estimate the contributions in the right-hand-side.

$$|K_1| \le \int_\Omega \varrho_\varepsilon \left| \vec{u}_\varepsilon - \vec{u} \right|^2 |\nabla_x \vec{u}|\ dx \le 2 \|\nabla_x \vec{u}\|_{L^\infty(\Omega;\mathbf{R}^9)} \int_\Omega \frac{1}{2}\ \varrho_\varepsilon \left| \vec{u}_\varepsilon - \vec{u} \right|^2\ dx.$$

$$|K_2| \le \left| \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \left(\vec{U} - \vec{u}_\varepsilon \right) \cdot \nabla_x \vartheta\ dx \right|$$

$$\le \|\nabla_x \vartheta\|_{L^\infty(\Omega;\mathbf{R}^3)} \left[2\overline{\rho} \int_\Omega \left| [s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\varrho,\vartheta)]_{ess} \right| |\vec{u} - \vec{u}_\varepsilon|\ dx \right.$$

$$\left. + \int_\Omega \left| [\varrho_\varepsilon \left(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\varrho,\vartheta) \right)]_{res} \right| |\vec{u} - \vec{u}_\varepsilon|\ dx. \right]$$

After Lemma 4.1 we have

$$\int_\Omega |[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)]_{ess}|\,|\vec u-\vec u_\varepsilon|\ dx \le \delta\,\|\vec u-\vec u_\varepsilon\|^2_{L^2(\Omega;\mathbf{R}^3}+C(\delta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx,$$

and using interpolation we get

$$\int_\Omega |[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\right)]_{res}|\,|\vec u-\vec u_\varepsilon|\ dx$$

$$\le \delta\,\|\vec u-\vec u_\varepsilon\|^2_{L^6(\Omega;\mathbf{R}^3)}+C(\delta)\,\|[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\right)]_{res}\|^2_{L^{6/5}(\Omega)}\,.$$

Using hypotheses (2.1)-(2.4) together with the (formal) property $t\to\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx \in L^\infty(0,T)$, we conclude that

$$\|[\varrho_\varepsilon\left(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\right)]_{res}\|^2_{L^{6/5}(\Omega)} \le C\left(\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx\right)^{5/3}.$$

So finally we end with

$$|K_2|\le \delta\,\|\vec u-\vec u_\varepsilon\|^2_{L^6(\Omega;\mathbf{R}^3)}+\mathcal{C}(\delta;\varrho,\vec u,\vartheta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx,$$

where the constant $\mathcal{C}$ depends on δ and the (formal) norms of $(\varrho,\vec u,\vartheta)$.

Using (3.29) (or (3.41)) we get

$$K_3=\int_\Omega \left(\varrho_\varepsilon\left(\partial_t\vec u+\vec u\cdot\nabla_x\vec u\right)\cdot(\vec u-\vec u_\varepsilon)\right)\ dx=\int_\Omega \frac{\varrho_\varepsilon}{\varrho}\ \left(\vec U-\vec u_\varepsilon\right)\left(\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\vec u)-p(\varrho,\vartheta)\right)dx$$

$$=\int_\Omega \frac{\varrho_\varepsilon-\varrho}{\varrho}\ (\vec u-\vec u_\varepsilon)\left(\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\vec u)-p(\varrho,\vartheta)\right)dx+\int_\Omega (\vec u-\vec u_\varepsilon)\left(\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\vec u)-p(\varrho,\vartheta)\right)dx.$$

Estimating the first integral as for K_2, we have

$$\left|\int_\Omega \left[\frac{\varrho_\varepsilon-\varrho}{\varrho}\ (\vec u-\vec u_\varepsilon)\left(\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\vec u)-p(\varrho,\vartheta)\right)\right]_{ess}\ dx\right|$$

$$\le \mathcal{C}(\delta;\varrho,\vec u,\vartheta)\|\,[\varrho_\varepsilon-\varrho]_{ess}\,\|^2_{L^2(\Omega)}+\delta\,\|\vec u-\vec u_\varepsilon\|^2_{L^2(\Omega;\mathbf{R}^3)}$$

$$\le \mathcal{C}(\delta;\varrho,\vec u,\vartheta)\left(\|\,[\varrho_\varepsilon]_{ess}\,\|^2_{L^{6/5}(\Omega)}+\|\,[1]_{ess}\,\|^2_{L^{6/5}(\Omega)}\right)+\delta\,\|\vec u-\vec u_\varepsilon\|^2_{L^6(\Omega;\mathbf{R}^3)}\,.$$

Integrating by parts in the second integral, we have also

$$\int_\Omega (\vec u-\vec u_\varepsilon)\left(\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\vec u)-p(\varrho,\vartheta)\right)\ dx$$

$$=\int_\Omega \left(\mathbb{S}(\vartheta,\nabla_x\vec u):\nabla_x\left(\vec u-\vec u_\varepsilon\right)-p(\varrho,\vartheta)\ \mathrm{div}_x\left(\vec u-\vec u_\varepsilon\right)\right)\ dx.$$

So, using the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$, we end as above with

$$|K_3|\le \int_\Omega \left(\mathbb{S}(\vartheta,\nabla_x\vec u):\nabla_x\left(\vec u-\vec u_\varepsilon\right)-p(\varrho,\vartheta)\ \mathrm{div}_x\left(\vec u-\vec u_\varepsilon\right)\right)\ dx$$

$$+\delta \left\|\vec{u}-\vec{u}_\varepsilon\right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)}+\mathcal{C}'(\delta;\varrho,\vec{u},\vartheta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx,$$

where the constant $\mathcal{C}'$ depends on δ and the (formal) norms of $(\varrho,\vec{u},\vartheta,\nabla_x\varrho,\nabla_x^2\vec{u},\nabla_x\vartheta)$.

Now we have

$$K_6=\int_\Omega \varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\partial_t\theta\ dx=\int_\Omega \varrho_\varepsilon\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx$$

$$+\int_\Omega \varrho_\varepsilon\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{res}\partial_t\theta\ dx,$$

where the second term is bounded as follows

$$\left|\int_\Omega \varrho_\varepsilon\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{res}\partial_t\theta\ dx\right|$$

$$\leq \|\partial_t\theta\|_{L^\infty(\Omega)}\left(\int_\Omega \Big[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big]_{res}\ dx+\|s(\varrho,\vartheta)\|_{L^\infty(\Omega)}\int_\Omega \Big[\varrho_\varepsilon\Big]_{res}\ dx\right).$$

$$\leq \mathcal{C}'(\delta;\varrho,\vec{u},\vartheta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx.$$

The remaining integral is bounded as follows

$$\int_\Omega \varrho_\varepsilon\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx=\int_\Omega\Big(\varrho_\varepsilon-\varrho\Big)\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx$$

$$+\int_\Omega \varrho\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx.$$

Using Taylor formula

$$\left|\int_\Omega\Big(\varrho_\varepsilon-\varrho\Big)\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx\right|\leq \mathcal{C}(\delta;\varrho,\vec{u},\vartheta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx.$$

Finally

$$\int_\Omega \varrho\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx$$

$$=\int_\Omega \varrho\Big[s(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)-s(\varrho,\vartheta)\Big]_{ess}\partial_t\theta\ dx$$

$$-\int_\Omega \varrho\Big[\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big]_{ess}\partial_t\theta\ dx$$

$$+\int_\Omega \varrho\Big[\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big]\partial_t\theta\ dx.$$

The first two integrals in the right-hand side can be estimated in the same way as before and we end with

$$-\int_\Omega \varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(\varrho,\vartheta)\partial_t\theta\ dx\leq \mathcal{C}(\delta;\varrho,\vec{u},\vartheta)\int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx$$

$$- \int_\Omega \varrho \Big[\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon - \varrho) + \partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon - \vartheta) \Big] \partial_t \theta \ dx.$$

Accordingly, we have also

$$K_7 = - \int_\Omega \varrho_\varepsilon \Big(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\varrho,\vartheta) \Big) \vec{u} \cdot \nabla_x \theta \ dx \leq \mathcal{C}(\delta; \varrho, \vec{u}, \vartheta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ dx$$

$$- \int_\Omega \varrho \Big[\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon - \varrho) + \partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon - \vartheta) \Big] \vec{u} \cdot \nabla_x \theta \ dx.$$

$$K_8 \leq \delta \left\| \vec{u} - \vec{u}_\varepsilon \right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}(\delta; \varrho, \vec{u}, \vartheta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ dx.$$

Finally we find

$$K_9 = \int_\Omega \Big(\left(1 - \frac{\varrho_\varepsilon}{\varrho} \right) \partial_t p(\varrho,\vartheta) - \frac{\varrho_\varepsilon}{\varrho} \ \vec{u}_\varepsilon \nabla_x p(\varrho,\vartheta) \Big) \ dx$$

$$\leq \int_\Omega \Big(\left(1 - \frac{\varrho_\varepsilon}{r} \right) \Big(\partial_t p(\varrho,\vartheta) + \vec{u} \nabla_x p(\varrho,\vartheta) \Big) \ dx + \int_\Omega p(\varrho,\vartheta) \mathrm{div}_x \vec{u}_\varepsilon \ dx$$

$$+ \int_\Omega \Big(\left(1 - \frac{\varrho_\varepsilon}{\varrho} \right) \nabla_x p(\varrho,\vartheta) \Big(\vec{u}_\varepsilon - \vec{u} \Big) \ dx,$$

and using the same argument used for K_2, we get

$$\left| \int_\Omega \Big(\left(1 - \frac{\varrho_\varepsilon}{\varrho} \right) \nabla_x p(\varrho,\vartheta) \Big(\vec{u}_\varepsilon - \vec{u} \Big) \ dx \right|$$

$$\leq \delta \left\| \vec{u} - \vec{u}_\varepsilon \right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}'(\delta; \varrho, \vec{u}, \vartheta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ dx,$$

for any $\delta > 0$.

We have then

$$\int_\Omega \Big(\left(1 - \frac{\varrho_\varepsilon}{\varrho} \right) \partial_t p(\varrho,\vartheta) - \frac{\varrho_\varepsilon}{\varrho} \ \vec{u}_\varepsilon \nabla_x p(\varrho,\vartheta) \Big) \ dx$$

$$\leq \int_\Omega \Big(\left(1 - \frac{\varrho_\varepsilon}{r} \right) \Big(\partial_t p(\varrho,\vartheta) + \vec{u} \nabla_x p(\varrho,\vartheta) \Big) \ dx + \int_\Omega p(\varrho,\vartheta) \mathrm{div}_x \vec{u}_\varepsilon \ dx$$

$$+ \delta \left\| \vec{u} - \vec{u}_\varepsilon \right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}(\delta; \varrho, \vec{u}, \vartheta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ dx.$$

Plugging all of the previous estimates into (4.19) we get

$$\int_\Omega \left(\frac{1}{2} \ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{u}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) + \varepsilon H^R(I_\varepsilon) \right) (\tau, \cdot) \ dx + \int_0^\tau \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu) \ d\Gamma \ d\nu \ dt$$

$$+ \int_0^\tau \int_\Omega \Big(\frac{\vartheta}{\vartheta_\varepsilon} \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{u}_\varepsilon - \mathbb{S}(\varrho, \vec{u}) : (\nabla_x \vec{u}_\varepsilon - \nabla_x \vec{u}) - \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{u} \Big) \ dx \ dt$$

$$+\int_0^\tau \int_\Omega \Big(\frac{\vec q(\varrho_\varepsilon,\vartheta_\varepsilon)\cdot\nabla_x\vartheta}{\vartheta_\varepsilon} - \frac{\vartheta}{\vartheta_\varepsilon}\ \frac{\vec q(\varrho_\varepsilon,\vartheta_\varepsilon)\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\Big)\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\frac{\Theta}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1}-\log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a_\varepsilon}^{(j)}(B_\varepsilon-I_\varepsilon)\ d\vec\omega\ d\nu\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\frac{\Theta}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1}-\log\frac{n(\tilde I_\varepsilon)}{n(\tilde I_\varepsilon)+1}\right]\sigma_{s_\varepsilon}^{(j)}(\tilde I_\varepsilon-I_\varepsilon)\ d\vec\omega\ d\nu\ dx\ dt,$$

$$\le \int_\Omega \frac12\left(\varrho_{0,\varepsilon}|\vec u_{0,\varepsilon}-\vec u(0,\cdot)|^2+\mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|\varrho(0,\cdot),\vartheta(0,\cdot)\right)+H^R(I_{0,\varepsilon})\right)\ dx$$

$$+\int_0^\tau\left[\delta\,\|\vec u-\vec u_\varepsilon\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)}+\mathcal{C}(\delta;\varrho,\vec u,\vartheta)\int_\Omega\Big(\frac12\ \varrho_\varepsilon|\vec u_\varepsilon-\vec u|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\Big)\ dx\right]\ dt$$

$$+\int_\Omega\Big(p(\varrho,\vartheta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x\vec u\ dx$$

$$+\int_\Omega\Big(\Big(1-\frac{\varrho_\varepsilon}{\varrho}\Big)\ \Big(\partial_t p(\varrho,\vartheta)+\vec u\cdot\nabla_x p(\varrho,\vartheta)\Big)\ dx$$

$$-\int_\Omega\varrho\Big(\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big)\Big(\partial_t\vartheta+\vec u\cdot\nabla_x\theta\Big)\ dx.$$

$$-\int_0^\tau\int_\Omega\left(\varepsilon s_\varepsilon^R\partial_t\vartheta+\vec q_\varepsilon^R\cdot\nabla_x\vartheta\right)\ dx\ dt. \tag{4.29}$$

We must now estimate the four last terms in the right-hand side. The last one is bounded as follows

$$\left|\int_0^\tau\int_\Omega\left(\varepsilon s_\varepsilon^R\partial_t\vartheta\right)\ dx\ dt\right|\le\int_0^\tau\int_\Omega\varepsilon H_\varepsilon^R|\partial_t\log\vartheta|\ dx\ dt+\int_\Omega\varepsilon E_\varepsilon^R|\partial_t\log\vartheta|\ dx\ dt$$

$$\le\|\partial_t\log\vartheta\|_{L^\infty(\Omega)}\left(\int_0^\tau\int_\Omega\varepsilon H_\varepsilon^R\ dx\ dt+e_0\right).$$

In the same stroke

$$\left|\int_\Omega\vec q_\varepsilon^R\cdot\nabla_x\vartheta\ dx\ dt\right|\le C\|\vartheta-\overline{\vartheta}\|_{L^\infty(\Omega)}\left(\int_\Omega\varepsilon H_\varepsilon^R\ dx\ dt+e_0\right),$$

provided that e_0 is small enough (see Theorems 3.1 and 3.2), then

$$\left|\int_\Omega\left(\varepsilon s_\varepsilon^R\partial_t\vartheta+\vec q_\varepsilon^R\cdot\nabla_x\vartheta\right)\ dx\ dt\right|$$

$$\le\mathcal{C}(\delta;\varrho,\vec u,\vartheta)\left(e_0+\int_\Omega\left[\frac12\ \varrho_\varepsilon|\vec u_\varepsilon-\vec u|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)+\varepsilon H_\varepsilon^R\right]\ dx\right). \tag{4.30}$$

Using the previous thermodynamical identities for H_Θ and the continuity equation for the target system, we get rid of the three other integrals (see [10]) by observing that

$$\mathcal{A}:=\int_\Omega\Big(p(\varrho,\vartheta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x\vec u\ dx+\int_\Omega\Big(\Big(1-\frac{\varrho_\varepsilon}{\varrho}\Big)\ \Big(\partial_t p(\varrho,\vartheta)+\vec u\cdot\nabla_x p(\varrho,\vartheta)\Big)\ dx$$

$$-\int_\Omega \varrho\Big(\partial_\varrho s(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta s(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big)\Big(\partial_t\vartheta+\vec u\cdot\nabla_x\theta\Big)\ dx$$

$$=\int_\Omega\Big(p(\varrho,\vartheta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x\vec u\ dx+\int_\Omega \varrho\,(\vartheta-\vartheta_\varepsilon)\,\partial_\vartheta s(\varrho,\vartheta)\Big(\partial_t\vartheta+\vec u\cdot\nabla_x\vartheta\Big)\ dx$$

$$-\int_\Omega\Big(\varrho-\varrho_\varepsilon\Big)\partial_\varrho p(\varrho,\vartheta)\mathrm{div}_x\vec u\ dx.$$

Finally the second term in the right-hand side rewrites as follows

$$\int_\Omega \varrho\,(\vartheta-\vartheta_\varepsilon)\,\partial_\vartheta s(\varrho,\vartheta)\Big(\partial_t\vartheta+\vec u\cdot\nabla_x\vartheta\Big)\ dx$$

$$=\int_\Omega \varrho\,(\vartheta-\vartheta_\varepsilon)\,\Big[\partial_t s(\varrho,\vartheta)+\vec u\cdot\nabla_x s(\varrho,\vartheta)\Big]\ dx-\int_\Omega\Big(\vartheta-\vartheta_\varepsilon\Big)\partial_\vartheta p(\varrho,\vartheta)\mathrm{div}_x\vec u\ dx$$

$$=\int_\Omega(\vartheta-\vartheta_\varepsilon)\,\Big[\frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vec u):\nabla_x\vec u-\frac{\vec q(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)-\mathrm{div}_x\Big(\frac{\vec q(\vartheta,\nabla_x\vartheta)}{\vartheta}\Big)\Big]\ dx$$

$$-\int_\Omega\Big(\vartheta-\vartheta_\varepsilon\Big)\partial_\vartheta p(\varrho,\vartheta)\mathrm{div}_x\vec u\ dx.$$

We deduce finally that

$$\mathcal{A}=\int_\Omega\Big(p(\varrho,\vartheta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_\varrho p(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta p(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big)\mathrm{div}_x\vec u\ dx.$$

$$+\int_\Omega(\vartheta-\vartheta_\varepsilon)\,\Big[\frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vec u):\nabla_x\vec u-\frac{\vec q(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)-\mathrm{div}_x\Big(\frac{\vec q(\vartheta,\nabla_x\vartheta)}{\vartheta}\Big)\Big]\ dx$$

Observing that

$$\left|\int_\Omega\Big(p(\varrho,\vartheta)-p(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_\varrho p(\varrho,\vartheta)(\varrho_\varepsilon-\varrho)-\partial_\vartheta p(\varrho,\vartheta)(\vartheta_\varepsilon-\vartheta)\Big)\mathrm{div}_x\vec u\ dx\right|$$

$$\le C\left\|\mathrm{div}_x\vec u\right\|_{L^\infty(\Omega)}\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\ dx,$$

we see that (4.29) reduces finally to

$$\int_0^\tau\int_\Omega\left(\frac{1}{2}\ \varrho_\varepsilon|\vec u_\varepsilon-\vec u|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)+\varepsilon H^R(I_\varepsilon)\right)(\tau,\cdot)\ dx\ dt$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\vartheta}{\vartheta_\varepsilon}\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec u_\varepsilon):\nabla_x\vec u_\varepsilon-\mathbb{S}(\varrho,\vec u):(\nabla_x\vec u_\varepsilon-\nabla_x\vec u)-\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec u_\varepsilon):\nabla_x\vec u\Big)\,dx\ dt$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\tilde\vartheta_\varepsilon-\vartheta}{\tilde\vartheta_\varepsilon}\mathbb{S}(\varrho_\varepsilon,\nabla_x\vec u_\varepsilon):\nabla_x\vec u\Big)\ dx\ dt+$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\vec q(\vartheta_\varepsilon,\nabla_x\vartheta_\varepsilon)\cdot\nabla_x\vartheta}{\vartheta_\varepsilon}-\frac{\vartheta}{\vartheta_\varepsilon}\ \frac{\vec q(\varrho_\varepsilon,\nabla_x\vartheta_\varepsilon)\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\Big)\ dx\ dt$$

$$+\int_0^\tau \int_\Omega \Big((\vartheta - \vartheta_\varepsilon) \frac{\vec{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta^2} + \frac{\vec{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x (\vartheta_\varepsilon - \vartheta)}{\vartheta} \Big) \ dx \ dt$$

$$\leq \int_\Omega \frac{1}{2} \left(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{u}(0,\cdot)|^2 + \mathcal{E} \left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | \varrho(0,\cdot), \vartheta(0,\cdot) \right) + \varepsilon H^R(I_{0,\varepsilon}) \right) \ dx$$

$$+\int_0^\tau \Big[\delta \, \|\vec{u} - \vec{u}_\varepsilon\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}'(\delta; \varrho, \vec{u}, \vartheta) \int_\Omega \mathcal{E} \left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) \ dx \Big] \ dt. \qquad (4.31)$$

Finally we can control the dissipative terms (the three last integrals in the left-hand side), by using verbatim the computations in ([10]) which lead to the final inequality

$$\int_\Omega \left(\frac{1}{2} \ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{u}|^2 + \mathcal{E} \left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) + \varepsilon H^R(I_\varepsilon) \right) (\tau, \cdot) \ dx + K_1 \int_0^\tau \int_\Omega \Big| \nabla_x \vec{u}_\varepsilon) - \nabla_x \vec{u} \Big|^2 \ dx \ dt$$

$$+K_2 \int_0^\tau \int_\Omega \Big| \nabla_x \vartheta_\varepsilon - \nabla_x \vartheta \Big|^2 \ dx \ dt + K_3 \int_0^\tau \int_\Omega \Big| \nabla_x \log \vartheta_\varepsilon - \nabla_x \log \vartheta \Big|^2 \ dx \ dt$$

$$\leq \int_\Omega \Big(\frac{1}{2} \left(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{u}(0,\cdot)|^2 + \mathcal{E} \left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | \varrho(0,\cdot), \theta(0,\cdot) \right) + \varepsilon H^R(I_{0,\varepsilon}) \right) \ dx$$

$$+K_3 \int_0^\tau \int_\Omega \Big(\frac{1}{2} \ \varrho_\varepsilon \left| \vec{u}_\varepsilon - \vec{u} \right|^2 + \mathcal{E} \left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) + \varepsilon H^R(I_\varepsilon) \Big) \ dx \ dt + K_4 e_0, \quad (4.32)$$

where the positive constants K_j depend on $(\varrho, \vec{u}, \vartheta, \overline{\varrho}, \overline{\varrho})$ through the norms involved in Theorems 3.1,3.2. Integrating the inequality in t we obtain the inequality

$$\int_0^\tau \int_\Omega \Big(\frac{1}{2} \ \varrho_\varepsilon \left| \vec{u}_\varepsilon - \vec{u} \right|^2 + \mathcal{E} \left(\varrho_\varepsilon, \vartheta_\varepsilon | \varrho, \vartheta \right) + \varepsilon H^R(I_\varepsilon) \Big) \ dx \ dt$$

$$\leq \mathcal{C} \left(e_0 + \int_\Omega \Big(\frac{1}{2} \left(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{u}(0,\cdot)|^2 + \mathcal{E} \left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | \varrho(0,\cdot), \theta(0,\cdot) \right) + \varepsilon H^R(I_{0,\varepsilon}) \right) \ dx \right), \qquad (4.33)$$

where the positive constant $\mathcal{C}$ has the same dependence as the K_j and depends also on T.

Lemma 4.2 *Suppose that $e_0 \leq C\varepsilon^2$ and the initial data of the primitive system and any of the target systems are close in the following sense*

$$\|\varrho_{0,\varepsilon} - \varrho_0\|_{L^2(\Omega)} \leq C\varepsilon, \|\vartheta_{0,\varepsilon} - \vartheta_0\|_{L^2(\Omega)} \leq C\varepsilon, \|\sqrt{\varrho_{0,\varepsilon}} \ (\vec{u}_{0,\varepsilon} - \vec{u})\|_{L^2(\Omega;\mathbb{R}^3)} \leq C\varepsilon.$$

Then the following estimates hold

$$\Big(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon \Big) \Big[[0,T] \times \overline{\Omega} \Big] \leq C\varepsilon^2, \qquad (4.34)$$

$$\text{ess} \sup_{t \in (0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \leq C\varepsilon^2, \qquad (4.35)$$

$$\text{ess} \sup_{t \in (0,T)} \|\varrho_\varepsilon - \varrho]_{ess}(t)\|_{L^2(\Omega)} \leq C\varepsilon, \qquad (4.36)$$

$$\text{ess}\sup_{t\in(0,T)} \|[\vartheta_\varepsilon - \vartheta]_{ess}(t)\|_{L^2(\Omega)} \le C\varepsilon, \tag{4.37}$$

$$\text{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\,(\vec{u}_\varepsilon(t) - \vec{u}(t))\,\|_{L^2(\Omega;\mathbb{R}^3)} \le C\varepsilon. \tag{4.38}$$

$$\text{ess}\sup_{t\in(0,T)} \|[E^R_\varepsilon - E^R(I)]_{ess}(t)\|_{L^2(\Omega)} \le C\varepsilon, \tag{4.39}$$

$$\text{ess}\sup_{t\in(0,T)} \|[\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\varepsilon, \tag{4.40}$$

$$\text{ess}\sup_{t\in(0,T)} \|[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\varepsilon, \tag{4.41}$$

$$\text{ess}\sup_{t\in(0,T)} \|[E^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\varepsilon, \tag{4.42}$$

$$\text{ess}\sup_{t\in(0,T)} \|[s^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\varepsilon. \tag{4.43}$$

Proof: Bound (4.34) follows after the proof of (4.33) and implies (4.35). Bounds (4.36),(4.37),(4.38) and (4.42) follow after (4.23), (4.26) and (4.33). Bounds (4.40) and (4.41) follow after (4.24) and finally (4.42) and (4.43) follow after (4.27).

Let us finally quote the following result which is a straightforward application of Proposition 5.2 of [9] (the proof is omitted)

Proposition 4.2 *Let* $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\vartheta_\varepsilon\}_{\varepsilon>0}\{I_\varepsilon\}_{\varepsilon>0}$ *three sequences of non-negative measurable functions such that*

$$\left[\varrho^{(1)}_\varepsilon\right]_{ess} \to \varrho^{(1)} \; weakly-(*) \; in \; L^\infty(0,T;L^2(\Omega)),$$

$$\left[\vartheta^{(1)}_\varepsilon\right]_{ess} \to \vartheta^{(1)} \; weakly-(*) \; in \; L^\infty(0,T;L^2(\Omega)),$$

$$\left[I^{(1)}_\varepsilon\right]_{ess} \to I^{(1)} \; weakly-(*) \; in \; L^\infty(0,T;L^2(\Omega)), \; a.e. \; in \; \mathcal{S}^2 \times \mathbb{R}_+,$$

where

$$\varrho^{(1)}_\varepsilon = \frac{\varrho_\varepsilon - \varrho}{\varepsilon}, \; \vartheta^{(1)}_\varepsilon = \frac{\vartheta_\varepsilon - \vartheta}{\varepsilon}, \; I^{(1)}_\varepsilon = \frac{I_\varepsilon - I}{\varepsilon}.$$

Suppose that

$$\text{ess}\sup_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \le C\varepsilon^2. \tag{4.44}$$

Let $G, G^R \in C^1(\overline{\mathcal{O}_{ess}})$ *be given functions. Then*

$$\frac{[G(\varrho_\varepsilon, \vartheta_\varepsilon)]_{ess} - G(\varrho,\vartheta)}{\varepsilon} \to \frac{\partial G(\varrho,\vartheta)}{\partial \varrho}\;\varrho^{(1)} + \frac{\partial G(\varrho,\vartheta)}{\partial \vartheta}\;\vartheta^{(1)},$$

weakly $-(*)$ *in* $L^\infty(0,T;L^2(\Omega))$*, and if we note*

$$\left[G^R(I_\varepsilon)\right]_{ess} := \left[G^R(I_\varepsilon(\cdot,\cdot,\vec{\omega},\nu))\right]_{ess} = G^R(I_\varepsilon)\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}, \; for \; a.a. \; (\vec{\omega},\nu) \in \mathcal{S}^2 \times \mathbb{R}_+,$$

we have
$$\frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(I)}{\varepsilon} \to \frac{\partial G(I)}{\partial I}\ I^{(1)},$$
weakly $-(*)$ *in* $L^\infty(0,T;L^2(\Omega))$, *a.e. in* $\mathcal{S}^2 \times \mathbb{R}_+$.

Moreover if $G, G^R \in C^2(\overline{\mathcal{O}_{ess}})$ *then*
$$\left\| \frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess} - G(\varrho,\vartheta)}{\varepsilon} - \frac{\partial G(\varrho,\vartheta)}{\partial \varrho}\ \left[\varrho^{(1)}\right]_{ess} - \frac{\partial G(\varrho,\vartheta)}{\partial \vartheta}\ \left[\vartheta^{(1)}\right]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$
and
$$\left\| \frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(I)}{\varepsilon} - \frac{\partial G(I)}{\partial I}\ \left[I^{(1)}\right]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$
for a.a. $(\vec{\omega},\nu) \in \mathcal{S}^2 \times \mathbb{R}_+$.

5 Convergence toward the target systems

We are now in position to prove that the *equilibrium diffusion target system* (3.28)-(3.31) and the *non-equilibrium diffusion target system* (3.40)-(3.43)) are the limit in a suitable sense, of the primitive system (4.3)-(4.11) when $\varepsilon \to 0$.

Namely the convergence result in the equilibrium case goes as follows

Theorem 5.1 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain of class* $C^{2,\nu}$. *Assume that the thermodynamic functions* p, e, s *satisfy hypotheses (2.1 - 2.4) with* $P \in C^1[0,\infty) \cap C^2(0,\infty)$, *and that the transport coefficients* μ, λ, κ, σ_a, σ_s *and the equilibrium function* B *comply with (2.5 - 2.9).*

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ *be a weak solution to the scaled radiative Navier-Stokes system (1.4 - 1.7) for* $(t,x,\vec{\omega},\nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, *supplemented with the boundary conditions (1.9 - 1.10) and the initial conditions* $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ *such that*
$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$
where $(\varrho_0, \vec{u}, \vartheta_0) \in H^3(\Omega)$ *are smooth functions (see Theorem 3.1) such that* (ϱ_0, ϑ_0) *belong to the set* $\mathcal{O}^H_{ess}$ *defined in (4.22) where* $\overline{\varrho} > 0$, $\overline{\varrho} > 0$, *are two constants and* $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0$, $\int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ dx = 0$.

Suppose also that
$$\vec{u}_{0,\varepsilon} \to \vec{u}_0 \ \ weakly - (*) \ in \ L^\infty(\Omega;\mathbb{R}^3).$$
Then up to subsequences
$$\varrho_\varepsilon \to \varrho \ \ weakly - (*) \ in \ L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$
$$\vec{u}_\varepsilon \to \vec{u} \ weakly - (*) \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \vartheta \ \ weakly - (*) \ in \ L^\infty(0,T;L^4(\Omega)),$$
where $(\varrho, \vec{u}, \vartheta)$ *is the smooth solution of the equilibrium decoupled system (3.28)-(3.31) on* $[0,T] \times \Omega$ *and* $I(t,x,\nu,\vec{\omega}) = B(\nu, \vartheta(t,x))$, *with initial data* $(\varrho_0, \vec{u}_0, \vartheta_0)$.

The analogous convergence result in the non-equilibrium case is

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.4) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.5 - 2.9).*

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system (1.4 - 1.7) for $(t, x, \vec{\omega}, \nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.9 - 1.10) and the initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \quad I_\varepsilon(0,\cdot) = I_0 + \varepsilon I^{(1)}_{0,\varepsilon},$$

where the functions $(\varrho_0, \vec{u}, \vartheta_0)$ and $x \to I_0(x, \vec{\omega}, \nu)$ belong to $H^3(\Omega)$ and are such that $(\varrho_0, \vartheta_0, E_R(I_0))$ belong to the set $\mathcal{O}_{ess}$ defined in (4.22) where $\overline{\varrho} > 0$, $\overline{\varrho} > 0$, $\overline{E_R} > 0$ are three constants and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\, dx = 0$, $\int_\Omega \vartheta^{(1)}_{0,\varepsilon}\, dx = 0$, $\int_\Omega I^{(1)}_{0,\varepsilon}\, dx = 0$.

Suppose also that

$$\vec{u}_{0,\varepsilon} \to \vec{u}_0 \ \ weakly-(*) \ in \ L^\infty(\Omega; \mathbb{R}^3).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \ weakly-(*) \ in \ L^\infty(0,T; L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \ weakly-(*) \ in \ L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \ weakly-(*) \ in \ L^\infty(0,T; L^4(\Omega)),$$

and

$$N_\varepsilon \to N \ \ weakly-(*) \ in \ L^\infty((0,T) \times \Omega),$$

where $N_\varepsilon = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon \, d\vec{\omega} \ \ d\nu$ and $(\varrho, \vec{u}, \vartheta, N)$ is the smooth solution of the Navier-Stokes-Rosseland system (3.40)-(3.43) on $[0,T] \times \Omega$ with initial data $(\varrho_0, \vec{u}_0, \vartheta_0, N_0)$.

The proof of these results is the matter of the last part of the paper.

5.1 Proof of Theorem 5.1

As the first two equations (4.3) and (4.4) of our model are similar to those of the Navier-Stokes-Fourier analyzed in [9], in the following we only sketch the essential points, insisting on the energy and radiative contributions.

Let us observe that after Theorem 2.1, bounds (2.7)-2.9) and relative entropy inequalities (4.29) and (4.33), the temperature ϑ_ε is bounded in $L^2(0,T; W^{1,2}(\Omega))$ then after extraction of a subsequence

$$\vartheta_\varepsilon \to \vartheta \ \ in \ L^2([0,T] \times \Omega). \tag{5.1}$$

5.1.1 Continuity and Momentum equations

For the continuity equation, one observes after (4.34) that

$$\int_0^T \left\| \nabla_x \vec{u}_\varepsilon + \nabla_x^T \vec{u}_\varepsilon - \frac{2}{3}\, \mathrm{div}_x \vec{u}_\varepsilon \mathbb{I} \right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C.$$

Using this fact together with (4.43) and (4.35), we get

$$\int_0^T \| \vec{u}_\varepsilon(t) - \vec{u}(t) \|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \, dt \leq C,$$

so passing to the limit after possible extraction of a subsequence, we have $\vec{u}_\varepsilon \to \vec{u}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$. In the same stroke $\varrho_\varepsilon \to \varrho$, weakly in $L^\infty(0,T;L^{5/3}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (4.3) which rewrites as (3.28), together with the boundary condition $\vec{u} \cdot n_x|_{\partial\Omega} = 0$, provided $\partial\Omega$ is regular.

For the momentum equation, one knows that due to possible strong time oscillations of the gradient component of velocity, one has only $\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon \to \overline{\varrho \vec{u} \otimes \vec{u}}$, weakly in $L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3))$, however one can show after the analysis of [9] (see [7]) that one can pass to the limit in the convective term and obtain

$$\int_0^T \int_\Omega \overline{\varrho\, \vec{u} \otimes \vec{u}} : \nabla_x \phi \; dx \; dt \to \int_0^T \int_\Omega \varrho\, \vec{u} \otimes \vec{u} : \nabla_x \phi \; dx \; dt.$$

Moreover after the hypotheses on pressure, ϑ_ε is bounded in $L^\infty((0,T);L^4(\Omega)) \cap L^2(0,T;L^6(\Omega))$, which implies that

$$\mathbb{S}_\varepsilon \to \mu(\vartheta)(\nabla_x \vec{u} + \nabla_x^t \vec{u}),$$

weakly in $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ for a $q > 1$. So using weak compactness arguments of [9] for effective viscous flux, we can also pass to the limit in momentum equation (4.4) and obtain (3.29).

5.1.2 Radiative transfer equation

We have shown in the previous sections that $I_\varepsilon \to I$, weakly in $L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+)$, and that $\vartheta_\varepsilon \to \vartheta$, weakly in $L^2(0,T;W^{1,2}(\Omega))$. As the radiative transfer equation (4.11) is linear in I, we can pass to the limit in the weak formulation of radiative transfer equation which gives

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \sigma_a(\vartheta)\,(B(\nu,\vartheta) - I)\,\psi \; d\vec{\omega} \; d\nu \; dx = 0,$$

for any test function $\psi \in C_c^\infty((0,T) \times \overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$ which is the weak formulation of the equation $I(t,x,\nu,\vec{\omega}) = B(\nu,\vartheta(t,x))$.

5.1.3 Entropy balance

We rewrite equation (4.6) in the form

$$\int_0^\tau \int_\Omega \Big((\varrho s + s^R)\partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \frac{\vec{q}}{\vartheta} \cdot \nabla_x \varphi \Big) \ dx \ dt + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon^R}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \ dx \ dt$$

$$+ \langle \varsigma^m ; \phi \rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} + \big\langle \varsigma_\varepsilon^R ; \phi \big\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} - \int_\Omega \big(\varrho s + s^R \big))(0,\cdot)\varphi(0,\cdot) \ dx$$

$$\leq \int_\Omega \big[\big(\varrho_{0,\varepsilon} s_{0,\varepsilon} + s^R_{0,\varepsilon} \big) - \big(\varrho_0 s_0 + s_0^R \big) \big] \varphi(0,\cdot) \ dx$$

$$- \int_0^\tau \int_\Omega \big\{ \varrho_\varepsilon (s_\varepsilon - s) + (\varrho_\varepsilon - \varrho) s + s_\varepsilon^R - s^R \big) \big\} \partial_t \varphi \ dx \ dt$$

$$- \int_0^\tau \int_\Omega \{ \varrho_\varepsilon (s_\varepsilon - s) \vec{u}_\varepsilon + (\varrho_\varepsilon \vec{u}_\varepsilon - \varrho \vec{u}) s \} \cdot \nabla_x \varphi \ dx \ dt$$

$$- \int_0^\tau \int_\Omega \left[\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} - \frac{\vec{q}}{\vartheta} \right] \cdot \nabla_x \varphi \ dx \ dt + \langle \varsigma_\varepsilon^m - \varsigma^m ; \phi \rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} .$$

for any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$.

Using Proposition 4.2, one computes first

$$\frac{1}{\varepsilon} \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon^R}{\vartheta_\varepsilon} \cdot \nabla_x \varphi \ dx \ dt \to \int_0^\tau \int_\Omega \frac{\vec{f}_1}{\vartheta} \cdot \nabla_x \varphi \ dx \ dt,$$

as $\varepsilon \to 0$, where $\vec{f}_1$ is given by formula (3.27).

In the same stroke, we find

$$\big\langle \varsigma_\varepsilon^R ; \phi \big\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} \to \int_0^\tau \int_\Omega \frac{\vec{f}_1 \cdot \nabla_x \vartheta}{\vartheta^2} \ \varphi \ dx \ dt.$$

as $\varepsilon \to 0$, by using once more Proposition 4.2.

After the conditions on the data and the estimates in Lemma 4.2 and using verbatim the techniques of [9](Chap. 5) one concludes that all of the integrals in the right hand side converge to zero as $\varepsilon \to 0$, which proves that the limit inequality (3.42) is obtained..

5.2 Proof of Theorem 5.2

Exactly as in the equilibrium limit, the temperature ϑ_ε is bounded in $L^2(0,T; W^{1,2}(\Omega))$ then (5.1) holds, moreover we can pass to the limit in the weak continuity equation (4.3) which rewrites as (3.40) and we can also pass to the limit in momentum equation (4.4) and obtain (3.41).

5.2.1 Radiative transfer equation

We can adapt the result of Bardos, Golse, Perthame and Sentis [3] (see also [13]).

As we consider the "grey hypothesis", we use the average notation I_ε instead of $N_\varepsilon := \int_0^\infty I_\varepsilon \, d\nu$ in all this subsection. We start with

$$\partial_t I_\varepsilon + \frac{1}{\varepsilon} \, \vec{\omega} \cdot \nabla_x I_\varepsilon = \sigma_{a,\varepsilon} \left(B_\varepsilon - I_\varepsilon \right) + \frac{1}{\varepsilon^2} \, \sigma_{s,\varepsilon} \left(\tilde{I}_\varepsilon - I_\varepsilon \right), \tag{5.2}$$

with $B_\varepsilon := B(\vartheta_\varepsilon)$, and

$$I_\varepsilon|_{t=0} = I_0, \tag{5.3}$$

where $\tilde{I}_\varepsilon = \frac{1}{4\pi} \int_{\mathcal{S}^2} I_\varepsilon \, \mathrm{d}\vec{\omega}$, $\sigma_{a,\varepsilon} = \sigma_a(\vartheta_\varepsilon)$, $\sigma_{s,\varepsilon} = \sigma_s(\vartheta_\varepsilon)$ and $B_\varepsilon = B(\nu, \vartheta_\varepsilon)$.

After [7] we see that

$$\|I_\varepsilon\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C(T) \left(1 + \|I_0\|_{L^\infty(\Omega \times \mathcal{S}^2)} \right).$$

Multiplying (5.2) by I_ε, integrating over the whole phase space and using (2.9), we get

$$\|\sigma_{a,\varepsilon}^{1/2} \left(B_\varepsilon - I_\varepsilon \right) \|_{L^2(\Omega \times \mathcal{S}^2)} \leq C\varepsilon, \tag{5.4}$$

$$\|\sigma_{s,\varepsilon}^{1/2} \left(\tilde{I}_\varepsilon - I_\varepsilon \right) \|_{L^2(\Omega \times \mathcal{S}^2)} \leq C\varepsilon, \tag{5.5}$$

and

$$\left\| \varepsilon \, \partial_t I_\varepsilon + \frac{1}{\varepsilon} \, \vec{\omega} \cdot \nabla_x I_\varepsilon \right\|_{L^2(\Omega \times \mathcal{S}^2)} \leq C. \tag{5.6}$$

Using the Fourier argument of [3] (see Lemma 3 in [3]) we also get that for any $T > 0$ $\left(\tilde{I}_\varepsilon^\alpha \right)^{1/\alpha}$ is bounded in $L^q(0, T; W^{\beta,q}(\Omega))$ where $q = \frac{2p}{p+1}$, $\alpha = 1 + \frac{1}{2p}$ and for any $\beta < \frac{p-1}{2p+1}$.

Integrating (5.2) over $\vec{\omega}$, we get first

$$\partial_t \, \widetilde{I_\varepsilon} + \frac{1}{\varepsilon} \, \mathrm{div}_x \, \widetilde{\vec{\omega} I_\varepsilon} = \sigma_{a,\varepsilon} \left(B_\varepsilon - \tilde{I}_\varepsilon \right), \tag{5.7}$$

and multiplying (5.2) by ω and integrating over $\vec{\omega}$, we also have

$$\partial_t \, \widetilde{\vec{\omega} I_\varepsilon} + \frac{1}{\varepsilon} \, \mathrm{div}_x \, (\vec{\omega} \widetilde{\otimes \vec{\omega}} I_\varepsilon) = - \left(\frac{1}{\varepsilon^2} \, \sigma_{s,\varepsilon} + \varepsilon \sigma_{a,\varepsilon} \right) \widetilde{\vec{\omega} I_\varepsilon}. \tag{5.8}$$

Then we get the equation

$$\partial_t \, \widetilde{I_\varepsilon} - \mathrm{div}_x \left(\frac{1}{\sigma_{s,\varepsilon} + \varepsilon \sigma_{a,\varepsilon}} \left[\varepsilon \partial_t \, \widetilde{\vec{\omega} I_\varepsilon} + \, \mathrm{div}_x \, (\vec{\omega} \widetilde{\otimes \vec{\omega}} I_\varepsilon) \right] \right)$$

$$= \sigma_{a,\varepsilon} \left(B_\varepsilon - \tilde{I}_\varepsilon \right) \quad \text{in } \mathcal{D}'((0,T) \times \Omega \times \mathcal{S}^2). \tag{5.9}$$

Using (5.6) and (2.9), we conclude that the sequence $\{\partial_t \, \widetilde{I_\varepsilon}\}_\varepsilon$ is bounded in $L^q(0, T; W^{-1,q}(\Omega))$.

Setting $J_\varepsilon := \left(\tilde{I_\varepsilon^\alpha}\right)^{1/\alpha}$, we deduce that

$$J_\varepsilon \in L^q([0,T]; W^{\beta,q}(\Omega)),$$
$$\|\tilde{I}_\varepsilon - J_\varepsilon\|_{L^q((0,T)\times\Omega)} \to 0 \ \text{ for } \varepsilon \to 0,$$

and

$$\partial_t \tilde{I}_\varepsilon \in L^q([0,T]; W^{-1,q}(\Omega)).$$

Applying a variant of the Aubin-Lions Lemma given in [3], there exists a subsequence $\widetilde{I_\varepsilon}$ converging in $L^q((0,T)\times\Omega)$.

Now we can pass to the limit in (5.2). In fact from (5.4) and (5.6) we see that there exists a $g \in L^2((0,T)\times\Omega\times\mathcal{S}^2)$ such that

$$(\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon})^{-1/2}\,\mathrm{div}_x\,(\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon) \to g \ \text{ weakly in } L^2((0,T)\times\Omega\times\mathcal{S}^2).$$

Multiplying by $(\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon})^{1/2}\, I_\varepsilon$ and using (2.7)-(2.9) and (5.1) we obtain

$$I_\varepsilon \mathrm{div}_x\ (\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon) \to g\sigma_s^{1/2} I \ \text{ weakly in } L^1((0,T)\times\Omega\times\mathcal{S}^2),$$

with $\sigma_s = \sigma_s(\vartheta)$.

Now we see from above that

$$(\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon})^{1/2}\, I_\varepsilon \to \sigma_s^{1/2} I \ \text{ weakly in } L^2((0,T)\times\Omega\times\mathcal{S}^2),$$

so

$$\frac{1}{2}\,\mathrm{div}_x\ (\vec{\omega}\otimes\vec{\omega}\ I_\varepsilon^2) \to g\sigma_s^{1/2} I \ \text{ weakly in } L^1((0,T)\times\Omega\times\mathcal{S}^2),$$

and that

$$\frac{1}{2}\,\mathrm{div}_x\ (\vec{\omega}\otimes\vec{\omega} I_\varepsilon^2) \to \frac{1}{2}\,\mathrm{div}_x\ (\vec{\omega}\otimes\vec{\omega} I^2) \ \text{ weakly in } \mathcal{D}'((0,T)\times\Omega\times\mathcal{S}^2).$$

Therefore

$$g\sigma_s^{1/2} I = \frac{1}{2}\,\mathrm{div}_x\ (\vec{\omega}\otimes\vec{\omega} I^2).$$

Exactly as in [3], one can now check that

$$\sigma_s^{-1/2}\tilde{g} = \frac{1}{3}\frac{1}{\sigma_s}\ \nabla_x I,$$

and therefore one can pass to the limit in the second term in the left hand side of (5.9)

$$\frac{1}{\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon}}\ \nabla_x(\widetilde{\vec{\omega}\otimes\vec{\omega}} I_\varepsilon) = \frac{1}{(\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon})^{1/2}}\frac{1}{(\sigma_{s,\varepsilon} + \varepsilon\sigma_{a,\varepsilon})^{1/2}}\ \nabla_x(\widetilde{\vec{\omega}\otimes\vec{\omega}} I_\varepsilon)$$
$$\to \sigma_s^{-1/2}\tilde{g} = \frac{1}{3}\frac{1}{\sigma_s}\ \nabla_x I. \tag{5.10}$$

As the term in the right hand side of (5.9) clearly converges to $\sigma_a(\vartheta)\,[B(\vartheta) - I]$, this finally proves that I satisfies the limit equation (3.43).

The argument of [3] shows finally that I satisfies the Dirichlet boundary condition $I|_{\partial\Omega} = 0$. In fact from the fact that $\vec{\omega}\cdot\nabla_x I_\varepsilon^2$ is bounded in $L^2((0,T)\times\Omega\times\mathbb{R}_+)$ we deduce that I_ε has a trace which holds at the limit.

5.2.2 Entropy balance

We rewrite equation (4.6) in the form

$$\int_0^\tau \int_\Omega \Big(\varrho s \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \frac{\vec{q}}{\vartheta} \cdot \nabla_x \varphi \Big) \, dx \, dt + \langle \varsigma^m ; \phi \rangle_{[\mathcal{M};C]([0,T \times \overline{\Omega})}$$

$$+ \int_0^\tau \int_\Omega \frac{S_E(I)}{\vartheta} \, \varphi \, dx \, dt - \int_\Omega \varrho_0 s_0 \varphi(0,\cdot) \, dx$$

$$\leq \int_0^\tau \int_\Omega \Big(\frac{S_E(I_\varepsilon)}{\vartheta_\varepsilon} - \frac{S_E(I)}{\vartheta} \Big) \varphi \, dx \, dt. + \int_\Omega \left[\varrho_{0,\varepsilon} s_{0,\varepsilon} - \varrho_0 s_0 \right] \varphi(0,\cdot) \, dx$$

$$- \int_0^\tau \int_\Omega \{ \varrho_\varepsilon (s_\varepsilon - s) + (\varrho_\varepsilon - \varrho) s \} \, \partial_t \varphi \, dx \, dt$$

$$- \int_0^\tau \int_\Omega \{ \varrho_\varepsilon (s_\varepsilon - s) \vec{u}_\varepsilon + (\varrho_\varepsilon \vec{u}_\varepsilon - \varrho \vec{u}) s \} \cdot \nabla_x \varphi \, dx \, dt$$

$$- \int_0^\tau \int_\Omega \left[\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} - \frac{\vec{q}}{\vartheta} \right] \cdot \nabla_x \varphi \, dx \, dt + \langle \varsigma_\varepsilon^m - \varsigma^m ; \phi \rangle_{[\mathcal{M};C]([0,T \times \overline{\Omega})} \, .$$

for any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$.

We first observe that the first term in the right-hand side converge to zero, by applying the same argument as [7](see Proposition 4.1) based on the average Lemma of Bournaveas and Perthame [4].

Finally, after the hypotheses made on the data and the estimates in Lemma 4.2 and using once more verbatim the techniques of [9](Chap. 5) one concludes that all of the remaining integrals in the right hand side converge to zero as $\varepsilon \to 0$, which proves that the limit inequality (3.42) is obtained.

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