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Means on scattered compacta

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MEANS ON SCATTERED COMPACTA

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Abstract. We prove that a separable Hausdorff topological space X containing a cocountable subset homeomorphic to $[0, \omega_1]$ admits no separately continuous mean operation and no diagonally continuous n-mean for $n \geq 2$.

In this paper we construct a scattered compact space admitting no continuous mean operation, thus answering Problem 5 of [4]. By a *mean operation* on a set X we understand any binary operation $\mu : X \times X \to X$ such that $\mu(x, x) = x$ and $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$. If, in addition, the mean operation is associative, then it is called a *semilattice operation*.

The mean operation is a partial case of an n-mean operation. A function $\mu : X^n \to X$ defined on the nth power of a space X is called an *n-mean operation* (or briefly an *n-mean*) if

(1) $\mu(x, \dots, x) = x$ for every $x \in X$ and
(2) μ is *S_n-invariant* in the sense that $\mu(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \mu(x_1, \dots, x_n)$ for any permutation σ of the set $\{1, \dots, n\}$ and any vector $(x_1, \dots, x_n) \in X^n$.

It is clear that a mean is the same as a 2-mean.

The problem of detecting topological spaces with (or without) a continuous mean is classical in Algebraic Topology, see [1], [2], [3], [6], [7], [10]. It particular, due to Aumann [1], we know that for every $n \geq 1$ the n-dimensional sphere admits no continuous mean. On the other hand, the 0-dimension sphere $S^0 = \{-1, 1\}$ trivially possesses such a mean. More generally, each zero-dimensional metrizable separable space, being homeomorphic to a subspace of the real line, admits a continuous semilattice operation.

On the other hand, there are non-metrizable scattered compact Hausdorff spaces admitting no separately continuous semilattice operation. The simplest example is the compactification $\gamma\mathbb{N}$ of the discrete space $\mathbb{N}$ of natural numbers whose remainder $\gamma\mathbb{N} \setminus \mathbb{N}$ is homeomorphic to the ordinal segment $[0, \omega_1]$. The existence of such a compactification $\gamma\mathbb{N}$ follows from the famous Parovichenko theorem [9] (saying that any compact space of weight $\leq \aleph_1$ is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$).

Another way to construct $\gamma\mathbb{N}$ is as follows. Consider a family $\mathcal{A} = (A_\alpha)_{\alpha<\omega_1}$ of infinite subsets of $\mathbb{N}$ such that $A_\alpha \subset^* A_\beta$ for any ordinals $\alpha < \beta$. The almost inclusion $A_\alpha \subset^* A_\beta$ means that $A_\alpha \setminus A_\beta$ is finite. Now, consider the subalgebra B of $\mathcal{P}(\mathbb{N})$ generated by $\mathcal{A} \cup \big\{\{n\}\big\}_{n\in\mathbb{N}}$. Then $\gamma\mathbb{N}$ is the space of ultrafilters on B.

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A bit stronger notion than the separate continuity is the diagonal continuity. A function $f : X^n \to Y$ is called *diagonally continuous* if for any map $g = (g_i)_{i=1}^n : X \to X^n$ whose components $g_i : X \to X$, $1 \le i \le n$, are constant or identity functions the composition $f \circ g : X \to Y$ is continuous. It is clear that for a function $f : X^n \to Y$ we get the implications:

$$\text{continuous} \Rightarrow \text{diagonally continuous} \Rightarrow \text{separately continuous}.$$

A subset A of a set X is called *cocountable* if its complement $X \setminus A$ is at most countable. The following theorem is the main result of this paper.

Theorem 1. *If a separable Hausdorff topological space X contains a cocountable subset homeomorphic to $[0, \omega_1]$, then for every $n \ge 2$ the space X admits no diagonally continuous n-mean $\mu : X^n \to X$.*

Proof. This theorem will be proved by induction on $n \ge 2$. More precisely, by induction we shall prove that X admits no diagonally continuous n-amean. A function $\mu : X^n \to X$ will be called an *almost n-mean operation* (briefly, an *n-amean*) if μ is S_n-invariant and the set $\{x \in X : x \ne \mu(x, \dots, x)\}$ is at most countable.

Since the space X is separable, we can assume that $[\omega, \omega_1] \subset X$ has countable dense complement $D = X \setminus [\omega, \omega_1]$ that we denote by ω. So $X = \omega \cup [\omega, \omega_1) \cup \{\omega_1\}$.

The following lemma will allow us to start the inductive proof of the theorem.

Lemma 1. *The space X admits no separately continuous 2-amean.*

Proof. Assume that $\mu : X^2 \to X$ is a separately continuous 2-amean on X.

Given two points $a, b \in X$ consider the closed subsets

$$b/a = \{x \in [\omega, \omega_1) : b = \mu(a, x)\} \quad \text{and} \quad {\downarrow}b = \{x \in [\omega, \omega_1) : \mu(b, x) = x\}$$

of $[\omega, \omega_1) \subset X$. Let

$$A = \{(a, b) \in \omega^2 : |b/a| = \aleph_1\} \text{ and } B = \{b \in \omega : |{\downarrow}b| = \aleph_1\}.$$

Find an ordinal $\alpha_0 \in [\omega, \omega_1) \subset X$ such that

- $\mu(\alpha, \alpha) = \alpha$ for all $\alpha \ge \alpha_0$;
- $b/a \subset [\omega, \alpha_0)$ for all $(a, b) \in \omega^2 \setminus A$;
- ${\downarrow}b \subset [\omega, \alpha_0)$ for all $b \in \omega \setminus B$.

If the set B has countable closure $\bar{B}$ in X, then we will additionally assume that $\bar{B} \cap [\omega, \omega_1) \subset [\omega, \alpha_0)$.

Consider the closed unbounded subset

$$C = [\alpha_0, \omega_1) \cap \bigcap_{(a,b) \in A} b/a \ \cap \bigcap_{b \in B} {\downarrow}b$$

in $[\omega, \omega_1)$ and also the open subset

$$W = \{x \in (\alpha_0, \omega_1) : \exists c \in C \ \ \mu(c, x) \ne x\}$$

of $[0, \omega_1)$. Observe that $W \supset C \setminus C_0$ where $C_0 = \{c \in C : \forall x \in C \ \mu(x, c) = c\}$ is a subset of C containing at most one point. So, W is uncountable.

Let W_0 stand for the dense open subset of W consisting of isolated points of W.

Claim 1. *Any point $\alpha \in W_0$ has a neighborhood $V_\alpha \subset X$ such that $\mu\big(\{\alpha\} \times V_\alpha\big) = \{\alpha\}$.*

Proof. Using the definition of W, find $c \in C$ with $\mu(c,\alpha) \neq \alpha$. Choose disjoint neighborhoods $U_{\mu(c,\alpha)}, U_\alpha \subset X$ of the points $\mu(c,\alpha)$ and α. Replacing U_α by a smaller neighborhood we can assume that $\mu(\{c\} \times U_\alpha) \subset U_{\mu(c,\alpha)}$ and $U_\alpha \cap [\omega,\omega_1] = \{\alpha\}$. Finally, by the separate continuity of the operation μ, find a neighborhood $V_\alpha \subset U_\alpha$ such that $\mu(\{\alpha\} \times V_\alpha) \subset U_\alpha$. We claim that $\mu(\alpha, a) = \alpha$ for all $a \in V_\alpha$. This is clear if $a = \alpha$. If $a \neq \alpha$, then $a \in \omega$ because $V_\alpha \cap [\omega,\omega_1] = \{\alpha\}$. If $b = \mu(a,\alpha) \in \omega$, then $\alpha_0 < \alpha \in b/a$ and consequently, $(a,b) \in A$. It follows from $c \in C$ and $(a,b) \in A$ that $c \in b/a$, which means that $\mu(a,c) = b$. The latter equality cannot hold because $\mu(c,a) \in \mu(\{c\} \times V_\alpha) \in U_{\mu(c,\alpha)}$ while $b = \mu(\alpha,a) \in \mu(\{\alpha\} \times V_\alpha) \subset U_\alpha$. This contradiction shows that $b = \mu(a,\alpha) \in [\omega,\omega_1] \cap U_\alpha = \{\alpha\}$ and hence $\mu(\alpha,a) = \mu(a,\alpha) = \alpha$. □

Claim 2. *The set B has uncountable closure $\bar{B}$ in X.*

Proof. Assuming that $\bar{B}$ is countable, we get $\bar{B} \cap [\omega,\omega_1) \subset [\omega,\alpha_0)$ by the choice of α_0. By Claim 1, each ordinal $\alpha \in W_0$ has a neighborhood $V_\alpha \subset X$ such that $\mu(\{\alpha\} \times V_\alpha) = \{\alpha\}$. Since $\alpha \notin \bar{B}$ and the set ω is dense in X, we can pick a point $v_\alpha \in \omega \cap V_\alpha \setminus \bar{B}$. By the Dirichlet Principle, for some point $v \in \omega$ the set $W_v = \{\alpha \in W_0 : v_\alpha = v\}$ is uncountable. It follows that $\mu(\alpha, v) = \mu(\alpha, v_\alpha) = \alpha$ for every $\alpha \in W_v$. Consequently, $v \in B$, which contradicts the choice of $v = v_\alpha \notin \bar{B}$ for $\alpha \in W_v$. □

Observe that for any $c \in C$ and any $b \in B$ we get $\mu(c,b) = c$. By the separate continuity of the amean μ, we get $\mu(c,b) = c$ for all $b \in \bar{B}$. Since C and $\bar{B} \cap [\omega,\omega_1)$ are closed uncountable subsets of $[0,\omega_1)$ the intersection $C \cap \bar{B}$ is uncountable and thus we can chose two distinct points $x, y \in C \cap \bar{B}$, for which we get $x = \mu(x,y) = \mu(y,x) = y$, which is a desired contradiction completing the proof of Lemma 1. □

The inductive step of the inductive proof of Theorem 1 is fulfilled in the following lemma.

Lemma 2. *If for some $n \geq 2$ the space X admits no diagonally continuous n-amean, then it admits no diagonally continuous $(n+1)$-amean.*

Proof. To derive a contradiction, assume that X admits a diagonally continuous $(n+1)$-amean $\mu : X^{n+1} \to X$.

For points $\vec{a} \in X^n$ and $b \in X$ consider the closed subsets

$$b/\vec{a} = \{x \in [\omega,\omega_1) : b = \mu(\vec{a}, x)\} \quad \text{and} \quad {\downarrow}\vec{a} = \{x \in [\omega,\omega_1) : \mu(\vec{a}, x) = x\}$$

of $[\omega,\omega_1)$. Let

$$A = \{(\vec{a}, b) \in \omega^n \times \omega : |b/\vec{a}| = \aleph_1\} \quad \text{and} \quad B = \{\vec{b} \in \omega^n : |{\downarrow}\vec{b}| = \aleph_1\}.$$

Find a countable ordinal $\alpha_0 \in [\omega,\omega_1)$ such that

- $\mu(\alpha,\dots,\alpha) = \alpha$ for every $\alpha \in [\alpha_0,\omega_1)$;
- $b/\vec{a} \subset [\omega,\alpha_0)$ for every $(\vec{a}, b) \in (\omega^n \times \omega) \setminus A$, and
- ${\downarrow}\vec{b} \subset [\omega,\alpha_0)$ for every $\vec{b} \in \omega^n \setminus B$.

It follows that

$$C = [\alpha_0,\omega_1) \cap \Big(\bigcap_{(\vec{a},b) \in A} b/\vec{a}\Big) \cap \Big(\bigcap_{\vec{b} \in B} {\downarrow}\vec{b}\Big)$$

is a closed unbounded subset of $[\omega, \omega_1)$.

Since the space X admits no diagonally continuous n-amean, the set

$$W = \{\alpha \in [\alpha_0, \omega_1) : \mu(\alpha, \dots, \alpha, \omega_1) \neq \alpha\}$$

is uncountable (in the opposite case the function $\nu : X^n \to X$, $\nu : (x_1, \dots, x_n) \mapsto \mu(x_1, \dots, x_n, \omega_1)$, is a diagonally continuous n-amean on X, which does not exist according to our assumption).

The diagonal continuity of the function μ guarantees that the set W is open in $[\alpha_0, \omega_1)$. Consequently, the set W_0 of all isolated points of W is uncountable too.

Claim 3. *Each point $\alpha \in W_0$ has a neighborhood $V_\alpha \subset X$ such that for any point $x \in \omega \cap V_\alpha$ there is a neighborhood $V'_\alpha \subset X$ of α such that $\mu\big(\{x\}^{n-1} \times \{V'_\alpha\} \times \{\alpha\}\big) = \{\alpha\}$.*

Proof. By the definition of $W \supset W_0 \ni \alpha$, the point $z = \mu(\alpha, \dots, \alpha, \omega_1)$ differs from α, which allows us to choose disjoint open neighborhoods U_z and U_α of the points z and α in X, respectively. Since α is an isolated point of $[\omega, \omega_1]$, we can additionally assume that $U_\alpha \cap [\omega, \omega_1] \subset \{\alpha\}$. It follows from $\alpha \geq \alpha_0$ that $\mu(\alpha, \dots, \alpha) = \alpha$. The diagonal continuity of the operation μ yields a neighborhood $V_\alpha \subset X$ of α such that for any $x \in \omega \cap V_\alpha$ we get $\mu(x, \dots, x, \alpha, \alpha) \in U_\alpha$ and $\mu(x, \dots, x, \alpha, \omega_1) \in U_z$. For every $x \in V_\alpha$ the separate continuity of μ yields a neighborhood $V'_\alpha \subset X$ of α such that for every $y \in V'_\alpha$ we get $\mu(x, \dots, x, y, \alpha) \in U_\alpha$ and $\mu(x, \dots, x, y, \omega_1) \in U_z$. Choose any $y \in V'_\alpha \cap \omega$. We claim that the point $u = \mu(x, \dots, x, y, \alpha) \in U_\alpha$ belongs to $[\omega, \omega_1]$. Assuming the converse, we conclude that $((x, \dots, x, y), u) \in A$ and hence $\mu(x, \dots, x, y, c) = u$ for all $c \in C$. On the other hand, the separate continuity of μ and the inclusion $\mu(x, \dots, x, y, \omega_1) \in U_z$ yields a point $c \in C$ with $\mu(x, \dots, x, y, c) \in U_z$. Then $u = \mu(x, \dots, x, y, c) \in U_z \cap U_\alpha = \emptyset$, which is a desired contradiction showing that $\mu(x, \dots, x, y, \alpha) = u \in [\omega, \omega_1] \cap U_\alpha = \{\alpha\}$. □

Claim 4. *There is a point $x \in \omega$ such that the set*

$$B(x) = \{y \in [\omega, \omega_1) : \forall c \in C \;\; \mu(x, \dots, x, y, c) = c\}$$

is uncountable.

Proof. Assume conversely that for every $x \in \omega$ the set $B(x)$ is at most countable. Then we can find an ordinal $\beta \in [\alpha_0, \omega_1)$ such that $[\beta, \omega_1) \cap \bigcup_{x \in \omega} B(x) = \emptyset$. By Claim 3, every ordinal $\alpha \in W_0 \cap [\beta, \omega_1)$ has a neighborhood $V_\alpha \subset X$ such that for each point $v \in \omega \cap V_\alpha$ there is a neighborhood $V'_\alpha \subset X$ of α such that $\mu(\{v\}^{n-1} \times V'_\alpha \times \{\alpha\}) = \{\alpha\}$. For every ordinal $\alpha \in W_0 \cap [\beta, \omega_1)$ choose a point $v_\alpha \in \omega \cap V_\alpha$. By the Dirichlet Principle, for some point $v \in \omega$ the set $W_v = \{\alpha \in W_0 \cap [\beta, \omega_1) : v_\alpha = v\}$ is uncountable. So, we can choose an ordinal $\alpha \in W_v \setminus B(v)$. For the ordinal α and the point $v = v_\alpha \in V_\alpha$ there is a neighborhood $V'_\alpha \subset X$ of α such that $\mu(\{v\}^{n-1} \times V'_\alpha \times \{\alpha\}) = \{\alpha\}$.

Since the set $B(v)$ is closed (by the separate continuity of μ) and does not contain α, we can choose a point $y \in \omega \cap V'_\alpha \setminus B(v)$. For this point y we get $\mu(v, \dots, v, y, \alpha) = \alpha \geq \alpha_0$, which implies $(v, \dots, v, y) \in B$ and $\mu(v, \dots, v, y, c) = c$ for all $c \in C$. The latter means that $y \in B(v)$, which contradicts the choice of y. □

By Claim 4, for some $x \in \omega$ the closed set $B(x)$ is uncountable. Then $C \cap B(x)$ is a closed unbounded set in $[\omega, \omega_1)$, which allows us to find two distinct points

$y, c \in C \cap B(x)$. For these points by the S_{n+1}-invariance of μ we get

$$c = \mu(v, \dots, v, y, c) = \mu(v, \dots, v, c, y) = y,$$

which is a desired contradiction, completing the proof of Lemma 2. □

By induction, Lemmas 1 and 2 imply that for every $n \geq 2$ the space X admits no diagonally continuous n-amean and hence no diagonally continuous n-mean. □

Since each separately continuous mean $\mu : X^2 \to X$ is diagonally continuous, (the proof of) Lemma 1 implies the following corollary answering Problem 5 in [4].

Corollary 1. *If a separable Hausdorff topological space X contains a cocountable subset homeomorphic to $[0, \omega_1)$, then X admits no separately continuous mean $\mu : X^2 \to X$.*

Problem 1. *Let X be a separable Hausdorff topological space X containing a cocountable subset homeomorphic to $[0, \omega_1)$. Does X admit a separately continuous n-mean $\mu : X^n \to X$ for some $n \geq 3$?*

By the *n-th symmetric power* $SP^n(X)$ of a topological space X we understand the quotient space of X^n by the equivalence relation $\sim$: $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ if there is a permutation σ of $\{1, \dots, n\}$ such that $(y_1, \dots, y_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. The space X is identified with the subspace $\big\{\{(x, \dots, x)\} : x \in X\big\}$ of $SP^n(X)$.

Observe that X is a retract of its nth symmetric power $SP^n(X)$ if and only if X admits a continuous n-mean. This observation combined with Theorem 1 implies:

Corollary 2. *If a separable Hausdorff topological space X contains a cocountable subset homeomorphic to $[0, \omega_1]$, then for every $n \geq 2$ the space X is not a retract of its n-th symmetric power $SP^n(X)$.*

The n-th symmetric power $SP^n(X)$ is a partial case of the n-th G-symmetric power $SP^n_G(X)$ where G is a subgroup of the symmetric group S_n. The space $SP^n_G(X)$ is the quotient space of X^n by the equivalence relation $\sim_G$: $(x_1, \dots, x_n) \sim_G (y_1, \dots, y_n)$ if there is a permutation $\sigma \in G$ of $\{1, \dots, n\}$ such that $(y_1, \dots, y_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. The space X is identified with the subspace $\big\{\{(x, \dots, x)\} : x \in X\big\}$ of $SP^n_G(X)$.

Problem 2. *Let X be a separable compact space containing a cocountable subset homeomorphic to $[0, \omega_1]$. Is X a retract of $SP^n_G(X)$ for some $n \geq 2$ and some non-trivial subgroup $G \subset S_n$?*

Let us recall that a topological space X is called *scattered* if each subspace $A \subset X$ has an isolated point.

Problem 3. *Assume that a scattered compact space X admits a continuous n-mean for some $n \geq 2$. Does X admit a continuous n-mean for every $n \geq 2$?*

If $\vee : X \times X \to X$ is a semilattice operation on a set X, then for every $n \geq 2$ the map $\mu : X^n \to X$, $\mu(x_1, \dots, x_n) = x_1 \vee \dots \vee x_n$ is an n-mean on X. So, a topological space admitting a continuous semilattice operation admits continuous n-means for all $n \geq 2$.

Problem 4. *Assume that a scattered compact space X admits a continuous n-mean for every $n \geq 2$. Does X admit a continuous semilattice operation?*

It is known that each separately continuous semilattice operation on a zero-dimensional compact space is jointly continuous, see [8, II.1.5].

Problem 5. *Assume a scattered compact space X admits a separately continuous n-mean. Does X admit a continuous n-mean?*

Problem 6. *Is a normal functor F a power functor if each (scattered) compact space X is a retract of $F(X)$?*

According to [4] and [5], another example of a scattered compact space admitting no separately continuous semilattice operation is the (one-point compactification of the) Mrówka space $\psi\mathbb{N}$. By definition, the *Mrówka space* is the Stone space of the Boolean algebra generated by $\mathcal{A}\cup\big\{\{n\}\big\}_{n\in\mathbb{N}}$ for some maximal almost disjoint family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$.

Problem 7. *Does the Mrówka space $\psi\mathbb{N}$ admit a (separately) continuous n-mean for some $n \geq 2$?*

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