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Stanislav Kračmar
Šárka Nečasová
Patrick Penel

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Une variante avec poids des inégalités d'injection

Stanislav Kračmar[a], Šárka Nečasová[b], Patrick Penel[c]

[a] *Czech Technical University, Technical Mathematics, Karlovo náměsti 13, 121 35 Prague 2, Czech Republic*

[b] *Czech Academy of Sciences, Institute of Mathematics, Žitná 25, 115 67 Praha 1, Czech Republic*

[c] *Université du Sud, Toulon-Var, Mathématique, BP 20132, 83957 La Garde Cedex, France*

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ABSTRACT

In this Note, for vector functions defined on unbounded domains of $\mathbb{R}^3$, we consider continuous embeddings of weighted homogeneous Sobolev spaces into weighted Lebesgue spaces. Sufficient conditions on power-type weights for the validity of the inequalities are investigated. Moreover, the related properties of the suitable approximation by smooth functions with a bounded support can be proved.

RÉSUMÉ

Dans cette Note, pour des fonctions vectorielles définies sur des domaines non bornés de $\mathbb{R}^3$, nous considérons des inégalités d'injection d'espaces de Sobolev homogènes avec poids dans des espaces de Lebesgue avec poids. Des conditions suffisantes pour justifier ces inégalités sont établies dans le cas de poids de type puissance. En outre nous vérifions les propriétés d'approximation par des fonctions indéfiniment différentiables à support borné.

1. Introduction and formulation of the main results

The homogeneous Sobolev spaces of vector functions $\mathbf{D}^{1,q}_w(\Omega)$ are appropriate for the analysis of systems of partial differential equations and boundary-value problems in unbounded exterior domains Ω of $\mathbb{R}^3$, like the complementary set of one or more compact sets Ω^c in $\mathbb{R}^3$. The control of a suitable behavior at large distances is required for the solution vector fields. So a fundamental role in our treatment is played by the choice of admissible radial weights w in the q-class of Muckenhoupt weights.

We are inspired by Galdi's presentation of Sobolev classical embedding inequalities (see his book [3] chapter II section 5) to provide the weighted embedding inequalities. Another approach by using full Sobolev spaces with radial weights can be found in the works of Amrouche, Girault and their collaborators see e.g. [2]; a generalization of Lemma II.5.2 of [3] in this functional setting is given by Alliot [1], see Proposition 3.8. Let us mention that there are several results on weighted full Sobolev spaces and embeddings, or even weighted embedding of homogeneous Sobolev spaces but with different weights. See [7, 4, 8, 10].

The following conditions $(A_1^\alpha)_q$, and $(A_2^\alpha)_q$ are preparatory and adapted to our analysis:

$$(A_1^\alpha)_q \qquad \left(\int_R^r \frac{\mathrm{d}\rho}{\rho^{\frac{2}{q-1}} w(\rho)^{\frac{1}{q-1}}} \right)^{q-1} \leq \begin{cases} c(q,\kappa)\cdot R^{-\alpha}, & \text{for some } \alpha > 0, \quad \text{for } 1 < q < 3 \\ c(q,\kappa) & \text{for } q = 3 \\ c(q,\kappa)\cdot r^{\alpha}, & \text{for some } \alpha > 0, \quad \text{for } q > 3 \end{cases}$$

$$(A_2^\alpha)_q \qquad \begin{cases} |\cdot|^{2-q-\alpha}(\ln|\cdot|)^{-q} & \in \mathbf{L}^1_w(\Omega), \quad 1<q<3 \\ |\cdot|^{2-q+\alpha}(\ln|\cdot|)^{-q} & \in \mathbf{L}^1_w(\Omega), \quad q \geq 3 \end{cases}$$

The conditions (A^α) we introduce above do not impose serious restriction on radial weights in the q-class of Muckenhoupt weights. For instance, when the weight is assumed to be a power type function $w_\kappa(|\mathbf{x}|) := (1+|\mathbf{x}|)^\kappa$ for some $\kappa > 0$, condition $(A_1^\alpha)_{1<q<3}$ is always true for $\alpha = \frac{3-q+\kappa}{q-1}$.

Let us fix some notations : for any $\mathbf{x}_0 \in \mathbb{R}^3$ all used parameters $R > 0$ will have the property $\Omega^c \subset B_R(\mathbf{x}_0)$, where $B_R(\mathbf{x}_0)$ denotes the $\mathbf{x}_0$-centered ball of radius R; we now set $\Omega^R(\mathbf{x}_0) := \Omega \setminus B_R(\mathbf{x}_0)$, $\Omega_R(\mathbf{x}_0) := \Omega \cap B_R(\mathbf{x}_0)$, and $\Omega_{R,r}(\mathbf{x}_0) := \Omega_r(\mathbf{x}_0) \setminus \Omega_R(\mathbf{x}_0)$ for a spherical shell with $0 < \delta(\Omega^c) < R < r$.

Our objective is to establish the following results, where we assume concrete radial weights of the form w_κ :

T1 **Theorem 1.** *(***on a weighted embedding inequality***) Let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Assume that $\mathbf{u}$ is given in $\mathbf{D}^{1,q}_w(\Omega)$, $1<q<3$, with the weight $w = w_\kappa$ and $\kappa < \frac{3-q}{2q}$. Let the constant vector $\mathbf{u}_0$ be defined in Lemma 1.*
Then, for any $\mathbf{x}_0 \in \mathbb{R}^3$ and for all $R > 0$, $(\mathbf{u}(\cdot) - \mathbf{u}_0)(|\cdot - \mathbf{x}_0|^{-1}) \in \mathbf{L}^q_w(\Omega^R(\mathbf{x}_0))$. Moreover, there exists $K_1 = K_1(q, \mathbf{x}_0) > 0$ such that

$$\left(\int_{\Omega^R(\mathbf{x}_0)} \left| \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}_0}{\mathbf{x} - \mathbf{x}_0} \right|^q w(|\mathbf{x}|)\, \mathrm{d}\mathbf{x} \right)^{1/q} \leq K_1 \, |\mathbf{u} - \mathbf{u}_0|_{1,q,\Omega^R(\mathbf{x}_0);\, w} \,. \tag{1}$$ eq17

If Ω is locally lipschitzian, denoting by $s(q) = \frac{3q}{3-q}$ the Sobolev exponent, there exists $K_2 = K_2(q) > 0$ such that

$$\|\mathbf{u} - \mathbf{u}_0\|_{s(q),\Omega;\, w} \leq K_2 \, |\mathbf{u}|_{1,q,\Omega;\, w}. \tag{2}$$ eq20

T2 **Theorem 2.** *(***another form of weighted embedding inequality***) Let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Assume that $\mathbf{u}$ is given in $\mathbf{D}^{1,q}_w(\Omega) \cap \mathbf{L}^q_{\nabla w}(\Omega)$, $1<q<3$, with the weight $w = w_\kappa$ and $\kappa < \frac{3-q}{q}$. Let the constant vector $\mathbf{u}_0$ be defined in Lemma 1.*
Then, for any $\mathbf{x}_0 \in \mathbb{R}^3$ and for all $R > 0$, $(\mathbf{u}(\cdot) - \mathbf{u}_0)(|\cdot - \mathbf{x}_0|^{-1}) \in \mathbf{L}^q_w(\Omega^R(x_0))$. Moreover, there exists $K_3 = K_3(q, \mathbf{x}_0) > 0$ such that

$$\left(\int_{\Omega^R(\mathbf{x}_0)} \left| \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}_0}{\mathbf{x} - \mathbf{x}_0} \right|^q w(|\mathbf{x}|)\, \mathrm{d}\mathbf{x} \right)^{1/q} \leq K_3 \, \left(|\mathbf{u} - \mathbf{u}_0|_{1,q,\Omega^R(\mathbf{x}_0);\, w} + \|\mathbf{u} - \mathbf{u}_0\|_{q,\Omega^R(\mathbf{x}_0);\, |\nabla w|} \right). \tag{3}$$ eq17a

If Ω is locally lipschitzian, denoting by $s(q)$ the same value as in Theorem 1, there exists $K_4 = K_4(q) > 0$ such that

$$\|\mathbf{u} - \mathbf{u}_0\|_{s(q),\Omega;\, w} \leq K_4 \left(|\mathbf{u}|_{1,q,\Omega;\, w} + \|\mathbf{u} - \mathbf{u}_0\|_{q,\Omega;\, |\nabla w|} \right). \tag{4}$$ eq21

T3 **Theorem 3.** *(***on the approximation by smooth functions, $\mathbf{1 \leq q < 3}$***) Let $\Omega \subset \mathbb{R}^3$ be a locally lipschitzian exterior domain, $\mathbf{u} \in \mathbf{D}^{1,q}_w(\Omega)$, $1 \leq q < 3$, where the weight $w = w_\kappa$ satisfies the conditions $(A_1^\alpha)_{1<q<3}$, and $(A_2^\alpha)_{1<q<3}$. Let $\mathbf{u}_0$ be the constant vector given by Lemma 1.*
Then $\mathbf{u}$ can be approximated in the semi-norm $|\cdot|_{1,q,\Omega;\, w}$ by functions from $C_0^\infty(\Omega)^3$ if and only if $\mathbf{u}$ has zero trace on the boundary $\partial\Omega$ and $\mathbf{u}_0 = \mathbf{0}$.

C3 **Corollary 1.** *(***the unweighted case, $\mathbf{1 \leq q < 3}$***) Let $\Omega \subset \mathbb{R}^3$ be a locally lipschitzian exterior domain. The unconditional version of Lemma 1 where $w \equiv 1$ and $\alpha = \frac{3-q}{q-1}$ gives the constant vector $\mathbf{u}_0$.*
Then functions $\mathbf{u} \in \mathbf{D}^{1,q}(\Omega)$, $1 \leq q < 3$, can be approximated in the semi-norm $|\cdot|_{1,q,\Omega;\, 1}$ by functions from $C_0^\infty(\Omega)^3$ if and only if $\mathbf{u}$ has zero trace on the boundary $\partial\Omega$ and $\mathbf{u}_0 = \mathbf{0}$.

Remark 1. *The corollary just shown improves the corresponding theorem in ([3], Theorem II.7.1), indeed that properties $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ and $\mathbf{u}_0 = \mathbf{0}$ are not only sufficient but also necessary for approximating functions from $\mathbf{D}^{1,q}(\Omega)$ by smooth functions with compact support. As it is explained in [3], one can also replace the zero trace $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$ by the condition $\psi\mathbf{u} \in \mathbf{W}^{1,q}_0(\Omega)$ for all $\psi \in C_0^\infty(\mathbb{R}^3)$ without assuming any regularity on $\partial\Omega^c$.*

T4 **Theorem 4.** *(**on the approximation by smooth functions, $\mathbf{q \geq 3}$**) Let $\Omega \subset \mathbb{R}^3$ be a locally lipschitzian exterior domain, $\mathbf{u} \in \mathbf{D}^{1;q}_w(\Omega)$, $q \geq 3$ where the weight $w = w_\kappa$ satisfies conditions $(A_1^\alpha)_{q\geq 3}$, and $(A_2^\alpha)_{q \geq 3}$. Then $\mathbf{u}$ can be approximated in the semi-norm $\|\cdot\|_{1,q,\Omega;\,w}$ by functions from $C_0^\infty(\Omega)^3$ if and only if $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$.*

2. Relevant preliminaries

We assume w a radial weight function in the q-class of Muckenhoupt weights, and $\mathbf{u} \in \mathbf{D}^{1;q}_w(\Omega)$, $1 \leq q < 3$, a given vector field. S^2 is the unit sphere of $\mathbb{R}^3$. Let us begin with the following lemma, which is crucial to estimate all surface integrals, and which gives explicitly this constant vector of $\mathbb{R}^3$ we denote by $\mathbf{u}_0$. This lemma can be considered as a generalization of Lemma II.5.2 [3] for radial weights.

L1 **Lemma 1.** *($1 < q < 3$) Under the condition $(A_1^\alpha)_{1<q<3}$, there exists a unique $\mathbf{u}_0 \in \mathbb{R}^3$ such that*

$$\int_{S^2} |\mathbf{u}(R,\varphi) - \mathbf{u}_0|^q \,\mathrm{d}\varphi \;\leq\; C_q\, R^{-\alpha}\, \|\nabla \mathbf{u}\|^q_{q,\Omega^R;\,w} \tag{5}$$ eq1

Proof. We consider the given function $\mathbf{u}$ smooth enough. For $r > R > \delta(\Omega^c)$, using Hölder inequality we have

$$|\mathbf{u}(r,.) - \mathbf{u}(R,.)|^q = \left| \int_R^r \partial_\rho \mathbf{u}(\rho,.)\,\mathrm{d}\rho \right|^q \leq \left(\int_R^r |\partial_\rho \mathbf{u}(\rho,.)|^q \rho^2 w(\rho)\,\mathrm{d}\rho \right) \cdot \left(\int_R^r \frac{\mathrm{d}\rho}{\rho^{\frac{2}{q-1}} w(\rho)^{\frac{1}{q-1}}} \right)^{q-1} \tag{6}$$ LL1

Therefore, under the condition $(A_1^\alpha)_{1<q<3}$ and from the annexe (formula (17)), we obtain

$$\int_{S^2} |\mathbf{u}(r,\varphi) - \mathbf{u}(R,\varphi)|^q \,\mathrm{d}\varphi \;\leq\; c\, R^{-\alpha}\, \|\nabla \mathbf{u}\|^q_{q,\Omega^R;\,w}$$

Now, as $R \to \infty$, $\mathbf{u}(R,.)$ strongly converges in $\mathbf{L}^q(S^2)$ to $\mathbf{u}^*(.)$. Put $\mathbf{u}_0 := \overline{\mathbf{u}^*} = \frac{1}{|S^2|}\int_{S^2} \mathbf{u}^*(\varphi)\,\mathrm{d}\varphi$, then from the annexe (formula (18)) we get $\|\mathbf{u}(r) - \mathbf{u}_0\|_{q,S^2} \to 0$ as $r \to \infty$ at least for a sequence of radial values $\{r_m\}_m$ that tends to ∞.

R2 **Remark 2.** *When $q = 1$, the same result holds : Indeed, from formula (6) we directly get $\int_{S^2} |\mathbf{u}(r,\varphi) - \mathbf{u}(R,\varphi)|\,\mathrm{d}\varphi \;\leq\; C_R\, \|\nabla \mathbf{u}\|_{1,\Omega^R;\,w}$, where $\frac{1}{\rho^2 w_\kappa(\rho)} < C_R$ also tending to zero as $R \to \infty$.*

Remark 3. *For any $\mathbf{x}_0 \in \mathbb{R}^3$, taking $R > 0$ large enough, we can prove that $\frac{\mathbf{u}(\cdot) - \mathbf{u}_0}{|\cdot - \mathbf{x}_0|} \in \mathbf{L}^q_w(\Omega^R(\mathbf{x}_0))$. This result with the associated Sobolev-type inequalities is treated in Section 3.*

3. Proofs of Theorems 1 and 2

Technically we follow the proof given in [3] when $w \equiv 1$. So, let us consider $\mathbf{g}_q(\mathbf{x}) := (\mathbf{x} - \mathbf{x}_0)|\mathbf{x} - \mathbf{x}_0|^{-q}$ and $\mathbf{U} := \mathbf{u} - \mathbf{u}_0$, $\mathbf{u}$ *being a smooth function*. By means of easy differential calculations and using a transparent notation for the integral $I_{\mathbf{g}_q \cdot \nabla |\mathbf{U}|^q\, w}$, we obtain both formulas

$$\int_{\Omega_{R,r}(x_0)} \operatorname{div}\left(\mathbf{g}_q(\mathbf{x})|\mathbf{U}(\mathbf{x})|^q\right) w(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} \;=\; (3-q)\int_{\Omega_{R,r}(x_0)} \left|\frac{\mathbf{U}(\mathbf{x})}{\mathbf{x} - \mathbf{x}_0}\right|^q w(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} + I_{\mathbf{g}_q\cdot\nabla|\mathbf{U}|^q\, w} \tag{7}$$ Identity1

$$= \left(\int_{\partial B_R(x_0)} + \int_{\partial B_r(x_0)} \right) \left(\mathbf{g}_q \cdot \mathbf{n}\, |\mathbf{U}|^q w\right) \mathrm{d}S + I_{\nabla w} \tag{8}$$ Identity2

where $I_{\nabla w} := -\int_{\Omega_{R,r}(x_0)} \mathbf{g}_q(\mathbf{x})\, |\mathbf{U}(\mathbf{x})|^q\, \nabla w(|\mathbf{x}|)\,\mathrm{d}\mathbf{x}$.
The first integral $\int_{\partial B_R(x_0)} \cdots$ is non-positive; let us denote the second integral $\int_{\partial B_r(x_0)} \cdots$ by $I_{\partial B_r}$: We apply Lemma 1 to see how its contribution tends to zero, as $r \to \infty$, even if $q = 1$,

$$|I_{\partial B_r}| \leq r^{1-q} w(r)\, c_q\, r^{-\alpha} \|\nabla \mathbf{u}\|^q_{q,\Omega^r(x_0);\,w} \tag{9}$$ Ibdry

We now estimate $I_{\mathbf{g}_q\cdot\nabla|\mathbf{U}|^q w}$ using Young inequality in the form $q\, a.b \leq \gamma_q a^q + (q-1)\gamma_q^{-1/(q-1)} b^{q/(q-1)}$ with $\gamma_q := \left[\frac{q}{3-q}\right]^{q-1}$, $1 < q < 3$, so $(q-1)\gamma_q^{-1/(q-1)} = (q-1)\frac{3-q}{q}$, we have

$$|I_{\mathbf{g}\cdot\nabla|\mathbf{U}|^q\, w}| \le \int_{\Omega_{R,r}(x_0)} q\,|\mathbf{g}_q||\mathbf{U}|^{q-1}|\nabla\mathbf{u}|\, w\,\mathrm{d}\mathbf{x} \tag{10}$$

$$\le \gamma_q\, \|\nabla\mathbf{u}\|^q_{q,\Omega_{R,r}(x_0);\, w} + \ (3-q)\frac{q-1}{q}\ \int_{\Omega_{R,r}(x_0)} \frac{|\mathbf{U}(\mathbf{x})|^q}{|\mathbf{x}-\mathbf{x}_0|^q} w(|\mathbf{x}|)\mathrm{d}\mathbf{x} \tag{11}$$

Note that the obtained inequality holds when $q=1$.
Then from (7)(8) and the previous inequality, we obtain

$$\frac{3-q}{q}\ \int_{\Omega_{R,r}(x_0)} \Big|\frac{\mathbf{U}(\mathbf{x})}{\mathbf{x}-\mathbf{x}_0}\Big|^q\ w(|\mathbf{x}|)\mathrm{d}\mathbf{x}\ \le |I_{\partial B_r}| + \gamma_q\, \|\nabla\mathbf{U}\|^q_{q,\Omega_{R,r}(x_0);\, w} + |I_{\nabla w}|. \tag{12}$$ estimate

We estimate $I_{\nabla w}$ as follows

$$\Big|\int_{\Omega_{R,r}(x_0)} \mathbf{g}_q|\mathbf{U}|^q\ \nabla w(|\mathbf{x}|)\ \mathrm{d}\mathbf{x}\Big| \le 2\kappa(1+|x_0|)\int_{\Omega_{R,r}(x_0)} \Big|\frac{\mathbf{U}(\mathbf{x})}{\mathbf{x}-\mathbf{x}_0}\Big|^q\ w(|\mathbf{x}|)\ \mathrm{d}\mathbf{x},$$

where we use that the power type weight is such that $\frac{|\nabla w|}{|w|}|x-x_0| \le \kappa(1+|x_0|)$. Then, from (12) as $r\to\infty$, we obtain

$$\int_{\Omega^R(x_0)} \Big|\frac{\mathbf{U}(\mathbf{x})}{\mathbf{x}-\mathbf{x}_0}\Big|^q\ w(|\mathbf{x}|)\mathrm{d}\mathbf{x}\ \le\ \frac{\gamma_q}{\kappa_q}\ \|\nabla\mathbf{U}\|^q_{q,\Omega^R(x_0);\, w} \tag{13}$$ Estimate

the first part of Theorem 1 is established. The constant $\frac{\gamma_q}{\kappa_q}$ we obtain is precisely $\big(\frac{q}{3-q}\big)^q\big(\frac{1}{1-\frac{2\kappa q}{3-q}}\big)(1+|x_0|)$.

The proof of the second inequality in Theorem 1 also is largely based on [3]. For $r > 2R > \delta(\Omega^c)$, we will split the proof into two steps, considering $\|\mathbf{U}\|_{s(q),\Omega_R\cup\Omega_{R,2r};\, w}\ \le\ \|(1-\varphi_{R/2})\mathbf{U}\|_{s(q);\, w} + \|\varphi_R(1-\varphi_r)\mathbf{U}\|_{s(q);\, w}$, always for $\mathbf{U}=\mathbf{u}-\mathbf{u}_0$, and asking for the limit when $r\to\infty$. We have denoted $\varphi_R(\mathbf{x}) = \varphi(|\mathbf{x}|/R)$, where $\varphi\in C^1(\mathbb{R})$ is a convenient nondecreasing function such that $\varphi(\xi)=0$ if $|\xi|\le 1$ and $\varphi(\xi)=1$ if $|\xi|\ge 2$.
For simplicity, we set $\mathbf{U}^{\#}(\mathbf{x}) := \varphi_R(\mathbf{x})(1-\varphi_r(\mathbf{x}))\ \mathbf{U}(\mathbf{x})$ and $\mathbf{U}^b(\mathbf{x}) := (1-\varphi_{R/2}(\mathbf{x}))\mathbf{U}(\mathbf{x})$, so $\mathbf{U}^{\#}\in\mathbf{W}^{1,q}_{0,w}(\Omega_{R,2r})$ and $\mathbf{U}^b\in\mathbf{W}^{1,q}_w(\Omega_R)$. Applying the usual Sobolev inequality, we have

$$\begin{aligned}
\|\mathbf{U}^{\#}\|_{s(q),\Omega_{R,2r};\, w} &\le c\,\|\nabla\mathbf{U}^{\#}\|_{q,\Omega_{R,2r};\, w}\\
&\le c\,\big(\,\|\mathbf{U}\|_{q,\Omega_{R,2R};\, w} + \|\mathbf{U}\,|\cdot|^{-1}\|_{q,\Omega_{r,2r};\, w} + \|\nabla\mathbf{U}\|_{q,\Omega_{R,2r};\, w}\big) \qquad (14)\\
\|\mathbf{U}^b\|_{s(q),\Omega_R;\, w} &\le c\,\|\nabla\mathbf{U}^b\|_{q,\Omega_R;\, w}\\
&\le c\,\big(\,\|\mathbf{U}\|_{q,\Omega_{R/2,R};\, w} + \|\nabla\mathbf{U}\|_{q,\Omega_R;\, w}\big) \qquad (15)
\end{aligned}$$

eq29

eq30

Over the two bounded spherical shells $\Omega_{\alpha R,2\alpha R}$, with $\alpha=\frac{1}{2}$ or 1, weighted or unweighted inequalities are the same, then we can use the classical inequality in the form given by [[3], (4.14)] to bound the norm $\|\cdot\|_{q,\Omega_{\alpha R,2\alpha R};\, w}$ by $|\cdot|_{1,q,\Omega^R;\, w} + \big(\int_{\partial\Omega_{\alpha R,2\alpha R}}|\cdot|^q\mathrm{d}S\big)^{1/q}$, then we apply Lemma 1 for all surface integrals.

The second term in (14) tends to zero as $r\to\infty$, to this end we first apply inequality (13) with Ω^r. In the first term it remains only $|\mathbf{U}|_{1,q,\Omega^R;\, w}$. Then from (14) we get $\|\mathbf{U}^{\#}\|_{s(q),\Omega^R;\, w} \le c\|\nabla\mathbf{U}\|_{q,\Omega^R;\, w}$. From (15) we also obtain $\|\mathbf{U}^b\|_{s(q),\Omega_R;\, w} \le c\|\nabla\mathbf{U}\|_{q,\Omega_R;\, w}$. This completes the proof of (2).

The proof of Theorem 2 follows the same line as in the proof of Theorem 1 except the term $I_{\nabla w}$,

$$|I_{\nabla w}| \le \kappa\int_{\Omega_{R,r}(x_0)} \Big|\frac{\mathbf{U}(\mathbf{x})}{\mathbf{x}-\mathbf{x}_0}\Big|^q\ w(|\mathbf{x}|)\mathrm{d}\mathbf{x} + \int_{\Omega_{R,r}(x_0)} |\mathbf{U}(\mathbf{x})|^q|\nabla w(|\mathbf{x}|)|\mathrm{d}\mathbf{x}.$$

Then

$$\Big(\frac{3-q}{q}-\kappa\Big)\Big(\int_{\Omega^R(x_0)} \Big|\frac{\mathbf{U}(\mathbf{x})}{\mathbf{x}-\mathbf{x}_0}\Big|^q\ w(|\mathbf{x}|)\mathrm{d}\mathbf{x}\Big)\ \le\ \gamma_q\,\|\nabla\mathbf{U}\|^q_{q,\Omega^R(x_0);\, w} + \int_{\Omega^R(x_0)} |\mathbf{U}(\mathbf{x})|^q|\nabla w(|\mathbf{x}|)|\mathrm{d}\mathbf{x}.$$

3. Proofs of Theorems 3 and 4

To justify the sufficiency, we follow Sobolev's ideas [9] for approximating functions $\mathbf{u}$ from $\mathbf{D}^{1;q}_w(\Omega)$ by compactly supported smooth functions. In order to create, for R large enough, a truncated function $\psi_R\mathbf{u}$ having a bounded support in Ω, we consider $\widetilde{\Omega}_R = \{\mathbf{x} \in \Omega : \exp(\sqrt{\ln R}) < |\mathbf{x}| < R\}$ and

$$\psi_R(\mathbf{x}) := \psi\left(\frac{\ln\ln|\mathbf{x}|}{\ln\ln R}\right) \text{ for } \mathbf{x} \in \widetilde{\Omega}_R, \quad \text{clearly chosen with } \frac{1}{2} < \frac{\ln\ln|\mathbf{x}|}{\ln\ln R} < 1,$$

where $\psi \in C^1(\mathbb{R})$ is a convenient non increasing function with $\psi(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$ and $\psi(\xi) = 0$ if $|\xi| \geq 1$. Note that, when $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, $\psi_R\mathbf{u} \in \mathbf{W}^{1,q}_0(\Omega)$ with the property $0 < |\nabla\psi_R(\mathbf{x})| \leq \frac{c}{\ln\ln R}\frac{1}{|\mathbf{x}|\ln|\mathbf{x}|}$ for $\mathbf{x} \in \widetilde{\Omega}_R$. As a consequence

$$\|\nabla\psi_R\mathbf{u}\|^q_{q,\widetilde{\Omega}_R;\,w} \leq \frac{c^q}{(\ln\ln R)^q}\int_{\exp(\sqrt{\ln R})}^{R} w(\rho)\int_{S^2}\frac{1}{(\rho\ln\rho)^q}|\mathbf{u}(\rho,.)|^q\rho^2\mathrm{d}S\,\mathrm{d}\rho$$

Then, if $1 < q < 3$, from Lemma 1 with $\mathbf{u}_0 = \mathbf{0}$, it follows that

$$\|\nabla\psi_R\mathbf{u}\|^q_{q,\widetilde{\Omega}_R;\,w} \leq \frac{C}{(\ln\ln R)^q}\int_{\exp(\sqrt{\ln R})}^{R}\frac{\rho^{-\alpha}\rho^{2-q}}{(\ln\rho)^q}\,w(\rho)\,\mathrm{d}\rho.$$

Under the condition $\quad (A^\alpha_2)_{1<q<3} \qquad (|\cdot|^{2-q-\alpha})(\ln|\cdot|)^{-q} \in L^1_w(\Omega),$ we get $\|\nabla\psi_R\mathbf{u}\|_{q,w} \to 0$ as $R \to \infty$ since $\frac{C}{(\ln\ln R)^q} \to 0$.

If $q = 1$, applying Remark 4 with constant C_R replaced by $C_{\exp(\sqrt{\ln R})}$, we have the same result.

If $q > 3$, from [3], Exercise 5.2, we get

$$\|\nabla\psi_R\mathbf{u}\|^q_{q,\widetilde{\Omega}_R;\,w} \leq \frac{C}{(\ln\ln R)^q}\int_{\exp(\sqrt{\ln R})}^{R}\frac{\rho^{\alpha}\rho^{2-q}}{(\ln\rho)^q}\,w(\rho)\,\mathrm{d}\rho.$$

Under our assumption $(A^\alpha_2)_{q>3}$, we again obtain $\|\nabla\psi_R\mathbf{u}\|_{q,w} \to 0$ as $R \to \infty$.
Then, given $\epsilon > 0$, we can find R large enough and $\mathbf{u}_{R,\epsilon} \in C^\infty_0(\Omega)$ such that $\|\mathbf{u}_{R,\epsilon} - \psi_R\mathbf{u}\|_{1,q,\Omega;\,w} < \epsilon$. So, taking into account also integrability of $\nabla\mathbf{u}$ in $\mathbf{L}^q_w(\Omega)$:

$$\begin{aligned}|\mathbf{u} - \mathbf{u}_{R,\epsilon}|_{1,q,\Omega;\,w} &\leq \|(1-\psi_R)\nabla\mathbf{u}\|_{q,\Omega;\,w} + \|\nabla\psi_R\mathbf{u}\|_{q,\widetilde{\Omega}_R;\,w} + \|\mathbf{u}_{R,\epsilon} - \psi_R\mathbf{u}\|_{1,q,\Omega;\,w}\\ &\leq 2\epsilon + \|\nabla\psi_R\mathbf{u}\|_{q,\widetilde{\Omega}_R;\,w} \leq 3\epsilon.\end{aligned} \tag{16}$$

Remark 4. *We need Conditions (A^α_2) because we must control the estimate of $\|\nabla\psi_R\mathbf{u}\|_{q,w}$ as $R \to \infty$: Knowing that Condition $(A^\alpha_1)_{1<q<3}$ holds for $w = w_\kappa$ with $\alpha \geq \frac{3-q+\kappa}{q-1}$ and looking for (A^α_2) in the simplest case we have $\alpha + q - 2 - \kappa \geq 1$ and then we are in the same situation as in [3]*

$$\|\nabla\psi_R\mathbf{u}\|^q_{q,\widetilde{\Omega}_R;w} \leq \frac{C}{(\ln\ln R)^q}\int_{\exp\sqrt{\ln R}}^{R}\frac{1}{\ln(\rho)^q\rho}\,\mathrm{d}\rho \leq \frac{C}{(q-1)(\ln\ln R)^q}\frac{1}{(\ln R)^{\frac{q-1}{2}}}\,.$$

It remains to prove the necessity, firstly to show the zero trace on $\partial\Omega$ of $\mathbf{u} \in \mathbf{D}^{1;q}_w(\Omega)$ when approximated in the norm $\|\nabla\cdot\|_{q,\Omega;\,w}$ by a sequence $\{\mathbf{u}^n\}_{n>0}$ with $\mathbf{u}^n \in C^\infty_0(\Omega)$, secondly to verify the relation $\mathbf{u}_0 = \mathbf{0}$. The first point is obvious because the $(q,\partial\Omega;\,w)$-norms of the traces of $\mathbf{u}$ and $\mathbf{u}^n$ are the same. To justify the second point, we note that $\{\mathbf{u}^n\}_{n>0}$ is a Cauchy sequence in $\mathbf{D}^{1,q}_{0,w}(\Omega)$ which converges in $\mathbf{L}^{s(q)}_w(\Omega)$ by means of Sobolev imbedding, and as the main technical ingredient we use the following convergence

$$\lim_{\delta\to 0}\frac{1}{2\delta R^2}\int_{R-\delta}^{R+\delta}\int_{S^2}\mathbf{u}(r,\varphi)r^2\mathrm{d}\varphi\,\mathrm{d}r = \int_{S^2}\mathbf{u}(R,\varphi)\mathrm{d}\varphi\,, \qquad \text{for a detailed proof see [5], [6].}$$

Remark 5. *As in [3] when $w \equiv 1$, the requirement that the constant vector $\mathbf{u}_0$ from Lemma 1 is $\mathbf{0}$ is not necessary if $q \geq 3$. On the other hand, we can improve the results of Theorems 3 and 4 even if the trace of $\mathbf{u}$ does not vanish, replacing $C^\infty_0(\Omega)$ by $C^\infty_0(\overline{\Omega})$.*

4. Annexe (classical properties)

We denote by $\mathbf{D}^{1,q}_w(\Omega)$ the following set of functions

$$\mathbf{D}^{1,q}_w(\Omega) := \{\mathbf{u}|\ \mathbf{u} \in \mathbf{L}^1_{loc,w}(\Omega), \nabla u_i \in \mathbf{L}^q_w(\Omega)\ 1 \le i \le 3\},$$

where w is in the q-class of Muckenhoupt weights. As usually by factorization with respect to constants we get the Banach spaces equipped with the topology $|\cdot|_{1,q,\Omega;\,w} := \|\nabla \cdot\|_{q,\Omega;\,w}$. These Banach spaces of classes of functions are sometimes denoted by the same notation. As it is clear from the context, in the previous sections we used the symbol $\mathbf{D}^{1,q}_w(\Omega)$ for the set of functions. We recall that Ω is unbounded in all directions, the global summability of $\mathbf{u}$ is lost and the behavior of $\mathbf{u}$ at large distances. For each q, $\mathbf{D}^{1,q}_{0,w}(\Omega)$ denotes the completion of the space $C^\infty_0(\Omega)^3$ under the norm $\|\nabla \cdot\|_{q,\Omega;\,w}$.

By $\mathbf{W}^{1,q}_w(\Omega_R)$,$\mathbf{W}^{1,q}_{0,w}(\Omega_R)$ we mean full Sobolev spaces with their usual norms, see [10].

Let ∇^* be the gradient operator on S^2, the unit sphere in $\mathbb{R}^3$: The following identity holds $|\nabla^*\mathbf{u}|^2 = r^2[|\nabla\mathbf{u}|^2 - |\partial_r\mathbf{u}|^2]$. It means that either $|\nabla\mathbf{u}|^q \ge |\partial_r\mathbf{u}|^q$ or $|\nabla\mathbf{u}|^q \ge r^{-q}|\nabla^*\mathbf{u}|^q$ $(1 \le q < \infty)$.

From the first inequality, we get

$$\|\nabla\mathbf{u}\|^q_{q,\Omega^R;w} \ge \|\partial_r\mathbf{u}\|^q_{q,\Omega_{R,r};w} \ge c\int_R^r \int_{S^2} |\partial_\rho\mathbf{u}|^q w(\rho)\rho^2 \mathrm{d}S\ \mathrm{d}\rho \tag{17}$$

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then the last integral is bounded when $\mathbf{u} \in \mathbf{D}^{1,q}_w(\Omega)$. Now, from the second inequality, we get

$$\begin{aligned}\|\nabla\mathbf{u}\|^q_{q,\Omega^R;w} \ge \|\nabla\mathbf{u}\|^q_{q,\Omega_{R,r};w} &\ge c\int_R^r \int_{S^2} \rho^{-q}|\nabla^*\mathbf{u}|^q w(\rho)\rho^2 \mathrm{d}S\ \mathrm{d}\rho \\ &\ge c\int_R^r \|\nabla^*\mathbf{u}\|^q_{q,S^2}\rho^{2-q}w(\rho)\mathrm{d}\rho \\ &\ge cc_w\int_R^r \|\mathbf{u} - \overline{\mathbf{u}}\,\|^q_{q,S^2}\rho^{2-q}w(\rho)\mathrm{d}\rho \qquad (18)\end{aligned}$$

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Here, for Ω regular enough, we have used a Friedrichs-Poincaré type inequality (so-called Wirtinger inequality) which holds in the absence of a zero value at the boundary if we substract from $\mathbf{u}$ its mean value. Then

$$\|\mathbf{u} - \overline{\mathbf{u}}\,\|_{q,\Omega_{R,r};w} \le C\|\nabla\mathbf{u}\|_{q,\Omega;w}$$

The property does make sense with $\nabla\mathbf{u} \in \mathbf{L}^q_{loc,w}(\Omega)$ only and for $1 \le q < \infty$.

If Ω is locally Lipschitzian and $\nabla\mathbf{u} \in \mathbf{L}^q_{loc,w}(\overline{\Omega})$, then $\mathbf{u} \in \mathbf{L}^q_{loc}(\overline{\Omega})$ also near the boundary $\partial\Omega = \partial\Omega^c$, see [8].

Concluding Remark 1. *Our purpose in [5] and [6] is to prove the existence of very weak solutions in weighted* $\mathbf{L}^q$*-Spaces to the Stokes and Navier–Stokes Equations formulated to describe the motion of a flow around a rotating rigid body. To deal with these problems, weight functions taken from the Muckenhoupt* q*-class (usually denoted* A_q*) of the form* w_κ *are convenient. Then we've had to define appropriate spaces and needed corresponding embedding theorems, this is the reason why we have studied the present embeddings. We consider these inequalities interesting by themselves.*

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References

A [1] F. Alliot:*Etude des equations stationnaires de Stokes et Navier - Stokes dans des domaines exterieures* PdD. thesis (1994).

AG [2] C. Amrouche, V. Girault: *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension* Czechoslovak Math. J. 44, 114, no.1, 109-140 (1994).

Ga [3] G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I: Linearized Steady Problems, Vol. II: Nonlinear Steady Problems.* Springer-Verlag, New York–Berlin–Heidelberg, 1998-2011.

GO [4] P. Gurka, B. Opic: *Continuous and compact imbeddings of weighted Sobolev spaces, II.* Czechoslovak Math. J. 39, 114, no.1, 78-94 (1989).

KNP1 [5] S. Kračmar, M. Krbec, Š. Nečasová, P. Penel, K. Schumacher: *Very Weak Solutions to the Rotating Stokes problem in weighted spaces.* In preparation (2012).

KNP2 [6] S. Kračmar, Š. Nečasová, P. Penel: *Very Weak Solutions to the Rotating Navier–Stokes problem in weighted spaces.* In preparation (2012).

K [7] A. Kufner: *Weighted Sobolev spaces.* Translated from the Czech. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1985.

Ne [8] J. Nečas: *Les Méthodes directes en théorie des Equations Elliptiques.* Masson et C^{ie}, Paris 1967.

So [9] S. L. Sobolev: *Denseness of finite fields in the space* $L_q^m(\mathbb{R}^n)$. Sib. Mat. Zb. 3, 673-682 (1963). [in Russian]

T [10] B. O. Turesson: *Nonlinear potential theory and weighted Sobolev spaces.* Lecture Notes in Mathematics, 1736. Springer-Verlag, Berlin, 2000