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Abstract

We survey some recent results on the asymptotic behavior for large time of solutions to models of complete fluid systems, meaning models including compressibility, viscosity, and/or heat conductivity of the fluids. We introduce several concepts of solutions and discuss the existence of global-in-time trajectories as well as general questions concerning well-posedness. Then we address dissipativity properties, in particular the existence of absorbing sets, and asymptotic compactness of trajectories. Finally, the existence of attractors, convergence to equilibria, and other qualitative aspects of the long time behavior are studied.

Keywords: Navier-Stokes-Fourier system; long-time behavior; weak solution

Contents

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1 Introduction

Many important features of the fluid dynamics, and, in particular, the phenomena associated with *turbulence*, occur on large time scales. The long time behavior of the infinite-dimensional dynamical systems generated by the equations describing fluids in motion belongs to one of the most intensively studied topics in mathematical fluid dynamics, see the nowadays classical reference materials by Babin and Vishik [3], Chepyzhov and Vishik [10], Constantin and Foias [13], Constantin, Foias, and Temam [14], Constantin et al. [15], Eckmann and Ruelle [21], Foias et al. [34], Ladyzhenskaya [40], [41], Qin [48], Temam [53], among others.

In accordance with the basic principles of classical continuum mechanics, the time evolution of a fluid is described in terms of three state variables:

- the mass density . . . $\varrho = \varrho(t, x)$,
- the absolute temperature . . . $\vartheta = \vartheta(t, x)$,
- the velocity field . . . $\mathbf{u} = \mathbf{u}(t, x)$,

where all these quantities are functions of the time t and the spatial position x belonging to a reference physical domain $\Omega \subset R^3$.

In the *Eulerian coordinate system* attached to the physical space rather than to individual streamlines, the motion of the fluid is governed by the field equations:

MASS CONSERVATION

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0; \tag{1.1}$$

MOMENTUM BALANCE

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}; \tag{1.2}$$

INTERNAL ENERGY BALANCE

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}. \tag{1.3}$$

The new quantities appearing in (1.1 - 1.3) are:

- the pressure .... $p = p(\varrho, \vartheta)$,
- the specific internal energy .... $e = e(\varrho, \vartheta)$,
- the viscous stress tensor .... $\mathbb{S} = \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u})$,
- the external volume force .... $\mathbf{f} = \mathbf{f}(t, x)$,
- the internal energy flux .... $\mathbf{q} = \mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)$.

Unless otherwise stated, we focus on the fluids confined to a bounded physical space $\Omega \subset R^3$, which boundary is both mechanically and thermally insulated. Of course, there are various choices of the boundary conditions that comply with this requirement; we consider the following:

IMPERMABILITY

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.4}$$

where $\mathbf{n}$ denotes the outer normal vector;

NO SLIP VS. COMPLETE SLIP

$$\mathbf{u}_{\mathrm{tan}}|_{\partial\Omega} = 0 \text{ or } [\mathbb{S} \cdot \mathbf{n}]_{\mathrm{tan}} = 0; \tag{1.5}$$

NO INTERNAL ENERGY FLUX

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.6}$$

The initial state of the fluid system is expressed by means of the standard

INITIAL CONDITIONS

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0. \tag{1.7}$$

Remark 1.1 *Strictly speaking, given the specific form of the field equations (1.1 - 1.3), the initial conditions should be stated in terms of* $\varrho(0,\cdot)$*,* $(\varrho\mathbf{u})(0,\cdot)$*, and* $\varrho e(\varrho,\vartheta)(0,\cdot)$*. However, we find it more convenient to use the original state variables.*

In this paper, we study the problem (1.1 - 1.7) in the general framework of the theory of infinite-dimensional dynamical systems. To this end, we need some preliminary material concerning well-posedness and stability results for global-in-time solutions. More specifically, we discuss the following topics:

1. **Physically admissible constitutive relations and global-in-time existence [Section 2].** The well-known problem when dealing with systems of nonlinear conservation laws like (1.1 - 1.3) is the lack of sufficiently strong *a priori* bounds to ensure that all terms in the weak formulation are at least equi-integrable. We identify a physically admissible class of constitutive equations as well as a suitable weak formulation so that the problem (1.1 - 1.7) possesses at least one weak solution for any choice of finite energy initial data.

2. **Well-posedness and stability [Section 3].** Except the well-studied case of smooth and/or small initial data, the problem of well-posedness for (1.1 - 1.7) remains largely open. We introduce the concept of *dissipative solution* based on the so-called relative entropy (energy) inequality. We show that the dissipative solutions enjoy the weak-strong uniqueness property, meaning, they coincide with a (hypothetical) strong solutions emanating from the same initial data on the maximal existence interval of the latter. We also make a short excursion in the theory of hyperbolic (inviscid) system, for which the method of convex integration, developed recently in the context of the incompressible Euler system by DeLellis and Székelyhidi [18], yields infinitely many weak solutions for a specific class of initial data.

3. **The Second law of thermodynamics and the asymptotic behavior [Section 4].** Having collected the necessary preliminary material, we address the problem of the long-time behavior of the global-in-time weak solutions. It turns out that the Second law of thermodynamics, implemented in the system through effective viscosity a heat conductivity, has a major impact on the asymptotic properties of solutions. If the system is driven by a conservative force $\mathbf{f} = \nabla_x F(x)$, all solutions tend to a unique equilibrium that can be specified from the knowledge of the initial data. In such a case, the set of equilibrium solutions represents and (unbounded) attractor for the problem (1.1 - 1.7). On the other hand, if $\mathbf{f} \neq \nabla_x F$, the trajectories become unbounded as $t \to \infty$.

4. **Special issues, perspectives [Section 5].** We conclude the paper by several remarks concerning some particular cases including the systems driven by a time-periodic external forces. The paper is finished by a short review of possible perspective for future research.

2 Global-in-time weak solutions

Since the pioneering work of Leray [43], the issue of global-in-time existence of regular solutions to problems in fluid mechanics based on the Navier-Stokes system has been amply addressed in a number of purely theoretical as well as applications oriented studies, see Antontsev, Kazhikhov, and Monakhov [2], Caffarelli, Kohn and Nirenberg [8], Kato [39], Ladyzhenskaya [40], [42], Lions [44], [45], Temam [52], among many others. The problem remains basically open, even in the "simple" incompressible case considered in the natural $3-D$ topology - no positive large data result, no counterexamples (see Fefferman [25], Tao [51]). Anticipating tacitly that the problem becomes even more delicate for the more complex system (1.1 - 1.7) we therefore restrict ourselves to the class of weak solutions.

2.1 Weak formulation

To begin, we specify our general hypotheses concerning the thermodynamics functions $p = p(\varrho, \vartheta)$ and $e = e(\varrho, \vartheta)$. The fact that p and e can be expressed as functions of the *thermostatic variables* ϱ, ϑ anticipates the commonly accepted hypothesis that on the time scales relevant for the macroscopic models the fluid attains at each instant the state of thermodynamic equilibrium, see Gallavotti [35], [36].

In the remaining part of the paper we therefore assume

Gibbs' relation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right) \tag{2.1}$$

where $s = s(\varrho, \vartheta)$ is the specific entropy,

thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \tag{2.2}$$

2.1.1 Entropy equation and the total energy balance

In accordance with Gibbs' relation (2.1), the internal energy balance (1.3) can be written in the form of the entropy production equation

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x \Big(\varrho s(\varrho, \vartheta) \mathbf{u} \Big) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \tag{2.3}$$

with the entropy production rate

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{2.4}$$

Moreover, multiplying the momentum balance (1.2) on $\mathbf{u}$ and adding the resulting expression to the internal energy equation (1.3) we obtain the total energy balance in the form

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right] + \mathrm{div}_x \left(p(\varrho) \mathbf{u} + \mathbf{q} \right) - \mathrm{div}_x (\mathbb{S} \cdot \mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u}. \tag{2.5}$$

It is worth noting that the equations (1.3), (2.3), and (2.5) are *equivalent* as soon as all quantities in (1.1 - 1.3) are smooth. Integrating (2.5) over the physical domain Ω and using the boundary conditions (1.4 - 1.6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x. \tag{2.6}$$

Thus, if $\mathbf{f} = 0$, the total energy of the system is a conserved quantity.

2.1.2 Entropy inequality and the weak formulation

We introduce a weak formulation of the problem (1.1 - 1.3) that consists of

- the equations (1.1), (1.2) understood in the sense of distributions;
- the internal energy equation (1.3) replaced by

ENTROPY PRODUCTION EQUATION

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x\Big(\varrho s(\varrho,\vartheta)\mathbf{u}\Big) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{2.7}$$

where

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right), \tag{2.8}$$

supplemented with

TOTAL ENERGY BALANCE

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ \mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x. \tag{2.9}$$

Similarly to (2.1), (2.2), the relations (2.7 - 2.9) are to be understood in the sense of distributions. Replacing (2.4) by an inequality should accommodate possible singularities of the weak solutions, the total energy balance (2.9) is added to the system to compensate the loss of information due to (2.8). It can be shown that (2.7), (2.8) give rise to the original internal energy balance (1.3) as soon as all quantities are smooth, see [30, Chapter 3]. As we shall see, the formulation based on the entropy production and the total energy balance is rather convenient as it already includes the so-called relative entropy inequality which is the main tool in many problems concerning the asymptotic behavior of the system.

2.2 Constitutive relations

Our choice of constitutive relations is partially dictated by the needs of the mathematical theory, more specifically, they represent a compromise between the underlying physics and the necessary *a priori* bounds they provide. The reader may consult [30, Chapter 1] for details.

2.2.1 Thermodynamic functions

As already mentioned in Section 2.1, the thermodynamic quantities p, e, and s are explicit functions of the thermostatic variables ϱ, ϑ, interrelated through Gibbs' equation (2.1) and satisfying the thermodynamic stability hypothesis (2.2). In addition, we impose a state equation typical for the molecular pressure p_M of monoatomic gases, namely

$$p_M(\varrho, \vartheta) = \frac{2}{3}\varrho e_M(\varrho, \vartheta), \tag{2.10}$$

which, in combination with (2.1) yields a universal formula

$$p_M(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) \tag{2.11}$$

for a certain function P. If P is linear, we immediately recover the perfect gas law $p_M = a\varrho\vartheta$.

In accordance with (2.10), we have

$$e_M(\varrho, \vartheta) = \frac{3}{2}\vartheta \left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right), \tag{2.12}$$

and, going back to (2.1),

$$s_M(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right), \text{ where } S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}. \tag{2.13}$$

Furthermore, the thermodynamic stability (2.2) imposes the following restrictions on P:

- $\partial_\varrho p > 0$ implies $P'(Z) > 0$;
- $\partial_\vartheta e > 0$ yields
$$\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0; \tag{2.14}$$
- the relation (2.14) implies $Z \mapsto \frac{P(Z)}{Z^{5/3}}$ is decreasing, we assume
$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0; \tag{2.15}$$
- in accordance with (2.13), the function $Z \mapsto S(Z)$ is decreasing, we assume
$$\lim_{Z\to\infty} S(Z) = 0, \tag{2.16}$$
in accordance with the Third law of thermodynamics, see Belgiorno [5], [6], Callen [9].

2.2.2 Thermal radiation

The effect of thermal radiation leads to a modification of the pressure as well as the other thermodynamic functions by "radiative" components. This can achieved by taking

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{2.17}$$

$$e(\varrho, \vartheta) = e_M(\varrho, \vartheta) + \frac{a}{\varrho}\vartheta^4, \tag{2.18}$$

and, in accordance with Gibbs' equation (2.1),

$$s(\varrho, \vartheta) = s_M(\varrho, \vartheta) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}. \tag{2.19}$$

The interested reader may consult the monograph of Eliezer, Ghatak and Hora [22] for other possibilities of state equations as well as a thorough discussion of their physical background.

2.2.3 Transport coefficients

We focus on *newtonian fluids*, for which the viscous stress obeys

NEWTON'S RHEOLOGICAL LAW

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{2.20}$$

where both the shear viscosity coefficient μ and the bulk viscosity coefficient η may depend on the thermostatic variables ϱ, ϑ. For technical reasons, however, we are able to handle only the case $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$. Such a hypothesis is relevant, for instance, in the case of gases, see Becker [4].

Similarly, the heat flux is given by

FOURIER'S LAW

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta. \tag{2.21}$$

The field equations (1.1 - 1.3), with $\mathbb{S}$ and $\mathbf{q}$ given by (2.20), (2.21), will be termed the *Navier-Stokes-Fourier system.*

2.3 Global-in-time weak solutions for the Navier-Stokes-Fourier system

We say that a triple $[\varrho, \vartheta, \mathbf{u}]$ is a *weak solution* of the Navier-Stokes-Fourier system in the space-time cylinder $(0,T)\times\Omega$ if:

- the density and the temperature are positive, $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ for a.a. $(t,x) \in (0,T)\times\Omega$,

$$\varrho \in C_{\rm weak}([0,T]; L^{5/3}(\Omega)),\ (\varrho\mathbf{u}) \in C_{\rm weak}([0,T]; L^{\beta}(\Omega; R^3)),$$

$$\vartheta \in L^\infty(0,T; L^s(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),\ \mathbf{u} \in L^2(0,T; W^{1,r}(\Omega; R^3)) \text{ for some } r, s, \beta > 1,$$

$$\mathbf{u}|_{\partial\Omega} = 0 \text{ or } \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0;$$

- the equation of continuity (1.1) is replaced by

$$\int_\Omega (\varrho + b(\varrho))(\tau,\cdot)\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega (\varrho_0 + b(\varrho_0))\,\varphi(0,\cdot)\ \mathrm{d}x \tag{2.22}$$

$$= \int_0^\tau \int_\Omega \Big[(\varrho + b(\varrho))\,\partial_t\varphi + (\varrho + b(\varrho))\,\mathbf{u}\cdot\nabla_x\varphi + (b(\varrho) - b'(\varrho)\varrho)\,\mathrm{div}_x\mathbf{u}\varphi \Big]\ \mathrm{d}x\ \mathrm{d}t$$

for any test function $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\tau \in [0,T]$, and any $b \in C^1[0,\infty)$, $b(0) = 0$, $b'(\varrho) = 0$ for all $\varrho \geq c_b$;

- the momentum equation (1.2) holds in the sense of distributions,

$$\int_\Omega \varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x \tag{2.23}$$

$$= \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi - \mathbb{S} : \nabla_x\varphi + \varrho\mathbf{f}\cdot\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega}; R^3)$, where $\varphi|_{\partial\Omega} = 0$ in the case of no-slip boundary conditions, and $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$ in the case of complete slip boundary conditions;

- the entropy balance (2.7), (2.8) is replaced by a family of integral inequalities

$$\int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x - \int_\Omega \varrho s(\varrho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \frac{\varphi}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t \tag{2.24}$$

$$\leq -\int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}\cdot\nabla_x\varphi}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \geq 0$, and a.a. $\tau \in [0,T]$;

- the total energy satisfies

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(\tau,\cdot)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) \right)\ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{2.25}$$

for a.a. $\tau \in [0,T]$.

Remark 2.1 *The integral identity (2.22) corresponds to the so-called renormalized version of the continuity equation (1.1) introduced by DiPerna and Lions [20]. Note that (2.22) reduces to the standard weak formulation of (1.1) if $b = 0$.*

As shown in [30, Chapter 2] a weak solution is strong, in particular the entropy balance is satisfied with an *equality* sign, as soon all quantities appearing in the weak formulation are smooth enough.

2.3.1 Hypotheses and the main existence result

In accordance with the previous discussion, we suppose that the functions p, e, s, μ, η, and κ satisfy the following list of hypotheses:

1. the functions p, e, s are given through (2.17 - 2.19), where

$$P \in C^1[0,\infty) \cap C^3(0,\infty),\ P(0)=0,\ P'(Z)>0,\ 0<\frac{\frac{5}{3}P(Z)-P'(Z)Z}{Z}<c \text{ for all } Z>0, \tag{2.26}$$

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^{5/3}}=p_\infty>0,\ \lim_{Z\to\infty}S(Z)=0; \tag{2.27}$$

2. the viscosity coefficients are continuously differentiable functions of the temperature such that

$$0<\underline{\mu}(1+\vartheta^\Lambda)\le\mu(\vartheta)\le\overline{\mu}(1+\vartheta^\Lambda),\ |\mu'(\vartheta)|<c \text{ for all } \vartheta\in[0,\infty),\ \frac{2}{5}<\Lambda\le 1, \tag{2.28}$$

$$0\le\eta(\vartheta)\le\overline{\eta}(1+\vartheta^\Lambda) \text{ for all } \vartheta\in[0,\infty); \tag{2.29}$$

3. the heat conductivity coefficient $\kappa=\kappa(\vartheta)$ satisfies

$$0<\underline{\kappa}(1+\vartheta^3)\le\kappa(\vartheta)\le\overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta\in[0,\infty). \tag{2.30}$$

We report the following existence result ([30, Chapter 3, Theorem 3.1]):

Theorem 2.1 *Let $\Omega\subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the thermodynamic functions p, e, s and the transport coefficients μ, η, κ satisfy the hypotheses [1. - 3.] stated at the beginning of this section. Let the initial data satisfy*

$$\varrho_0\in L^{5/3}(\Omega),\ \vartheta_0\in L^\infty(\Omega),\ \varrho_0,\vartheta_0>0, \mathbf{u}_0\in W^{1,\infty}(\Omega;R^3),\ \textit{and let } \mathbf{f}\in L^\infty((0,T)\times\Omega;R^3). \tag{2.31}$$

Then the Navier-Stokes-Fourier system (2.22 - 2.25) admits a weak solution $[\varrho,\vartheta,\mathbf{u}]$ on an arbitrary time interval $(0,T)$. More specifically, the weak solution enjoys the following properties:

$$\varrho\ge 0 \textit{ a.a. in } (0,T)\times\Omega,\ \varrho\in C([0,T];L^1(\Omega))\cap L^\infty(0,T;L^{5/3}(\Omega))\cap L^\delta((0,T)\times\Omega) \tag{2.32}$$

for a certain $\delta>\frac{5}{3}$;

$$\vartheta>0 \textit{ a.a. in } (0,T)\times\Omega,\ \vartheta\in L^\infty(0,T;L^4(\Omega))\cap L^2(0,T;W^{1,2}(\Omega)), \tag{2.33}$$

$$\vartheta^3,\log(\vartheta)\in L^2(0,T;W^{1,2}(\Omega)); \tag{2.34}$$

$$\mathbf{u}\in L^2(0,T;W_0^\alpha(\Omega;R^3)),\ \alpha=\frac{8}{5-\Lambda},\ \varrho\mathbf{u}\in C_{\rm weak}(0,T;L^{5/4}(\Omega;R^3)). \tag{2.35}$$

Remark 2.2 *For the purpose of the dynamical system theory, it is more convenient to work with global-in-time solutions. We say that* $[\varrho, \vartheta, \mathbf{u}]$ *is a weak solution of the Navier-Stokes-Fourier system in* $[0, \infty) \times \Omega$ *if (2.22 - 2.25) are satisfied on any* $[0, T]$, $T > 0$ *finite. The existence theorem (Theorem 2.1) remains valid on* $[0, \infty)$.

The hypotheses of Theorem 2.1 are not optimal. The interested reader may consult [30, Chapter 3] for possible extensions.

A complete proof of Theorem 2.1 is lengthy and very technical. The main building blocks are:

- the general existence theory for the compressible barotropic Navier-Stokes system developed by Lions [45], in particular the identity satisfied by the effective viscous pressure;
- the concept of oscillation defect measure introduced in [26];
- the extension of the theory to the case of variable transport coefficients [27].

3 Well-posedness and stability

We shortly address the issue of well-posedness and stability of the Navier-Stokes-Fourier system in the framework of weak solutions. To this end, we introduce a thermodynamic potential called *ballistic energy*:

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)\Big), \tag{3.1}$$

cf. Ericksen [23].

The following two properties of H_Θ playing a crucial role in the study of stability are a straightforward consequence of the hypothesis of thermodynamics stability (2.2):

- $\varrho \mapsto H_\Theta(\varrho, \Theta)$ is a strictly convex function of ϱ;
- $\vartheta \mapsto H_\Theta(\varrho, \vartheta)$ is decreasing for $\vartheta < \Theta$ and increasing for $\vartheta > \Theta$ for any ϱ.

3.1 Relative entropy and dissipative solutions

Motivated by the previous studies of Dafermos [16], Germain [37], we introduce the relative entropy (energy) functional:

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta)\right) \, \mathrm{d}x. \tag{3.2}$$

In accordance with the previously observed properties of the function H_Θ, the function $\mathcal{E}$ represents a kind of (non-symmetric) distance between the triples $[\varrho, \vartheta, \mathbf{u}]$ and $[r, \Theta, \mathbf{U}]$.

3.1.1 Relative entropy inequality

Writing the relative entropy in the form

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right) = \sum_{j=1}^{5} I_j,$$

where

$$I_1 = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right) \, \mathrm{d}x,$$

$$I_2 = - \int_\Omega \varrho \mathbf{u} \cdot \mathbf{U} \ {\rm d}x,$$

$$I_3 = \int_\Omega \varrho \left(\frac{1}{2} |\mathbf{U}|^2 - \frac{\partial H_\Theta (r, \Theta)}{\partial \varrho} \right) \ {\rm d}x,$$

$$I_4 = - \int_\Omega \Theta \varrho s(\varrho, \vartheta) \ {\rm d}x,$$

$$I_5 = \int_\Omega \left(\frac{\partial H_\Theta (r, \Theta)}{\partial \varrho} r + H_\Theta (r, \Theta) \right) \ {\rm d}x,$$

we realize that $\mathcal{E}$ can be used to measure the time evolution of the distance between a weak solution $[\varrho, \vartheta, \mathbf{u}]$ and an arbitrary trio of smooth functions $[r, \Theta, \mathbf{U}]$ satisfying the natural constraint

$$r > 0, \ \Theta > 0, \ \mathbf{U}|_{\partial \Omega} = 0 \text{ if } \mathbf{u} \text{ satisfies the no-slip boundary condition, } \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ in the case of complete slip.} \tag{3.3}$$

Indeed expressing the time evolution of each integral $I_1 - I_5$ by means of some of the integral identities (2.22 - 2.25) we arrive at

RELATIVE ENTROPY INEQUALITY

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{t=\tau} \tag{3.4}$$
$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ {\rm d}x \ {\rm d}t$$
$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ {\rm d}t \text{ for a.a. } \tau \geq 0,$$

where the remainder reads

$$\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \tag{3.5}$$
$$= \int_0^\tau \int_\Omega \varrho (\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \ {\rm d}x \ {\rm d}t + \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\mathbf{U} - \mathbf{u} \Big) \cdot \nabla_x \Theta \ {\rm d}x \ {\rm d}t$$
$$+ \int_0^\tau \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) - p(\varrho, \vartheta) {\rm div}_x \mathbf{U} + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \Big) \ {\rm d}x \ {\rm d}t$$
$$- \int_0^\tau \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{U} \cdot \nabla_x \Theta + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ {\rm d}x \ {\rm d}t$$
$$+ \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \ {\rm d}x \ {\rm d}t + \int_0^\tau \int_\Omega \varrho \mathbf{f} \cdot (\mathbf{u} - \mathbf{U}) \ {\rm d}x \ {\rm d}t,$$

cf. [31] for details.

3.1.2 Dissipative solutions

Motivated by DiPerna and Lions [44] we say that $[\varrho, \vartheta, \mathbf{u}]$ is a *dissipative solution* of the Navier-Stokes-Fourier system if:

- $[\varrho, \vartheta, \mathbf{u}]$ belong to the regularity class specified through (2.32 - 2.35);
- $[\varrho, \vartheta, \mathbf{u}]$ satisfy the relative entropy inequality for a.a. $\tau \in [0, T]$ for any trio of test functions $[r, \Theta, \mathbf{U}]$ satisfying (3.3).

It is clear from the previous discussion that any weak solution of the Navier-Stokes-Fourier system is a dissipative solution.

3.2 Weak-strong uniqueness

A remarkable feature of the dissipative solutions is that they comply with the principle of *weak-strong uniqueness.* We focus on the case of no-slip boundary condition for the velocity. Consider regular initial data

$$\varrho(0, \cdot) = \varrho_0, \ \inf_{\Omega} \varrho_0 > 0, \ \varrho_0 \in W^{3,2}(\Omega), \tag{3.6}$$

$$\vartheta(0, \cdot) = \vartheta_0, \ \inf_{\Omega} \vartheta_0 > 0, \ \vartheta_0 \in W^{3,2}(\Omega), \tag{3.7}$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0 \in W^{3,2}(\Omega; R^3), \tag{3.8}$$

satisfying the natural compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0, \ \nabla_x \vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \nabla_x p(\varrho_0, \vartheta_0)|_{\partial\Omega} = \mathrm{div}_x \mathbb{S}(\varrho_0, \vartheta_0, \nabla_x \mathbf{u}_0) + \varrho_0 \mathbf{f}|_{\partial\Omega}. \tag{3.9}$$

Theorem 3.1 *Under the hypotheses of Theorem 2.1, suppose that* $[\varrho, \vartheta, \mathbf{u}]$ *is a dissipative solution to the Navier-Stokes-Fourier system emanating from the initial data satisfying (3.6 - 3.9). Let*

$$\tilde{\varrho}, \ \tilde{\vartheta} \in C([0, T] : W^{3,2}(\Omega)), \ \tilde{\mathbf{u}} \in C([0, T]; W^{3,2}(\Omega; R^3))$$

be a strong solution of the same problem defined on an existence interval $[0, T_{\max})$*,* $0 < T < T_{\max}$.

Then

$$\varrho = \tilde{\varrho}, \ \vartheta = \tilde{\vartheta}, \ \mathbf{u} = \tilde{\mathbf{u}} \ \textit{in} \ [0, T] \times \Omega.$$

Although the main idea - taking $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ as test functions in the relative entropy inequality (3.4) - is rather straightforward, the complete proof of Theorem 3.1 is rather tedious and technical, see [28, Theorem 6.2] and [31, Theorem 2.1].

Remark 3.1 *The existence of local-in-time strong solutions for the Navier-Stokes-Fourier system was proved by Valli [54], [55], Valli-Zajaczkowski [56]. Global existence for "small" data was established by Matsumura and Nishida [46], [47].*

3.3 Conditional regularity

Another application based on the relative entropy inequality is the following conditional regularity result, see [32, Theorem 2.1]:

Theorem 3.2 *Under the hypotheses of Theorem 2.1, assume that* $[\varrho, \vartheta, \mathbf{u}]$ *is a dissipative solution of the Navier-Stokes-Fourier system in* $(0,T) \times \Omega$*, emanating from the initial data belonging to the class specified in (3.6 - 3.9), and such that*

$$\text{ess} \sup_{(0,T)\times\Omega} |\nabla_x \mathbf{u}| < \infty.$$

Then $[\varrho, \vartheta, \mathbf{u}]$ *is a classical solution of the problem in* $(0,T) \times \Omega$*.*

Remark 3.2 *Other conditional regularity results are available in the literature, see Fan, Jiang, and Ou [24], Jiang and Ou [38]. The main advantage of Theorem 3.2 is its applicability in the situation when the weak solutions are known to exist globally in time.*

3.4 Inviscid fluids

According to the present state of the art, it is still not clear whether or not the use of *weak solutions* in the context of viscous fluids is really necessary. In contrast with the inviscid fluids, where the shock waves effectively appear no matter how smooth and small the data are, similar singularities have not been observed (rigorously shown) for solutions of the Navier-Stokes-Fourier system. On the other hand, the problem of well-posedness, and, in particular, uniqueness in the class of weak solutions remains largely open even for the viscous fluids. In order to explore some unexpected properties the weak solutions might possess, we make a short excursion in the theory of inviscid fluids, which is inspired by the recent results by DeLellis and Székelyhidi [17], [18], [19].

3.4.1 Euler-Fourier system

We consider the inviscid analogue of the system (1.1 - 1.3), namely

Euler-Fourier system

$$\partial_t \varrho + \text{div}_x(\varrho \mathbf{u}) = 0; \tag{3.10}$$

$$\partial_t(\varrho \mathbf{u}) + \text{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho\vartheta) = 0; \tag{3.11}$$

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \text{div}_x(\varrho\vartheta\mathbf{u})\right) - \Delta\vartheta = -\varrho\vartheta\text{div}_x\mathbf{u}, \tag{3.12}$$

where, for the sake of simplicity, we have taken the pressure and the internal energy of a monoatomic gas. Although the model is rather inconsistent - the viscosity vanishes while the gas is still heat conductive - similar equations were used in certain applications in meteorology, see Wilcox [57]. We restrict ourselves to the periodic spatial domain

$$\Omega = \left([-1,1]|_{\{-1;1\}} \right)^3 \tag{3.13}$$

to avoid the technical problem connected with the presence of a kinematic boundary.

From the mathematical viewpoint, the system (3.10 - 3.12) is hyperbolic-parabolic and fits in the category of problems studied by Serre [49], [50]. In particular, the problem admits regular solutions for regular initial data on a possibly short time interval, see Alazard [1], Serre [49]. As (3.10-3.12) shares similar structure with the compressible isentropic Euler system, shock waves are likely to develop in a finite time even for smooth and small initial data.

3.4.2 Infinitely many weak solutions for smooth initial data

The following rather surprising result was proved by an adaptation of the method of convex integration developed by DeLellis and Székelyhidi [18], see [12, Theorem 3.1]:

Theorem 3.3 *Let $T > 0$ and let the initial data satisfy*

$$\varrho_0 \in C^3(\Omega), \vartheta_0 \in C^2(\Omega), \mathbf{u}_0 \in C^3(\Omega; R^3),\ \varrho_0(x) > \underline{\varrho} > 0,\ \vartheta_0(x) > \underline{\vartheta} > 0 \text{ for any } x \in \Omega. \tag{3.14}$$

Then the initial-value problem (3.10 - 3.13) admits infinitely many weak solutions in $(0,T) \times \Omega$ belonging to the class:

$$\varrho \in C^2([0,T] \times \Omega),\ \partial_t \vartheta \in L^p(0,T;L^p(\Omega)),\ \nabla_x^2 \vartheta \in L^p(0,T;L^p(\Omega;R^{3\times 3})) \text{ for any } 1 \leq p < \infty,$$
$$\mathbf{u} \in C_{\rm weak}([0,T];L^2(\Omega;R^3)) \cap L^\infty((0,T)\times\Omega;R^3),\ {\rm div}_x \mathbf{u} \in C^2([0,T]\times\Omega).$$

Remark 3.3 *Given its regularity, the solution $[\varrho, \vartheta, \mathbf{u}]$ is "almost" strong in the sense that it is only the momentum equation (3.11) that is satisfied in the weak sense, while (3.10), (3.12) hold pointwise in $(0,T) \times \Omega$.*

At least some of the solutions obtained in Theorem 3.3 are apparently "non-physical". In particular, it can be shown, exactly as for the Navier-Stokes-Fourier system, that the weak solutions of (3.10 - 3.14) obey the weak-strong uniqueness principle, as a matter of fact they satisfy the relative entropy inequality (3.4) with $\mathbb{S} = 0$, as long as they conserve the total energy, specifically,

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{3}{2}\varrho\vartheta \right)(\tau,\cdot)\ {\rm d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{3}{2}\varrho_0\vartheta_0 \right)\ {\rm d}x \text{ for a.a. } \tau \in (0,T). \tag{3.15}$$

Thus the solutions obtained in Theorem 3.3 must violate (3.15) in the sense that they "produce" energy, or, in the terminology of Section 3.1.2, they are not dissipative.

On the other hand, however, even the satisfaction of (3.15) does not guarantee uniqueness as shown in the following result (see [12, Theorem 4.2]):

Theorem 3.4 *Let $T > 0$ and let the initial data ϱ_0, ϑ_0 be given such that*

$$\varrho_0, \vartheta_0 \in C^2(\Omega),\ \ \varrho_0(x) > \underline{\varrho} > 0,\ \ \vartheta_0(x) > \underline{\vartheta} > 0 \text{ for any } x \in \Omega. \tag{3.16}$$

Then there exists a velocity field $\mathbf{u}_0$,

$$\mathbf{u}_0 \in L^\infty(\Omega; R^3),$$

such that the problem (3.10 - 3.13), with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$, admits infinitely weak solutions in $(0,T) \times \Omega$ satisfying the energy balance (3.15).

4 Long-time behavior

In the remaining part of the paper we focus on the viscous fluids described by means of the Navier-Stokes-Fourier system (2.22 - 2.25) on a bounded regular domain Ω, where the velocity field satisfies the no-slip condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{4.1}$$

From the point of view of dynamical systems theory, the Navier-Stokes-Fourier system features two seemingly contradictory properties:

- the system is conservative as the *total energy* is a conserved quantity, cf. (2.6);
- the system is dissipative as the *mechanical energy* is irreversibly converted to heat, cf. (2.3), (2.4).

For the sake of simplicity, the only driving mechanism is represented by the external volume force $\mathbf{f}$. We discuss the following three cases:

- the "truly" conservative case; we suppose $\mathbf{f} = \nabla_x F$, where $F = F(x)$ is a time-independent potential;
- the time independent non-conservative case $\mathbf{f} = \mathbf{f}(x) \neq \nabla_x F$;
- the general time dependent forcing $\mathbf{f} = \mathbf{f}(t, x)$.

4.1 Equilibrium states

The equilibrium solutions are usually characterized by the following properties:

- they minimize the entropy production among all admissible trajectories;
- they maximize the entropy among all admissible states of the system;
- any trajectory stabilizes for $t \to \infty$ to an equilibrium.

With our choice of conservative boundary conditions, in particular (4.1), the entropy production vanishes for equilibria, which yields

$$\tilde{\mathbf{u}} = 0, \ \tilde{\vartheta} = \tilde{\vartheta}(t)$$

whenever $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ is an equilibrium solution.

Under these circumstances, the momentum equation (1.2) reduces to

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho}\mathbf{f},$$

while the equation of continuity (1.1) yields $\tilde{\varrho} = \tilde{\varrho}(x)$ is independent of time. However, as the total entropy is constant at equilibrium we may infer that:

1. the system admits equilibrium solutions only if $\mathbf{f} = \nabla_x F$, with $F = F(x)$;
2. any equilibrium is of the form $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} = 0]$, where $\tilde{\varrho}$, $\tilde{\vartheta} =$ const are independent of time;
3. $\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F$.

For the rest of this part we shall therefore assume that

$$\mathbf{f}(t,x) = \nabla_x F(x), \ F \in W^{1,\infty}(\Omega; R^3). \tag{4.2}$$

Consequently, the system admits two conserved quantities:

- the total *mass*

$$\int_\Omega \varrho(t,\cdot) \ \mathrm{d}x = M_0 > 0;$$

- the total *energy*

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right)(t,\cdot) \ \mathrm{d}x = E_0;$$

where both M_0 and E_0 are determined by the initial data.

In what follows, we want to avoid the equilibrium densities $\tilde{\varrho}$ with vacuum - vanishing on some non-empty subset of Ω. To this end, we make an extra structural restriction concerning the pressure, namely

$$\frac{\partial p(0, \tilde{\vartheta})}{\partial \varrho} > 0 \text{ for any } \tilde{\vartheta} > 0. \tag{4.3}$$

We conclude by the following result, see [33, Chapter 4, Theorem 4.1]:

Theorem 4.1 *Let $\Omega \subset R^3$ be a bounded domain with Lipschitz boundary. Suppose that the thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$ are continuously differentiable for $[\varrho, \vartheta] \in [0, \infty) \times (0, \infty)$ satisfying Gibbs' relation (2.1), the thermodynamic stability hypothesis (2.2), and the structural condition (4.3). Let $F \in W^{1,\infty}(\Omega)$.*

Then for given $M_0 > 0$, E_0 there is at most one equilibrium solution $[\tilde{\varrho}, \tilde{\vartheta}]$ in the class of Lipschitz functions such that

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F \text{ in } \Omega, \int_\Omega \tilde{\varrho} \ \mathrm{d}x = M_0, \int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \right) \ \mathrm{d}x = E_0. \tag{4.4}$$

In addition, $\tilde{\vartheta} = \text{const} > 0$, $\tilde{\varrho}$ is strictly positive in Ω, and

$$\int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) \ \mathrm{d}x \geq \int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x$$

for any measurable functions ϱ, ϑ satisfying

$$\int_\Omega \varrho \ \mathrm{d}x = M_0, \int_\Omega (\varrho e(\varrho, \vartheta) - \varrho F) \ \mathrm{d}x = E_0.$$

4.1.1 Lyapunov functional and stabilization to equilibria

On condition that the equilibrium density $\tilde{\varrho}$ is strictly positive, we can take the trio $[r = \tilde{\varrho}, \Theta = \tilde{\vartheta}, \mathbf{U} = 0]$ as test functions in the relative entropy inequality (3.4) obtaining

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, 0 \right) \right]_{t=0}^{t=\tau} \tag{4.5}$$

$$+ \int_0^\tau \int_\Omega \frac{\tilde{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t \leq 0.$$

Thus the relative entropy provides a Lyapunov functional in the potential case $\mathbf{f} = \nabla_x F$. A natural conjecture therefore is that any trajectory will stabilize to and equilibrium for $t \to \infty$ as stated in the following theorem, see [33, Chapter 4, Theorem 4.5]:

Theorem 4.2 *In addition to the hypotheses of Theorem 2.1, suppose that the pressure* p *satisfies (4.3) and* $\mathbf{f} = \nabla_x F$, $F = F(x)$. *Let* $[\varrho, \vartheta, \mathbf{u}]$ *be a weak solution of the Navier-Stokes-Fourier system (2.22 - 2.25) in* $[0, \infty) \times \Omega$, *with the velocity field* $\mathbf{u}$ *satisfying the no-slip boundary condition (4.1).*

Then

$$\varrho(t, \cdot) \to \tilde{\varrho} \text{ in } L^{5/3}(\Omega),$$
$$\vartheta(t, \cdot) \to \tilde{\vartheta} \text{ in } L^{4}(\Omega),$$

and

$$(\varrho\mathbf{u})(t, \cdot) \to 0 \text{ in } L^{5/4}(\Omega; R^3),$$

as $t \to \infty$, *where* $[\tilde{\varrho}, \tilde{\vartheta}]$ *is the unique equilibrium solution satisfying*

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x = M_0, \ \int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \right) \ \mathrm{d}x = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) - \varrho_0 F \right) \ \mathrm{d}x = E_0.$$

4.2 Global attractor in the potential case

The previous discussion indicates that the set of equilibria represents an *attractor* for the global-in-time solutions to the Navier-Stokes-Fourier system. More precisely, our goal is to show that *all* global trajectories tend to equilibria at a *uniform speed* depending only on the size of the initial data. We report the following result, see [33, Chapter 5, Theorem 5.1]:

Theorem 4.3 *Under the hypotheses of Theorem 4.2, consider a family of weak solutions* $[\varrho, \vartheta, \mathbf{u}]$ *of the Navier-Stokes-Fourier system (2.22 - 2.25) in* $(0, \infty) \times \Omega$ *such that*

$$\int_\Omega \varrho \ \mathrm{d}x \geq M_0, \ \int_\Omega \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x \leq E_0, \ \operatorname{ess} \inf_{\tau > 0} \int_\Omega \varrho s(\varrho, \vartheta)(\tau, \cdot) \ \mathrm{d}x \geq S_0, \tag{4.6}$$

where $M_0 > 0$, E_0, $S_0 > 0$ *are given numbers.*

Then for any given $\varepsilon > 0$, *there exists* $T = T(\varepsilon)$ *such that*

$$\sup_{t > T(\varepsilon)} \|\varrho(t, \cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} < \varepsilon,$$

$$\operatorname{ess} \sup_{t > T(\varepsilon)} \|\vartheta(t, \cdot) - \tilde{\vartheta}\|_{L^4(\Omega)} < \varepsilon,$$

$$\sup_{t > T(\varepsilon)} \|(\varrho \mathbf{u})(t, \cdot)\|_{L^{5/4}(\Omega; R^3)} < \varepsilon$$

whenever $[\varrho, \vartheta, \mathbf{u}]$ *belongs to the class (4.6), where* $[\tilde{\varrho}, \tilde{\vartheta}]$ *is the equilibrium solution specified in Theorem 4.2.*

Thus, in accordance with the pessimistic prognosis of the Second law of thermodynamics, any solution of an energetically closed dissipative system stabilizes to equilibrium in the long run:

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu
Rudolph Clausius, 1822-1888

The set of equilibria in the conservative case $\mathbf{f} = \nabla_x F(x)$ is a genuine attractor, it attracts sets given by (4.6) with a uniform speed. On the other hand, clearly, the set of *all* equilibrium solutions is unbounded.

4.3 Nonconservative driving forces

We turn attention to non-conservative driving forces $\mathbf{f} = \mathbf{f}(t, x)$. Let us start with the following results describing a dichotomy in the class of global-in-time solutions ([33, Chapter 5, Theorem 5.2]):

Theorem 4.4 *In addition to the hypotheses of Theorem 2.1, suppose that the pressure p satisfies (4.3), and the no-slip boundary condition (4.1) is prescribed. Furthermore, let*

$$\mathbf{f} \in L^\infty((0,\infty)\times\Omega; R^3)$$

be given.

Then for any weak solution $[\varrho, \vartheta, \mathbf{u}]$ *of the Navier-Stokes-Fourier system (2.22 - 2.25) in* $(0,\infty)\times\Omega$ *one of the following alternatives occurs:*

1. *Either*
$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(\tau,\cdot)\ \mathrm{d}x \to \infty \ \textit{for}\ \tau \to \infty; \tag{4.7}$$

2. *or there exists a constant* E_∞ *such that*
$$E(\tau) \leq E_\infty \ \textit{for a.a.}\ \tau \in (0,\infty). \tag{4.8}$$

In the latter case, each sequence of times $\tau_n \to \infty$ *contains a subsequence (not relabeled) such that the time shifts*

$$\mathbf{f}_n(t,x) = \mathbf{f}(t+\tau_n, x)$$

satisfy

$$\mathbf{f}_n \to \nabla_x F \ \textit{weakly-(*) in}\ L^\infty((0,T)\times\Omega;R^3),\ T>0\ \textit{arbitrary finite},$$

where the limit potential

$$F = F(x),\ F \in W^{1,\infty}(\Omega),$$

may depend on the choice of the sequence $\{\tau_n\}_{n=1}^\infty$.

As in immediate consequence, we obtain the following:

Corollary 4.1 *Under the hypotheses of Theorem 4.4, suppose that*

$$\mathbf{f} = \mathbf{f}(x),\ \mathbf{f} \neq \nabla_x F.$$

Then

$$E(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)(\tau,\cdot)\ \mathrm{d}x \to \infty \ \textit{for}\ \tau \to \infty;$$

for any solution $[\varrho, \vartheta, \mathbf{u}]$ *of the Navier-Stokes-Fourier system driven by* f.

Intuitively, the energy "blow up" alternative (4.7) occurs if the mechanical energy supplied to the system through $\mathbf{f}$ is irreversibly converted to heat. One would therefore expect the internal energy component to become infinite as well. This is indeed the case provided we strengthen slightly our hypotheses concerning the shear viscosity coefficient μ, see [33, Chapter 5, Corollary 5.3]:

Theorem 4.5 *In addition to the hypotheses of Theorem 4.3, suppose that the shear viscosity coefficient μ is a continuously differentiable function of ϑ such that*

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta), \ |\mu'(\vartheta)| < c \text{ for all } \vartheta \in [0,\infty).$$

Then for any weak solution $[\varrho, \vartheta, \mathbf{u}]$ of the Navier-Stokes-Fourier system (2.22 - 2.25) in $(0,\infty)\times\Omega$ we have:

1. *Either*
$$\int_\Omega \varrho e(\varrho,\vartheta)(\tau,\cdot)\ \mathrm{d}x \to \infty \text{ for } \tau \to \infty;$$
2. *or there exists a constant E_∞ such that*
$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)\ \mathrm{d}x \leq E_\infty \text{ for a.a. } \tau \in (0,\infty).$$

4.3.1 Solutions with unbounded energy

Using Theorem 4.4 we can easily construct examples of driving forces for which the energy blows up, namely

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)\ \mathrm{d}x \to \infty \text{ for } \tau \to \infty.$$

The reader may consult [33, Chapter 5, Section 5.2.2] for details.

1. $\mathbf{f}$ almost periodic (periodic) in t but not constant in t;
2. $\mathbf{f}$ asymptotically almost periodic (periodic),
$$\sup_{x\in\Omega} |\mathbf{f}(t,x) - \mathbf{g}(t,x)| \to 0 \text{ as } t \to \infty,$$
where $\mathbf{g}$ is almost periodic non-constant in time;
3. for any non-empty open subset B of Ω, there is $\mathbf{f} \in C^\infty((0,T)\times\Omega; R^3)$,
$$\text{supp}[\mathbf{f}(t,\cdot)] \subset B \text{ for any } t > 0, \ \|\mathbf{f}(t,\cdot)\|_{C^m(\Omega;R^3)} \to 0 \text{ as } t \to \infty, \ m = 1,2,\cdot$$
such that
$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)\ \mathrm{d}x \to \infty \text{ for } \tau \to \infty.$$
for any solution of the Navier-Stokes-Fourier system driven by $\mathbf{f}$.

4.4 Systems driven by oscillatory forces

In the light of the above results, it may seem that almost "any" external force $\mathbf{f}$ drives the energy of an energetically isolated fluid system to infinity unless $\mathbf{f}$ is represented by a gradient of a time independent scalar potential. Still the scenario described by the second alternative in Theorem 4.4 allows for certain interesting exceptions as, for instance, the rapidly oscillating driving forces we discuss in the present section.

Chepyzhov, Pata, and Vishik [11] examined the long time behavior and attractors for the standard 2D *incompressible* Navier-Stokes system driven by an external force of the form

$$\mathbf{f} = \mathbf{g}_1 + \varepsilon^{-\alpha}\mathbf{g}_2\left(\frac{t}{\varepsilon}\right).$$

They show that the so-called trajectory attractors $\mathcal{A}_\varepsilon$ converge for $\varepsilon \to 0$, in a certain sense, to the attractor $\mathcal{A}_0$ of the system driven by $\mathbf{g}_1$. Thus highly oscillating perturbations do not influence the asymptotic behavior of solutions though they may be of substantially large amplitude.

Pursuing this philosophy, we consider generally unbounded driving forces in the form

$$\mathbf{f}(t,x) = t^\delta \omega(t^\beta)\mathbf{w}(x), \tag{4.9}$$

where ω is a bounded function with uniformly bounded time averages (see the hypothesis (4.10) in the following theorem).

We have the following result, see [7, Theorem 1]:

Theorem 4.6 *Under the hypotheses of Theorem 2.1, suppose that*

$$\mathbf{f}(t,x) = t^\delta \omega(t^\beta)\mathbf{w}(x),$$

where

$$\omega \in L^\infty(R),\ \omega \neq 0,\ \sup_{\tau>0}\left|\int_0^\tau \omega(t)\mathrm{d}t\right| < \infty. \tag{4.10}$$

and the parameters β, δ satisfy

$$\delta > 0,\ \beta > 2 + 2\delta \ \textit{or}\ \delta \leq 0,\ \beta > 2 + \delta. \tag{4.11}$$

Then any weak solution $[\varrho, \vartheta, \mathbf{u}]$ of the Navier-Stokes-Fourier system (2.22 - 2.25) in $(0,\infty)\times\Omega$ satisfies

$$\varrho(t,\cdot) \to \tilde{\varrho} \ \textit{in}\ L^{5/3}(\Omega)\ \textit{as}\ t \to \infty,$$

$$\vartheta(t,\cdot) \to \tilde{\vartheta} \ \textit{in}\ L^{4}(\Omega)\ \textit{as}\ t \to \infty,$$

$$(\varrho\mathbf{u})(t,\cdot) \to 0 \ \textit{in}\ L^{5/4}(\Omega;R^3)\ \textit{as}\ t \to \infty,$$

where $\tilde{\varrho}$, $\tilde{\vartheta}$ are positive constants,

$$\tilde{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho \ \mathrm{d}x.$$

Remark 4.1 *The hypothesis (4.11) is probably not optimal. On the other hand, however, certain acceleration of oscillations is needed since otherwise we would end up in the situation described in (4.7).*

5 Examples of the long-time behavior for non-conservative systems, conclusion

The non-conservative fluid systems, where the energy is either supplied or exchanged with the outer "world", offer more interesting possibilities of the asymptotic behavior than we have observed in energetically insulated systems.

5.1 Time periodic motion

Following [29], we consider the Navier-Stokes-Fourier system driven by a *time periodic* external force $\mathbf{f}$,

$$\mathbf{f} \in L^\infty(R \times \Omega; R^3), \ \mathbf{f}(t+\omega; \cdot) = \mathbf{f}(t, \cdot), \ \omega > 0, \ \text{for a.a. } t \in R. \tag{5.1}$$

Similarly to the preceding part, we consider the no-slip boundary condition for the velocity,

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{5.2}$$

In order to obtain non-trivial results, we replace the conservative no-flux condition (1.6) by

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = d(\vartheta - \Theta_0)|_{\partial\Omega}, \ d \in L^\infty(\partial\Omega), \ \Theta_0 \in L^1(\partial\Omega). \tag{5.3}$$

Unlike (1.6), the boundary condition (5.3) allows for the exchange of thermal energy through the boundary that may preclude the scenario indicated in (4.7).

5.1.1 Weak formulation

The weak formulation (2.22 - 2.25) must be modified in order to accommodate the periodicity of solutions in time as well as the boundary condition (5.3). Denote

$$S^1 = [0, \omega]|_{\{0,\omega\}}$$

the "flat" sphere in the time variable. We say that $[\varrho, \vartheta, \mathbf{u}]$ is a weak *time periodic* solution to the Navier-Stokes-Fourier system with the period $\omega > 0$ if the following holds:

- the density and the temperature are positive, $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ for a.a. $(t,x) \in S^1 \times \Omega$,

$$\varrho \in C_{\rm weak}(S^1; L^{5/3}), \ (\varrho\mathbf{u}) \in C_{\rm weak}(S^1; L^\beta(\Omega; R^3)),$$

$$\vartheta \in L^\infty(S^1; L^4(\Omega)) \cap L^2(S^1; W^{1,2}(\Omega)), \ \mathbf{u} \in L^2(S^1; W^{1,2}_0(\Omega; R^3));$$

- the equation of continuity reads

$$\int_{S^1} \int_\Omega \Big[(\varrho + b(\varrho))\, \partial_t \varphi + (\varrho + b(\varrho))\, \mathbf{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho)\varrho)\, {\rm div}_x \mathbf{u} \varphi \Big] \ {\rm d}x \ {\rm d}t = 0 \tag{5.4}$$

  for any test function $\varphi \in C^1(S^1 \times \overline{\Omega})$, and any $b \in C^1[0,\infty)$, $b(0) = 0$, $b'(\varrho) = 0$ for all $\varrho \geq c_b$;

- the momentum equation is satisfied in the sense of distributions,

$$\int_{S^1}\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi - \mathbb{S} : \nabla_x\varphi + \varrho\mathbf{f}\cdot\varphi \Big)\ \mathrm{d}x\ \mathrm{d}t = 0 \tag{5.5}$$

for any $\varphi \in C^1(S^1 \times \overline{\Omega}; R^3)$, where $\varphi|_{\partial\Omega} = 0$;

- the entropy balance is replaced by a family of integral inequalities

$$\int_{S^1}\int_\Omega \frac{\varphi}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t - \int_{S^1}\int_{\partial\Omega} \frac{d}{\vartheta}(\vartheta - \Theta_0)\varphi\ \mathrm{d}S_x \tag{5.6}$$

$$\leq - \int_{S^1}\int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}\cdot\nabla_x\varphi}{\vartheta} \right)\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C^1(S^1 \times \overline{\Omega})$, $\varphi \geq 0$;

- the total energy balance reads

$$\int_{S^1}\left[\partial_t\psi \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)\ \mathrm{d}x \right]\ \mathrm{d}t = \int_{S^1}\psi\left[\int_{\partial\Omega} d(\vartheta - \Theta_0)\ \mathrm{d}S_x - \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x \right]\mathrm{d}t \tag{5.7}$$

for any $\psi \in C^\infty(S^1)$.

5.1.2 Existence of time periodic solutions

The following result was proved in [29, Theorem 2.1]:

Theorem 5.1 *In addition to the hypotheses of Theorem 4.3, suppose that the shear viscosity coefficient μ is a continuously differentiable function of ϑ and satisfies*

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta),\ |\mu'(\vartheta)| < c \text{ for all } \vartheta \in [0,\infty).$$

Let $\mathbf{f}$ be given by (5.1), and let the functions d, Θ_0 appearing in (5.3) satisfy

$$d(x) \geq \underline{d} > 0,\ \Theta_0(x) \geq \underline{\Theta} > 0 \text{ for a.a. } x \in \partial\Omega.$$

Then for any $M_0 > 0$, the Navier-Stokes-Fourier system (5.4 - 5.7) admits a time-periodic weak solution $[\varrho,\vartheta,\mathbf{u}]$ such that

$$\int_\Omega \varrho\ \mathrm{d}x = M_0.$$

One may expect other results of similar character for problems with non-conservative boundary conditions. In particular the existence of bounded absorbing sets and global attractors.

5.2 Concluding remarks

As we have seen in the preceding sections, the long-time dynamics of energetically isolated fluid systems is rather simple and quite well understood. As an inevitable consequence of the Second law of thermodynamics, any system driven by a conservative external force approaches, as time tends to infinity, a single equilibrium determined uniquely by the total mass and energy of the fluid. In case the fluid is driven by a nonconservative force but its physical boundary still remains insulated, the energy is irreversibly transfered into heat resulting in the blow-up of the total energy of the system at infinite time. Interestingly enough, the rapidly oscillating external forces may escape this scenario and drive the system to a stable equilibrium with finite energy. The results we obtained are quite general and applicable to a vast class of the weak solutions without any restriction on the size of the data. On the other hand, however, we should keep in mind that the technical hypotheses imposed on the specific form of constitutive equations were dictated by the needs of the *existence* theory and many of the previous results can be possibly shown to hold in a much more general setting.

Thus, rather surprisingly and in a sharp contrast with the commonly accepted scenario that attractors with complicated topological structure are the right objects to capture the behavior of fluid systems (cf., for instance, Eckmann and Ruelle [21]) we have seen that the energetically insulated fluid systems evolve to well-determined, and completely predictable states in accordance with the Second law of thermodynamics. This apparent "paradox" should be seen from the perspective of the following arguments:

- The time scales relevant in the real world that are related to turbulence are significantly shorter than the "infinite" time at which the extrapolation of the Second law of thermodynamics inevitably leads to the so-called heat death of the universe. Last, but not least, the present theory is limited by the principles of classical mechanics and hence any kind of metaphysical conclusion of this type lies definitely out of its reach.
- Fluids can transfer heat; however, the fraction of the kinetic energy of the fluid that is irreversibly dissipated into heat is relatively small or even negligible compared with the effects of convection. In addition, the thermal conductivity of certain fluids (gases) may be very small, too.
- There is no physical system perfectly insulated from the outer world; to the contrary, it is a characteristic feature of many dissipative systems that they can effectively dispose the entropy via external "cooling", similarly to the example of the time periodic motion we discussed in Section 5.1.
- The thermal and mechanical effects may act independently, at least on observable time scales, leading to isothermal or isentropic systems, where the mechanical motion is completely separated from thermal effects.

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