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Abstract

Some new classes of compacta K are considered for which $C(K)$ endowed with the pointwise topology has a countable cover by sets of small local norm–diameter.

1 Introduction

A topological notion introduced in [8] plays an important role in the study of topological and renorming properties of Banach spaces [11]: If $(X, \mathcal{T})$ is a topological space and ϱ is a metric on X, we say that it has a *countable cover by sets of small local ϱ–diameter* if for every $\varepsilon > 0$ we can write $X = \bigcup_{n\in\mathbb{N}} X_n$ in such a way that for any $n \in \mathbb{N}$ and every $x \in X_n$ there exists a $\mathcal{T}$–open set U such that $x \in U$ and ϱ–diam$(U \cap A) < \varepsilon$. Here we consider some compacta K such that $C(K)$, endowed with the pointwise topology, has a countable cover by sets of small local ϱ–diameter when ϱ is the norm–metric, or $C_p(K)$ has **SLD** for short. Let us recall that given a topology τ, coarser than the norm topology on a Banach space X, we say that X has τ–Kadets when τ and its norm topology coincide in the unit sphere. If $C(K)$ has a pointwise Kadets equivalent norm then $C_p(K)$ must have **SLD** [8], that in turn implies that $C_p(K, \{0, 1\})$ is σ–discrete.

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Whether any of the converse implications holds is a well-known open problem. However M. Raja has shown, roughly speaking, that **SLD** is *very close* to the existence of a pointwise Kadets renorming, namely from [11] it follows that if X is a Banach space endowed with a topology τ, coarser than the norm topology, then (X, τ) has the property **SLD** for the norm if, and only if, there exists a non negative symmetric homogeneous τ–lower semicontinuous function (that may be not convex) F on X with $\|\cdot\| \leq F \leq 3\|\cdot\|$ such that the norm topology and τ coincide on the set $\{x \in X: \ F(x) = 1\}$. In [12] it is proved that $C_p(K \times L)$ has **SLD** whenever $C_p(L)$ has it and $C_p(K)$ has a pointwise Kadets norm [12]. Let us mention that if X is a Banach space such that (X, weak) has the **SLD** property for the norm and the bidual of $(X, \|\cdot\|)$ is strictly convex then X has a locally uniformly rotund equivalent norm [10]. (A norm $\|\cdot\|$ in a Banach space is locally uniformly rotund if $\lim_k \|x_k - x\| = 0$ whenever $\lim_k \|(x_k + x)/2\| = \lim_k \|x_k\| = \|x\|$.) Despite no topological characterization has been obtained for those K's such that $C_p(K)$ has **SLD**, some light on this questions has been shed in some particular classes of compacta [4], [5], [2], [6].

In this note we present two classes of compact spaces K for which the spaces $C_p(K)$ have **SLD**. It is well-known that every compact space is a continuous image of a 0-dimensional compact space. In turn, a 0-dimensional space can be regarded as a subspace of a Cantor cube 2^S which can be identified with the power-set of a fixed set S. Consequently, a 0-dimensional compact space carries a partial ordering, which is just the inclusion relation. It is natural to ask that the partially ordered compact space has the property that for every two elements x and y there exists their infimum $\inf\{x, y\}$. Moreover, it is natural to expect that the operation $\langle x, y\rangle \mapsto \inf\{x, y\}$ is continuous. Once this happens, we speak about *compact semilattices*.

One should not expect positive topological properties of $C(K)$ spaces, where K is an arbitrary compact semilattice, since this class contains 1-point compactifications of trees studied in [4]. We prove, however, that for a fairly large class of compact semilattices K the space $C_p(K)$ is **SLD**.

2 Preliminaries

A *semilattice* is a partially ordered set $\langle S, \leq\rangle$ which contains the minimal element (always denoted by 0) and in which every pair of elements x, y has the greatest lower bound, denoted by $x \wedge y$. The element $x \wedge y$ is sometimes called the *meet of* x, y and S is sometimes called a *meet semilattice*. Some authors do not require the existence of the minimal element, we do it since we are going to consider compact semilattices in which the minimal element always exists. A semilattice S is *topological* if it carries a Hausdorff topology with respect to which $\wedge$ is continuous.

A *filter* in a semilattice S is a subset $F \subseteq S$ (possibly empty) satisfying

$$\{x \in S \colon a \wedge b \leq x\} \subseteq F$$

for every $a, b \in F$. A filter F is *principal* if it is of the form

$$[p, \to) = \{x \in S \colon p \leq x\}.$$

Later on, we shall use some standard (although not trivial) properties of compact 0-dimensional semilattices. For details we refer to one of the books [7] or [3].

In particular, we shall need the following algebraic notion. An element p of a semilattice S is *compact* if for every $A \subseteq S$ with $\sup A = p$ there exists a finite $A_0 \subseteq A$ such that $\sup A_0 = p$. In particular, 0 is a compact element.

The following fact will be used later without explicit reference:

Proposition 2.1. *Let K be a compact 0-dimensional semilattice. Then*

(1) A principal filter $[p, \to)$ is a clopen set if and only if p is a compact element.

(2) Given $a, b \in K$ such that $a \not\leq b$, there exists a compact element p such that $p \leq a$ and $p \not\leq b$.

(3) Clopen principal filters and their complements generate the topology of K.

(4) Given a nonempty clopen set $A \subseteq K$, every minimal element of A is compact.

We now make few comments concerning a Stone-like duality for semilattices. Namely, given a compact semilattice K, denote by $\mathbb{S}(K)$ the family of all clopen filters in K. By Proposition 2.1(1), every nonempty element F of $\mathbb{S}(K)$ can be identified with its vertex p, which is a compact element such that $F = [p, \to)$. Recall that the empty set is a filter. Thus, $\mathbb{S}(K)$ can be identified with the set of all positive compact elements of K plus the "artificial" element ∞. Observe that $\mathbb{S}(K)$, treated as the family of filters, is a semilattice in which the meet of $F, G \in \mathbb{S}(K)$ is $F \cap G$. If $F = [p, \to)$, $G = [q, \to)$, then either $F \cap G = [r, \to)$, where $r = \sup\{p, q\}$ or $F \cap G = \emptyset$.

It turns out that this operation is reversible, namely, if $\langle S, \wedge, 0 \rangle$ is a semilattice (considered without any topology) then one can define $K(S)$ to be the family of all filters in S endowed with inclusion. The space $K(S)$ is compact 0-dimensional when endowed with the topology inherited from the Cantor cube $\mathcal{P}(S)$, the power-set of S. The duality (proved in [7]) says that $K(S)$ is canonically isomorphic to K whenever $S = \mathbb{S}(K)$. More precisely, given $x \in K$, define $\hat{x} = \{p \in \mathbb{S}(K) \colon x \in p\}$. Then $\hat{x}$ is a filter in $\mathbb{S}(K)$; in other words, $\hat{x} \in K(\mathbb{S}(K))$. It turns out that all elements of $K(\mathbb{S}(K))$ are of this form.

The second part of this note is devoted to compact distributive lattices. Suppose that K is a compact semilattice with the unique maximal element 1. Then, by compactness, K is a complete lattice, that is, for every $A \subseteq K$ the $\sup A$ and $\inf A$ exist. In fact, $\inf A$ is the limit of the net $\{\inf S\}_{S \in [A]^{<\omega}}$, where $[A]^{<\omega}$ denotes the family of all finite subsets of A. On the other hand, $\sup A$ is the infimum of the set of all upper bounds of A (this set contains 1, therefore is nonempty). We shall denote $\sup\{x, y\}$ by $x \vee y$

(sometimes it is called the *join of* x and y). It is natural to ask when the operation $\vee$ is continuous. Once it happens, we say that K is a *compact lattice*. A lattice $\langle K, \wedge, \vee \rangle$ is *distributive* if it satisfies $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for every $a, b, c \in K$. The notion of a filter in a lattice is the same as in a semilattice. Note that a lattice with the reversed ordering is again a lattice (where the meet is exchanged with the join). Let us call it the *reversed lattice*. A filter in the reversed lattice will be called an *ideal*. Important for us is the notion of a prime filter. Namely, a filter F is called *prime* if it is nonempty and its complement is a nonempty ideal. As we are interested in compact 0-dimensional distributive lattices K, we shall work with the family of all clopen prime filters in K, denoted by $\mathbb{P}(K)$. This is justified by the following:

Proposition 2.2. *Let K be a compact 0-dimensional distributive lattice. Then for every $a, b \in K$ with $a \not\leq b$ there exists a clopen prime filter $P \subseteq K$ such that $a \in P$ and $b \notin P$.*

Fix a distributive lattice L. A subset $G \subseteq L$ is called *convex* if it is of the form $G = I \cap F$, where I is an ideal and F is a filter. For convenience, we allow here that $I = L$ or $F = L$, therefore every ideal and every filter are convex sets. Given $a, b \in L$ we define the interval $[a, b] = \{x \in L \colon a \wedge b \leq x \leq a \vee b\}$. This is the minimal convex set containing a, b. Assume now that L is a compact distributive lattice. Given two disjoint closed convex sets A, B, given $a_0 \in A$, there always exist $a_1 \in A$, $b_1 \in B$ such that $a_1 \in [a_0, b_1]$ and $a_1 \in [x, b_1]$. $b_1 \in [a_1, y]$ holds for every $x \in A$, $y \in B$. Furthermore, $A \cap [a_1, b_1] = \{a_1\}$ and $B \cap [a_1, b_1] = \{b_1\}$. The pair $\langle a_1, b_1 \rangle$ is called a *gate* between A and B. The notion of a gate between convex sets is actually defined for a bigger class of compact spaces, called *compact median spaces*. The existence of gates follows from the following fact: Given a family $\mathcal{F}$ consisting of closed convex sets with $\bigcap \mathcal{F} = \emptyset$, there exist $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$. For details we refer to the book [13].

3 Modest semilattices

In this section we show that $\mathcal{C}_p(K, 2)$ is σ-discrete whenever K is a compact totally disconnected semilattice satisfying certain condition.

Namely, we call a semilattice K *modest*, if it is totally disconnected and for every compact element $p \in K$ the set p^- of all immediate predecessors of p is finite. Recall that x is an *immediate predecessor* of p if $x < p$ and no y satisfies $x < y < p$.

Below we give two natural examples of modest semilattices.

Proposition 3.1. *Every compact totally disconnected distributive lattice is a modest semilattice.*

Proof. Let $\langle L, \wedge, \vee, 0, 1 \rangle$ be such a lattice. By definition, both operations $\wedge$ and $\vee$ are continuous, therefore L is a topological semilattice. Fix $p \in L$ compact and suppose $\{x_n\}_{n \in \omega} \subseteq p^-$ is such that $x_n \vee x_m = p$ for every $n < m < \omega$. Let y be an accumulation point of $\{x_n\}_{n>0}$. From Proposition 2.1 $y \not\geq p$, therefore there exists a clopen prime filter U such that $p \in U$ and $y \notin U$. Find $k < \ell$ such that $x_k, x_\ell \in K \setminus U$. As $K \setminus U$ is an ideal, $p = x_k \vee x_\ell \in K \setminus U$, a contradiction. This shows that $x \geq p$ and hence p is not a compact element. □

Example 3.2. Let T be a finitely-branching tree and consider $\alpha T = T \cup \{\infty\}$, where ∞ is an additional element satisfying $\infty > t$ for every $t \in T$. The tree T can be regarded as a locally compact space, where a neighborhood of $t \in T$ is of the form $(s, t]$, where $s < t$. Define the topology on αT so that it becomes the one-point compactification of T. Define an operation $\wedge$ on αT as follows: let $s \wedge t = \max(s, t)$ whenever s and t are comparable, and let $s \wedge t = \infty$ otherwise. It is easy to check that this is a continuous semilattice operation. The property that T is finitely-branching is equivalent to the fact that αT is a modest semilattice.

Theorem 3.3. *Let K be a modest semilattice. Then $\mathcal{C}_p(K, 2)$ is σ-discrete.*

Proof. Throughout this proof, we shall consider $\mathbb{S}(K)$ as the set of all positive compact elements, plus an artificial element $\infty \notin K$ which corresponds to the empty clopen filter. So the meet operation on $\mathbb{S}(K)$ is actually the supremum in K. Given $p, q \in \mathbb{S}(K)$ we shall denote by $p \cdot q$ the meet of p and q in $\mathbb{S}(K)$, which equals either $\sup\{p, q\}$ in K or ∞ in case $\{p, q\}$ is not bounded from above in K.

Fix $f \in C(K)$. We say that f has a *jump* at $p \in \mathbb{S}(K)$ if $p \neq \infty$ and $f(p) \neq f(q)$ for every $q \in p^-$.

Claim 3.4. *Assume $f \in C(K, 2)$. If f is not constant then f has a jump at some $p \in \mathbb{S}(K)$.*

Proof. Let p be a minimal element of K such that $f(p) \neq f(0)$. Then $p \in \mathbb{S}(K)$ by Proposition 2.1(4) and clearly f has a jump at p. □

We shall say that $f \in C(K)$ has a *relative jump* at $p \in \mathbb{S}(K)$ *with respect to* $q \in \mathbb{S}(K) \cup \{0\}$ if $q < p$ and $f(x) \neq f(p)$ for every $x \in p^- \cap [q)$. Note that a relative jump with respect to 0 is just a jump.

Fix $f \in C(K, 2)$. Let $L_0(f) \subseteq \mathbb{S}(K)$ be the subsemilattice generated by all $p \in \mathbb{S}(K)$ such that f has a jump at p.

By induction, we define $L_n(f)$ to be the subsemilattice of $\mathbb{S}(K)$ generated by $L_{n-1}(f)$ together with all $p \in \mathbb{S}(K)$ such that f has a relative jump at p with respect to some $q \in L_{n-1}(f)$.

Claim 3.5. *For every $f \in C(K, 2)$ there exists $n \in \omega$ such that $L_n(f) = L_{n+1}(f)$.*

Proof. Let $\varphi\colon K \to F$ be a continuous epimorphism onto a finite semilattice such that f is constant on each fiber of φ. Note that $\mathbb{S}(F) \subseteq \mathbb{S}(K)$, after the obvious identification (the pre-image of a compact element is compact). We shall prove by induction that $L_n(f) \subseteq \mathbb{S}(F)$ for every $n \in \omega$.

Notice that a relative jump with respect to 0 is just a jump. Thus, set $L_{-1}(f) = \{\infty\}$, in order to start the induction.

Assume now that $L_{n-1}(f) \subseteq \mathbb{S}(F)$ and fix $p \in L_n(f)$ such that f has a relative jump at p with respect to $q \in L_{n-1}(f)$. Find $t \in F$ such that $p \in \varphi^{-1}(t)$. Let $u = \min \varphi^{-1}(t) \in \mathbb{S}(F)$. We need to show that $p = u$. For suppose that $u < p$. Find $r \in p^- \cap [u)$. Notice that $q \leq u$, because $[q)$ is the union of some fibers of φ and p, u are in the same fiber. It follows that f does not have a relative jump at p with respect to q, because $f(p) = f(r)$. This is a contradiction, which shows that $L_n(f) \subseteq \mathbb{S}(F)$. □

Let σ denote the set of all finite sequences of natural numbers. Given $f \in C(K, 2)$, let $n(f)$ be the minimal $n \in \omega$ such that $L_{n+1}(f) = L_n(f)$. By Claim 3.5, this is well defined. Furthermore, let $s(f)(i) = |L_i(f)|$ for $i \leq n(f)$. So $s(f) \in \sigma$.

This provides a decomposition of $C(K, 2)$ into countably many pieces, indexed by σ. Namely, we shall prove that given $s \in \sigma$, the set

$$C_s = \{f \in C(K, 2) : s(f) = s\}$$

is discrete with respect to the pointwise convergence topology. In order to avoid too many indices, we set $L(f) = L_{n(f)}(f)$.

The following claim is crucial.

Claim 3.6. *Let $f \in C(K, 2)$ and let $\varphi\colon K \to S$ be the continuous semilattice homomorphism induced by the inclusion $L(f) \subseteq \mathbb{S}(K)$ (i.e. $S = \mathbb{S}(L(f))$ and φ is the dual map to this inclusion). Then f is constant on each fiber of φ.*

Proof. Note that for each $t \in S$ the minimal element of $\varphi^{-1}(t)$ belongs to $L(f)$.

Suppose that f is not constant on $G = \varphi^{-1}(t)$ and let $q = \min G$. Then f has a jump at some compact element $p \in G$, $p > q$. This jump is relative with respect to q. But $q \in L_n(f)$ for some $n \leq n(f)$ so $p \in L_{n+1}(f) \subseteq L(f)$. We conclude that q is the minimal element of some fiber of φ, a contradiction. □

Fix $f \in C(K, 2)$ and define

$$M_f = L(f) \cup \bigcup_{p \in L(f)} p^-.$$

Since K is modest, M_f is finite. Assume $g \in C(K, 2)$ is such that $f \restriction M_f = g \restriction M_f$ and $s(f) = s(g)$. We claim that $f = g$.

For this aim, we shall prove by induction that $L_n(g) = L_n(f)$ for every $n \leq n(f) = n(g)$. This is trivial for $n = -1$. In fact, this also holds for $n = 0$, because given $p \in L_0(f)$ for which f has a jump, g has the same values as f in the finite set $\{p\} \cup p^-$. It follows that g has a jump at p. Hence $L_0(g) \subseteq L_0(f)$ and consequently $L_0(g) = L_0(f)$ because these sets have the same finite cardinality.

Now assume that $L_{n-1}(g) = L_{n-1}(f)$ and fix $p \in L_n(f)$ such that f has a relative jump at p with respect to some $q \in L_{n-1}(f)$. Then $q \in L_{n-1}(g)$ and g has exactly the same values as f on the finite set $\{p\} \cup (p^- \cap [q])$. It follows that g has a relative jump at p with respect to q, i.e. $p \in L_n(g)$. This shows that all generators of $L_n(f)$ are contained in $L_n(g)$. Thus $L_n(f) \subseteq L_n(g)$ and consequently $L_n(f) = L_n(g)$, again because of the same finite cardinality.

Finally, by Claim 3.6, we conclude that $g = f$, because M_f intersects each fiber of the quotient map induced by $L(f)$. □

Actually, the proof above provides a recipe for the minimal semilattice quotient for a given function into $\{0, 1\}$.

4 Finite-dimensional compact lattices

In this section we prove that $C(K)$ has the **SLD** whenever K is a finite-dimensional compact lattice. Recall that a lattice K is *finite-dimensional* if $K \subseteq L_1 \times \cdots \times L_n$, where each L_i is a compact totally ordered space (called briefly a *compact line*). Note that every compact totally ordered space is a continuous image of a 0-dimensional one, therefore we can restrict attention to 0-dimensional lattices.

A result from [2] says that $C(K)$ has a pointwise Kadets renorming (which implies **SLD**) whenever K is a product of compact lines. Up to now, it is not known whether the same result holds for K being a closed sublattice of a finite product of compact lines. This question remains open, however we prove a weaker result concerning the **SLD** property.

From now on L_i, $1 \leq i \leq n$, will stand for totally ordered spaces that are zero–dimensional compact for their order topology. Let $\leq$ be the *product order* in $\prod_{i=1}^n L_i$, i.e. $(a_i)_{i=1}^n \geq (b_i)_{i=1}^n$ whenever $a_i \geq b_i$ for any $i \leq n$; this space endowed with it and the product topology is a zero–dimensional compact distributive lattice.

Until the the end of the proof of Proposition 4.7 we assume that K is a fixed sublattice of $\prod_{i=1}^n L_i$, endowed with a topology $\mathcal{T}$ for which it is a compact distributive lattice that, by compactness, must be the subspace topology of $\prod_{i=1}^n L_i$.

Given $x = (x_i)_{i=1}^n \in \prod_{i=1}^n L_i$ let $\pi_i(x) = x_i$, and $\varrho_i(x) = (x_j)_{j \neq i}$, $1 \leq i \leq n$. If $1 \leq i \leq n$, and $p \in \mathbb{P}(L_i)$ we write $\tilde{p} := \left(\prod_{j<i} L_j \times p \times \prod_{j>i} L_j\right) \cap K$. It is clear that $\tilde{p} \in \mathbb{P}(K)$.

Theorem 4.1. *$C(K)$ has the property* **SLD** *for the pointwise topology.*

It will be deduced from some results.

Lemma 4.2. *For $1 \leq i \leq n$ let p, $q \in \mathbb{P}(L_i)$ with $p \supseteq q$, if $\langle a, b \rangle$ is a gate between $\langle \tilde{p}, \tilde{q} \rangle$ we have $\pi_i(a) < \pi_i(b)$; and for any $c \in K$ we have $c \notin \tilde{q} \cup (K \setminus \tilde{p})$ whenever $\pi_i(a) < \pi_i(c) < \pi_i(b)$. In particular $K \setminus \tilde{p} = \{x \in K : \pi_i(x) \leq \pi_i(a)\}$ and $\tilde{q} = \{x \in K : \pi_i(x) \geq \pi_i(b)\}$. Moreover for any $c \in K$ we have*

(1) *if* $\pi_i(c) \geq \pi_i(b)$ and $\pi_j(a) < \pi_j(b)$ *then* $\pi_j(c) \geq \pi_j(b)$ $1 \leq j \leq n$.

Proof. For the first part, from $\pi_i((a \vee c) \wedge b) = \pi_i(c)$ we get $(a \vee c) \wedge b \in [a, b] \setminus \{a, b\}$. Then, since $\langle a, b\rangle$ is a gate we have $(a \vee c) \wedge b \notin \tilde{q} \cup (K \setminus \tilde{p})$. But $\tilde{q}$ is a filter so $c \notin \tilde{q}$. Similar arguments show that $c \notin K \setminus \tilde{p}$.

To show (1) observe that since $\langle a, b\rangle$ is a gate we have $a < b$. Suppose that c and j witness that (1) is false. We have $c \in \tilde{q}$ and $a \vee (b \wedge c) \in \tilde{q}$. Since $\langle a, b\rangle$ is a gate we have $a \vee (b \wedge c) = b$, so $\pi_j(b) = \pi_j(a) \vee (\pi_j(b) \wedge \pi_j(c))) = \pi_j(a) \vee \pi_j(c)$ which contradicts that $\pi_j(a) < \pi_j(b)$ and $\pi_j(c) < \pi_j(b)$. □

Lemma 4.3. *Let a, $b \in K$ be such that $\{x \in K \cap [a, b] : \ \pi_i(a) < \pi_i(x) < \pi_i(b)\} = \emptyset$, then $\{x \in K : \ \pi_i(a) < \pi_i(x) < \pi_i(b)\} = \emptyset$.*

Proof. Since $(a \vee x) \wedge b \in [a, b]$ for any $x \in K$ and $\pi_i((a \vee x) \wedge b) = \pi_i(x)$ the assertion follows. □

If $1 \leq i \leq n$, p, $q \in \mathbb{P}(L_i)$ with $p \supseteq q$, $\langle a, b\rangle$ is a gate between $\langle \tilde{p}, \tilde{q}\rangle$ and a, b belong to a subset $S \subseteq K$ we will say that f (i, ε)*–leaps* (resp. *jumps*) at $\langle \tilde{p}, \tilde{q}\rangle$, or at $\langle a, b\rangle$ or within S whenever it ε–leaps (resp. jumps) at $\langle \tilde{p}, \tilde{q}\rangle$.

Definition 4.4. *Given $f \in C(K)$, $1 \leq i \leq n$, and $\varepsilon > 0$ we will say that f (i, ε)–*leaps *m* times *if there exists $\{\langle p_{i,j}, q_{i,j}\rangle\}_{j=1}^m$, $p_{i,j}$, $q_{i,j} \in \mathbb{P}(L_i)$ such that $\tilde{p}_{i,1} \supseteq \tilde{q}_{i,1} \supsetneqq \tilde{p}_{i,2} \supseteq \tilde{q}_{i,2} \supsetneqq \cdots \supsetneqq \tilde{p}_{i,m} \supseteq \tilde{q}_{i,m}$; and*

(i) f ε–leaps at each $(\tilde{p}_{i,j}, \tilde{q}_{i,j})$, $1 \leq j \leq m$;

(ii) if f ε–jumps at $\tilde{p}$ for some $p \in \mathbb{P}(L_i)$, $1 \leq i \leq n$, then there exists j, $1 \leq j \leq m$, such that $p = p_{i,j} = q_{i,j}$.

From Lemma 4.6 below it follows that this definition makes sense.

Remark 4.5. According to Lemma 4.2 given $1 \leq i \leq n$ and p_k, $q_k \in \mathbb{P}(L_i)$, $k = 1$, 2, such that $\tilde{p}_1 \supseteq \tilde{q}_1 \supsetneqq \tilde{p}_2 \supseteq \tilde{q}_2$, and $\langle a_j, b_j\rangle$ gates between $\langle \tilde{p}_j, \tilde{q}_j\rangle$, $j = 1$, 2, we have $\pi_i(a_1) < \pi_i(b_1) \leq \pi_i(a_2)$.

Lemma 4.6. *Given $f \in C(K)$ and $\varepsilon > 0$ the set of all $p \in \mathbb{P}(L_i)$ such that f (i, ε)–jumps at $\tilde{p}$ is finite and so is the set of all $m \in \mathbb{N}$ such that f (i, ε)–leaps m–times.*

Proof. By compactness for any $1 \leq k \leq n$ it is possible to choose $\{\alpha_{k,j}\}_{j=0}^{\ell_k}$, $\alpha_{k,j} \in L_k$, such that the $[\alpha_{k,j-1}, \alpha_{k,j}]$'s cover L_k and the oscillation of f on each $K \cap \prod_{k=1}^n [\alpha_{k,j-1}, \alpha_{k,j}]$ is strictly less than ε; those sets cover K. Let p_m, $q_m \in \mathbb{P}(L_i)$, $m = 1$, 2, such that f (i, ε)–leaps at each $(\tilde{p}_m, \tilde{q}_m)$, with $\tilde{p}_1 \supset \tilde{q}_1 \supsetneqq \tilde{p}_2 \supset \tilde{q}_2$, and this is witnessed by the gates $\langle a_m, b_m\rangle$, $m = 1$, 2. We have that a_m and b_m cannot be in the same $\prod_{k=1}^n [\alpha_{k,j-1}, \alpha_{k,j}]$. To finish the proof it is enough to apply Remark 4.5 and (1). □

Fix $f \in C(K)$ and $\varepsilon > 0$ until the end of the proof of Proposition 4.7. Given $1 \leq i \leq n$ let $m_i = m_i(f)$ the maximum number of times that f (i,ε)–leaps, so let $\{\langle p_{i,j}, q_{i,j}\rangle\}_{j=1}^{m_i}$ satisfying (i) and (ii) of Definition 4.4. Let $C_k := \prod_{i=1}^n [a_{k,i}, b_{k,i}] \cap K$, $1 \leq k \leq \ell$, $a_{k,i}$, $b_{k,i} \in L_i$, any covering of K made up by sets such that each C_k is either included or disjoint of every one of the sets $K \setminus \tilde{p}_{i,j}$, $\tilde{q}_{i,j}$, $1 \leq j \leq m_i$, $1 \leq i \leq n$. From now on we will fix k and, for simplicity, we will write $C_k = \prod_{i=1}^n [a_i, b_i] \cap K$. Let the sequence $p(n)$ defined inductively by $p(1) = 3$, $p(n+1) = 8p(n) + 6$, $n \in \mathbb{N}$.

Proposition 4.7. $\operatorname{osc}(f, C_k) \leq p(n)\varepsilon$.

It will be a consequence of some lemmata.

Lemma 4.8. *Given u, $v \in K$, $1 \leq i \leq n$, U, V open sets in L_i with $|f(u) - f(v)| > \varepsilon$, $\varrho_i(u) = \varrho_i(v)$, $\pi_i(u) < \pi_i(v)$ and $\pi_i(u) \in U$, $\pi_i(v) \in V$, there exist a, $b \in [u,v] \cap K$ with $\pi_i(a) \in U$, $\pi_i(b) \in V$, $\pi_i(a) < \pi_i(b)$, such that $\langle a, b\rangle$ witnesses an (i,ε)–leap of f.*

Proof. Since $S := \{x \in K : \ \varrho_i(x) = \varrho_i(u)\}$ is a 0–dimensional totally ordered compact space, the right (resp. left) isolated points are dense in it. From the continuity of f we get two open sets U, V in L_i and a gate $\langle a, b\rangle$ that witnesses an ε–leap of the restriction of f to S. Lemma 4.3 shows that $\langle a, b\rangle$ is a gate in K too. □

Lemma 4.9. *Let a, $b \in C_k$, if $M := \max f([a,b] \cap K)$ and $m := \min f([a,b] \cap K)$ then $[M, m] \setminus f([a,b] \cap K)$ contains no interval of length bigger that ε.*

Proof. By contradiction let r, $s \in \mathbb{R}$ such that $r < s$, $s - r > \varepsilon$ and $[r,s] \cap f([a,b] \cap K) = \{r, s\}$; We must have that $D := \{i : \ 1 \leq i \leq n, \ \pi_i(a) \neq \pi_i(b)\}$ is non empty. We argue by induction over $\ell := \operatorname{card} D$. Let v, $w \in [a,b] \cap K$, such that $f(v) = s$, $f(w) = r$. When $\ell = 1$ the set $[v, w]$ is totally ordered. If, for instance, $v < w$, set $v_0 := \sup\{y \in [v,w] \cap K : \ f(y) \geq s\}$ and $w_0 := \inf\{y \in [v_0, w] \cap K : \ f(y) \leq r\}$. Thus $f(v_0) - f(w_0) \geq s - r > \varepsilon$ and $[v_0, w_0] \cap K = \{v_0, w_0\}$. Lemma 4.3 shows that f (i,ε)–jumps at (v_0, w_0). A contradiction.

Assume that the assertion holds for k, $1 \leq k < n$, and $\ell = k + 1$. Fix i, $1 \leq i \leq n$, such that $\pi_i(v) \neq \pi_i(w)$, and, for instance, $\pi_i(v) < \pi_i(w)$. Now set v_0 such that $\pi_i(v_0) = \sup\{\pi_i(y) : \ y \in [v,w] \cap K, \ f(y) \geq s\}$ and w_0 such that $\pi_i(w_0) = \inf\{\pi_i(y) : y \in [v_0, w] \cap K, \ f(y) \leq r\}$. From the choice of (v_0, w_0) and Lemma 4.3 if follows that $\{x \in K : \ \pi_i(x) \leq \pi_i(v_0)\}$ and $\{x \in K : \ \pi_i(x) \geq \pi_i(w_0)\}$ are a proper clopen ideal and a filter respectively, whose union is K. On the other hand the sets

$$\{x \in [v_0, w_0] \cap K : \ \pi_i(x) = \pi_i(v_0)\} \text{ and } \{x \in [v_0, w_0] \cap K : \ \pi_i(x) = \pi_i(w_0)\}$$

are nonempty disjoint closed convex subsets of K, then there exists a gate $\langle \alpha, \beta\rangle$ between them, it is clear that it must be a gate between the clopen ideal and filter above and $\pi_i(\alpha) = \pi_i(v_0)$, $\pi_i(\beta) = \pi_i(w_0)$. We get that the cardinal of the set $\{j : \ 1 \leq j \leq n, \ \pi_j(\alpha) \neq \pi_j(v_0)\}$ is not bigger than k so, according to the inductive hypothesis applied to $[\alpha, v_0]$ we have $f(\alpha) \geq s$; the same argument shows that $f(\beta) \leq r$. Then we have found an ε–jump which is a contradiction. □

Proof of Proposition 4.7. If a, $b \in C_k$ and $\{i: \ 1 \le i \le n, \pi_i(a) \ne \pi_i(b)\}$ has cardinal k, we will show by induction on k that $|f(a) - f(b)| < p(k)\varepsilon$. Assume that $k = 1$. By contradiction let $1 \le i \le n$ with $\varrho_i(a) = \varrho_i(b)$ and $|f(a) - f(b)| > 3\varepsilon$. According to Lemma 4.9 there exists $\xi \in [a, b] \cap K$ such that $|(1/2)(f(a) + f(b)) - f(\xi)| < \varepsilon$. Then $|f(a) - f(\xi)| > \varepsilon$ and $|f(b) - f(\xi)| > \varepsilon$. From Lemma 4.8 we get two different (i, ε)–leaps. A contradiction.

Assume that the assertion holds for k, $1 \le k < n$, and $\ell = k + 1$. By contradiction suppose that $|f(a) - f(b)| > p(k + 1)\varepsilon$ and $\ell = k + 1$. Since $p(k + 1)\varepsilon < |f(a) - f(b)| \le |f(a) - f(a \vee b)| + |f(a \vee b) - f(b)|$ we may assume that $a < b$ and $|f(a) - f(b)| > (1/2)p(k + 1)\varepsilon$. From Lemma 4.9 we get $c \in [a, b] \cap K$ such that $|f(a) - f(c)| > (1/2)((1/2)p(k+1) - 1)\varepsilon$ and $|f(c) - f(b)| > (1/2)((1/2)p(k+1) - 1)\varepsilon$. We claim that this implies that f (i, ε)–leaps in $[a, c]$ and in $[c, b]$ which contradicts the maximality of m_i. Let us show it in $[a, c]$. Since L_i is 0–dimensional, the continuity of f allows us to assume that $\pi_i(a)$ (resp. $\pi_i(c)$) is right (resp. left) isolated. Then $\{x \in [a, b]: \ \pi_i(x) = \pi_i(a)\}$ and $\{x \in [a, b]: \ \pi_i(x) = \pi_i(c)\}$ are nonempty clopen convex sets; let $\langle u, v\rangle$ a gate between them, that should be a gate between the clopen ideal and filter $\{x \in [a, b]: \ \pi_i(x) \le \pi_i(a)\}$ and $\{x \in [a, c]: \ \pi_i(x) \ge \pi_i(c)\}$ too. It is clear that $\pi_i(u) = \pi_i(a)$, $\pi_i(v) = \pi_i(c)$ and the cardinal of the sets $\{j: \ 1 \le j \le n, \ \pi_j(u) \ne \pi_j(a)\}$ and $\{j: \ 1 \le j \le n, \ \pi_j(v) \ne \pi_j(c)\}$ are not bigger than k so, according to the inductive hypothesis $|f(a) - f(u)| \le p(k)\varepsilon$ and $|f(v) - f(c)| \le p(k)\varepsilon$. Then $|f(u) - f(v)| > ((1/2)((1/2)p(k+1) - 1) - 2p(k))\varepsilon \ge \varepsilon$ so $\langle u, v\rangle$ witnesses an (i, ε)–leap. □

Proof of Theorem 4.1 Given $\varepsilon > 0$ we can write $C(K) = \bigcup_{p=1}^{\infty} C_p$ in such a way that for any $p \in \mathbb{N}$ there exist m_i, $1 \le i \le n$ such that $m_i(f) = m_i$, $1 \le i \le n$, for al $f \in C_p$. For a $f \in C_p$ these leaps are witnessed by filters and gates $\langle a_{i,j}, b_{i,j}\rangle$, $1 \le j \le m_i$, $1 \le i \le n$, with $|f(a_{i,j}) - f(b_{i,j})| > \varepsilon$. According to Proposition 4.7 if

$$g \in \{h \in C_p: \ |h(a_{i,j}) - h(b_{i,j})| > \varepsilon, \ 1 \le j \le m_i, 1 \le i \le n\}$$

then $\operatorname{osc}(g, K \cap \prod_{i=1}^{n}[a_i, b_i]) \le p(n)\varepsilon$. To finish apply [9, Theorem 4.(iii)]. □

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